Scuola Internazionale Superiore
di Studi Avanzati
Area of Mathematics
Ph.D. in Geometry

The Hodge-Deligne polynomials of some moduli spaces of coherent systems

Supervisor:
Emeritus Professor
Peter E. Newstead

Candidate:
Matteo Tommasini

October 25, 2012

Thesis submitted in partial fulfillment of the requirements for the degree of Doctor Philosophiæ

Academic Year 2011/2012

Per Elisa

Oh, and if I ever want proof
Then I find it in you;
Oh, yeah I honestly do,
In you I find proof.

Coldplay, “Proof”

There was a moment's expectant pause whilst panels slowly came to life on the front of the console. Lights flashed on and off experimentally and settled down into a businesslike pattern. A soft low hum came from the communication channel.

"Good morning," said Deep Thought at last.

"Er...Good morning, O Deep Thought," said Loonquawl nervously, "do you have...er, that is..."

"An answer for you?" interrupted Deep Thought majestically. "Yes. I have."

The two men shivered with expectancy. Their waiting had not been in vain.

"There really is one?" breathed Phouchg.

"There really is one," confirmed Deep Thought.

"To Everything? To the great Question of Life, the Universe and Everything?"

"Yes."

Both of the men had been trained for this moment, their lives had been a preparation for it, they had been selected at birth as those who would witness the answer, but even so they found themselves gasping and squirming like excited children.

"And you're ready to give it to us?" urged Loonquawl.

"I am."

"Now?"

"Now," said Deep Thought.

They both licked their dry lips.

"Though I don't think," added Deep Thought, "that you're going to like it."

"Doesn't matter!" said Phouchg. "We must know it! Now!"

"Now?" inquired Deep Thought.

"Yes! Now..."

"All right," said the computer and settled into silence again. The two men fidgeted. The tension was unbearable.

"You're really not going to like it," observed Deep Thought.

"Tell us!"

"All right," said Deep Thought. "The Answer to the Great Question..."

"Yes...!"

"Of Life, the Universe and Everything..." said Deep Thought.

"Yes...!"

"Is..." said Deep Thought, and paused.

"Yes...!"

"Is..."

"Yes...!!!...?"

"*Forty-two*," said Deep Thought, with infinite majesty and calm.

Douglas Adams, "The Hitchhiker's Guide to the Galaxy"

Contents

Introduction

This Ph.D. thesis is devoted to the computation of the Hodge-Deligne polynomials of some moduli spaces of stable coherent systems.

During the last 2 decades coherent systems on algebraic curves have been widely studied in algebraic geometry, mainly because they are a very powerful tool in order to understand Brill-Noether theory for vector bundles. In its turn, Brill-Noether theory has an important role to play in understanding the geometric structure of the moduli spaces of curves.

For any smooth irreducible projective complex curve C, a coherent system on C (see [KN]) is a pair (E, V) where E is an algebraic vector bundle on C and V is a linear subspace of the space of global sections of E. To any such object one can associate a triple (n, d, k) where n, d are the rank and the degree of E and k is the dimension of V. This notion can be generalized to any projective scheme (see, for example [LP] and [He]).

To construct a space which parametrizes coherent systems on an algebraic curve (see [KN]), one has to fix the invariants (n, d, k) and also a stability parameter α in $\mathbb{R}_{\geq 0}$. Having fixed these data, one can give a natural scheme structure to the sets

- $G(\alpha; n, d, k)$ which parametrizes isomorphism classes of α-stable coherent systems of type (n, d, k);
- $\widetilde{G}(\alpha; n, d, k)$ which parametrizes classes of S-equivalent α-semistable coherent systems of type (n, d, k).

$\widetilde{G}(\alpha; n, d, k)$ is a projective scheme and $G(\alpha; n, d, k)$ is an open subscheme of it. If n, d and k are relative coprime and α is generic, then the 2 schemes coincide (see [BGMMN, definition 1.2]).

Having fixed a triple (n, d, k), it is known that the stability condition does not vary in open intervals and that there exist only finitely many critical values where the stability condition changes. Therefore, for every triple (n, d, k) there are only finitely many distinct moduli spaces of stable coherent systems, parametrized by open intervals. In particular, for every (n, d, k) there are strong relations between the Brill-Noether locus $B(n, d, k)$ and the moduli space $G(\alpha; n, d, k)$ for α near zero (this is described in [BGMN, proposition 2.5 and §2.3]).

Regrettably, in general there is no good explicit description of the moduli spaces near zero, while those associated to a parameter α very large are easy to study (see [BO] for $k < n$ and [BGMN] for $k \geq n$).

Because of their connection with Brill-Noether loci, in the last 20 years moduli spaces of coherent systems have been studied quite intensively (see [BGMN], [BGMMN], [LN], [LN2], just to cite some recent papers). As we said, having fixed (n, d, k), the moduli space for α large is easy to understand; this led S. B. Bradlow, L. Brambila-Paz, O. García-Prada, V. Mercat, V. Muñoz and P. E. Newstead, to start a programme of research to try to pull back information from that moduli space to the moduli space for α small by crossing every intermediate critical value. Until now the information that they were able to obtain (sometimes with additional restrictions on the curve C and/or on the triple (n, d, k)) were mainly about non-emptiness, irreducibility, Picard groups, Poincaré polynomials and first and second homotopy groups of some of those moduli spaces.

The aim of this thesis is that of trying to get more refined invariants for at least some of those moduli spaces. In particular, we compute the so called Hodge-Deligne polynomials of some of those spaces. For any (reduced) scheme X, these are polynomials of 2 variables u, v where the coefficients are the dimensions of the graded pieces associated to a standard filtration of the cohomology groups of X. This filtration was originally introduced by P. Deligne in [D1], [D2] and [D3]; these polynomials can be defined for every algebraic scheme, not necessarily smooth, irreducible or projective. If the moduli spaces are smooth, irreducible and projective and one evaluates those polynomials on the same variable $u = v =: t$, one gets the usual Poincaré polynomials.

A good motivation to compute those polynomials is given by analogous computations performed for some spaces of holomorphic triples in [MOVG], [MOVG2] and [M].

For every triple (n, d, k) and for every critical value α_c for it, the only polynomials that one needs to compute are those arising from the subschemes $G^+(\alpha_c; n, d, k) \subset G(\alpha_c^+; n, d, k)$ and $G^-(\alpha_c; n, d, k) \subset G(\alpha_c^-; n, d, k)$. $G^+(\alpha_c; n, d, k)$, respectively $G^-(\alpha_c; n, d, k)$, parametrizes all coherent systems (E, V) that are α_c^+-stable but α_c^--unstable (respectively, α_c^--stable but α_c^+-unstable), so one needs to get a geometric description of these subschemes in order to compute the corresponding Hodge-Deligne polynomials.

Each of these subschemes will be divided into subschemes according to some invariants. The most important ones will be the length r of any α_c-Jordan-Hölder filtration and the type $\underline{t}$ of the α_c-canonical filtration. By "type of the α_c-canonical filtration" we mean the following: we can associate to every α_c-semistable coherent system an α_c-*canonical* filtration of length $1 \leq s \leq r$:

$$0 \subset (E_1, V_1) \subset \cdots \subset (E_s, V_s) = (E, V)$$

such that every $(E_i, V_i)/(E_{i-1}, V_{i-1})$ is the maximal subobject of $(E,V)/(E_{i-1},\ V_{i-1})$ that is the direct sum of α_c-stable coherent systems with the same α_c-slope as (E,V). To any such filtration we can associate a unique vector $\underline{t} = (t_1, \cdots, t_s) \in \mathbb{N}^s$, called the type of the filtration of (E,V), where t_i is defined as the length of any α_c-Jordan-Hölder filtration of $(E_i, V_i)/(E_{i-1}, V_{i-1})$. In particular, an object (E,V) will have unique α_c-Jordan-Hölder filtration if and only if its α_c-canonical filtration is of type $(1, \cdots, 1)$ (r times). Another invariant that we will have to take into account will be the presence of isomorphic coherent systems in the graded.

For technical reasons, this gives us explicit results only for low values of n and k. In particular, we get complete results in the cases when $n = 2, 3$ and $k = 1$ (for every value of d). We get almost complete results in the case when $n = 2$ and $k = 2$. Moreover, we get almost complete results in the case $n = 4, k = 1$. In this case the polynomial that one would like to obtain is the sum of 50 polynomials associated to various subschemes. At the moment we are able to compute 42 of those polynomials; for the remaining 8 it is not currently possible to get explicit formulae.

The structure of the thesis is as follows: we decided to divide it into 2 parts, putting in the second one most of the detailed computations, so that all the basic ideas and the main results are grouped into the first part. To be more precise, the first part of the thesis has the following structure:

- *Chapters 1, 2 and 3*: we describe some of the known results on coherent systems and their moduli spaces; we state and prove some results on α_c-Jordan-Hölder and α_c-canonical filtrations and some technical lemmas on pullbacks and sums of families of extensions of coherent systems. Moreover, we define non-degenerate extensions and we describe necessary and sufficient conditions for having such type of extensions.

- *Chapter 4*: we give a result of cohomology and base change for families of coherent systems. We use that result in order to prove some useful propositions on the existence of universal families of extensions of families of coherent systems in the spirit of [L]. We prove analogous results for non-splitting extensions and for non-degenerate extensions.

- *Chapter 5*: we describe how we can parametrize all classes of equivalence of non-split exact sequences indexed by any binary tree (see that chapter for the details).

- *Chapter 6*: we describe how we can parametrize the coherent systems (E,V)'s with a graded of fixed type, in the case when (E,V) has unique α_c-Jordan-Hölder filtration of length 3 or 4; we managed to get complete results except for the 4 subcases where the second and third object of the graded are isomorphic.

- *Chapter 7*: we summarize how we can parametrize all the coherent systems (E,V)'s with a graded of fixed type, in the case when (E,V) has not a unique α_c-Jordan-Hölder filtration of length 3 or 4. We are able to get complete results except in the case of

α_c-canonical filtration of type (1,2,1), where we are able to get a pointwise description but not a global one except for few subcases.

- *Chapter 8*: we describe the basic literature on Hodge-Deligne theory; we recall some known results and we state some easy lemmas about the Hodge-Deligne polynomials of some moduli spaces; we will need to use those results in the explicit computations of the next chapters.

- *Chapter 9*: we summarize all the explicit results that we get on the Hodge-Deligne polynomials of the moduli spaces $G(\alpha; n, d, k)$ when the pair (n, k) has values $(2, 1), (3, 1), (4,$ $1)$ and $(2, 2)$. In the first 2 cases we are able to compute every polynomial, while in the other 2 cases we get only partial results. The formulae for $n = 2$ and $n = 3$ can be obtained directly from the analogous computations in [M] for the moduli spaces of triples, while those for $n = 4$ are new in the literature, as far as we know.

In the second part of the thesis we describe in detail how we were able to get the results stated in chapters 7 and 9. To be more precise:

- *Chapter 10*: we describe the schemes parametrizing all the (E, V)'s in $G^+(\alpha_c; n, d, k)$ or in $G^-(\alpha_c; n, d, k)$ such that their α_c-canonical filtration is of type $(1, 2)$ or $(2, 1)$.

- *Chapter 11*: we describe the schemes parametrizing all the (E, V)'s in $G^+(\alpha_c; n, d, k)$ or in $G^-(\alpha_c; n, d, k)$ such that their α_c-canonical filtration is of type $(3, 1)$ or $(1, 3)$.

- *Chapter 12*: we describe the schemes parametrizing all the (E, V)'s in $G^+(\alpha_c; n, d, k)$ or in $G^-(\alpha_c; n, d, k)$ such that their α_c-canonical filtration is of type $(2, 1, 1)$, $(1, 2, 1)$ or $(1, 1, 2)$.

- *Chapter 13*: we compute the Hodge-Deligne polynomials of all the moduli spaces $G(\alpha; 2,$ $d, 1)$. We give a more explicit formula for such polynomials when α is small and d is large enough.

- *Chapters 14*: we perform analogous computations for all the moduli spaces $G(\alpha; 3, d, 1)$.

- *Chapters 15*: we compute 42 of the 50 polynomials associated to various subschemes of $G^+(\alpha_c; 4, d, 1)$ and $G^-(\alpha_c; 4, d, 1)$. We are therefore able to get explicit formulae for $G(\alpha; 4, d, 1)$ when the stability parameter is very large with some restrictions on d.

- *Chapters 16*: we perform computations analogous to those of chapter 13 for the moduli spaces $G(\alpha; 2, d, 2)$. Since in the literature not all the polynomials that we need are known at the moment, the formulae that we get are not complete.

Part I

Chapter 1

Coherent systems: objects, families and moduli spaces

We recall in this chapter some results that we will need to use about coherent systems. If not otherwise stated, we will always work over a fixed complex smooth irreducible projective curve C of genus $g \geq 2$ and $\mathcal{O} = \mathcal{O}_C$ will denote the structure sheaf of C.

Definition 1.0.1. ([KN],[LP]) A *coherent system* (E, V) of type (n, d, k) consists of an algebraic vector bundle E over C, of rank n and degree d, and a linear subspace $V \subseteq H^0(E)$ of dimension k. An equivalent definition that is often used in the literature is the following. A coherent system of type (n, d, k) is a triple $(E, \mathbb{V}, \phi)$ where E is as before, $\mathbb{V}$ is a vector space of dimension k and $\phi : \mathbb{V} \otimes \mathcal{O} \to E$ is a sheaf map such that the induced morphism $H^0(\phi) : \mathbb{V} \to H^0(E)$ is injective. The vector space $V \subseteq H^0(E)$ is then the image $H^0(\phi)(\mathbb{V})$. If we don't assume that $H^0(\phi)$ is injective, we will say that the triple $(E, \mathbb{V}, \phi)$ is a *weak coherent system* of type (n, d, k) (actually, this was the original definition of coherent system in [LP], but this notation is no more in use).

Remark 1.0.1. In fact, both [KN] and [LP] allow E to be any coherent sheaf in the definition; for α-semistable coherent systems on smooth curves (see definition 1.0.4 below), E is necessarily locally free, so this makes no difference. [He] allows E to be any coherent sheaf and his coherent systems correspond to our weak coherent systems.

Definition 1.0.2. ([KN]) Given a coherent system (E, V), a coherent subsystem of it is a pair (E', V') such that E' is a subbundle of E and $V' \subseteq V \cap H^0(E')$.

Definition 1.0.3. Given 2 coherent systems (E, V) and (E', V') (not necessarily of the same type), a morphism $(E', V') \to (E, V)$ is a morphism of vector bundles $\alpha : E' \to E$ such that $H^0(\alpha)(V') \subset V$. An isomorphism is any morphism α as before, that is also invertible and such that $H^0(\alpha)$ induces an isomorphism from V' to V. Given 2 weak coherent systems $(E, \mathbb{V}, \phi)$ and $(E', \mathbb{V}', \phi')$, a morphism $(E', \mathbb{V}', \phi') \to (E, \mathbb{V}, \phi)$ is any pair (α, β) where α is a morphism of vector bundles $E' \to E$ and $\beta : \mathbb{V}' \to \mathbb{V}$ is a linear map such that there is a commutative diagram:

$$\begin{array}{ccc} \mathbb{V}' \otimes \mathcal{O} & \xrightarrow{\phi'} & E' \\ {\scriptstyle \beta\otimes\mathrm{id}}\downarrow & \curvearrowright & \downarrow{\scriptstyle \alpha} \\ \mathbb{V} \otimes \mathcal{O} & \xrightarrow[\phi]{} & E. \end{array}$$

An isomorphism of weak coherent systems is a pair (α, β) such that both α and β are invertible.

By setting $\beta := H^0(\alpha)$ we get that two coherent systems are isomorphic if and only if the corresponding weak coherent systems are isomorphic. Moreover, the type of a (weak) coherent system is invariant under isomorphisms.

Definition 1.0.4. For every parameter $\alpha \in \mathbb{R}$ and for every coherent system (E, V) of type (n, d, k), we define the α-slope of (E, V) as

$$\mu_\alpha(E, V) := \frac{d}{n} + \alpha \, \frac{k}{n}.$$

We say that (E, V) is α-stable if

$$\mu_\alpha(E', V') < \mu_\alpha(E, V)$$

for all proper subsystems (E', V') (i.e. those such that $(0, 0) \subsetneq (E', V') \subsetneq (E, V)$). The notion of α-semistability is obtained by replacing the strict inequality before by a weak inequality. A coherent system is α-polystable if it is the direct sum of α-stable coherent systems of the same α-slope.

There is an analogous definition for weak coherent systems (see [KN]), but we will not need to use it. We shall simply recall that a weak coherent system of type (n, d, k) is α-semistable (respectively, α-stable) if and only if it is (the evaluation map of) an α-semistable (respectively, α-stable) coherent system of type (n, d, k) (see [KN, lemma 2.5]).

In general, the quotient of a coherent system by a coherent subsystem is defined only as a weak coherent system. When both coherent systems are α-semistable of the same slope, then the quotient coherent system does exists. To be more precise, we have:

Proposition 1.0.1. *([KN, corollary 2.5.1]) The α-semistable coherent systems of any fixed α-slope form a noetherian and artinian abelian category in which the simple objects are precisely the α-stable coherent systems. In particular, the following statements hold.*

(i) For any α-semistable coherent system (E, V) there exists an α-Jordan-Hölder filtration of it, i.e. a filtration:

$$0 = (E_0, V_0) \subset (E_1, V_1) \subset \cdots \subset (E_r, V_r) = (E, V) \tag{1.1}$$

such that the quotients $(Q_i, W_i) := (E_i, V_i)/(E_{i-1}, V_{i-1})$ *for* $i = 1, \cdots, r$ *are all* α*-stable with the same* α*-slope as* (E, V)*.*

(ii) If (E, V) *is an* α*-stable coherent system, then* $End(E, V) \simeq \mathbb{C}$*.*

Conversely, if (E, V) has an α-Jordan-Hölder filtration, then it is necessarily α-semistable and it is strictly α-stable if and only if the filtration has length 1. For simplicity, we will write α-JHF in order to denote any Jordan-Hölder filtration at α. In general, the Jordan-Hölder filtration of an α-semistable coherent system (E, V) is not unique, but the graded object

$$\mathrm{gr}_\alpha(E, V) := \bigoplus_{i=1,\cdots,r} (Q_i, W_i)$$

associated to it is uniquely determined up to isomorphisms. In particular, the length r of the filtration does not depend on the filtration chosen. Then we can give the following definition.

Definition 1.0.5. For every α-semistable coherent system (E, V) of type (n, d, k) we define $r_\alpha(E, V)$ as the length of any α-JHF of (E, V). If α is clear from the context, we will simply write $r(E, V)$. Two α-semistable coherent systems (E, V) and (E', V') are said to be S-equivalent if $\mathrm{gr}_\alpha(E, V) = \mathrm{gr}_\alpha(E', V')$.

For every scheme S, let us denote by π_S the projection $C \times S \to S$; for any point s in S we write C_s for $C \times \{s\}$.

Definition 1.0.6. ([BGMMN, definition A.6]) A *family of coherent systems of type* (n, d, k) *on* C *parametrized by a scheme* S is a pair $(\mathcal{E}, \mathcal{V})$ where

- $\mathcal{E}$ is a rank n vector bundle on $C \times S$ such that $\mathcal{E}_s = \mathcal{E}|_{C_s}$ has degree d for all s in S;
- $\mathcal{V}$ is a locally free subsheaf of $\pi_{S*}\mathcal{E}$ of rank k, such that the fibers $\mathcal{V}_s$ map injectively to $H^0(\mathcal{E}_s)$ for all s in S.

Another definition that appears in the literature is the following:

Definition 1.0.7. ([KN, definition 2.5]) A *family of coherent systems of type* (n, d, k) *on* C *parametrized by a scheme* S is a triple $(\mathcal{E}, \mathcal{V}, \phi)$ where:

- $\mathcal{E}$ is a rank n coherent sheaf on $C \times S$, flat over S;
- $\mathcal{V}$ is a locally free sheaf on S of rank k;
- $\phi : \pi_S^*(\mathcal{V}) \to \mathcal{E}$ is a morphism of $\mathcal{O}_{C\times S}$-modules,

such that for all s in S the fiber of ϕ over s gives rise to a coherent system of type (n, d, k) on $C_s \simeq C$.

In particular, this implies that for every s in S the sheaf $\mathcal{E}_s$ is locally free at (c, s) for all $c \in C$, so by [N, lemma 5.4] we get that $\mathcal{E}$ is locally free. So the second definition implies the first one. Conversely, for every family as in the first definition, one can easily associate a family according to the second definition by considering the map ϕ of global sections (see also [KN, §3.5]). Therefore, we will use without any distinction either the first or the second definition.

Remark 1.0.2. There is a more general notion of coherent system and of family of coherent systems that is used in [LP] and in [He]. In the case when their base X is a projective curve C and we have a condition of flatness (see [He, §1.3] and [LP]), we get that the notion of "flat family of coherent systems on $X \times S/S$" in [He] coincides with the notion of "family of coherent systems" over S described in the previous definitions.

Definition 1.0.8. A morphism of families of coherent systems $(\mathcal{E}', \mathcal{V}', \phi') \to (\mathcal{E}, \mathcal{V}, \phi)$ parametrized by a scheme S is any pair of morphisms (α, β) where α is a morphism of vector bundles $\mathcal{E}' \to \mathcal{E}$ over $C \times S$ and β is a morphisms of vector bundles $\mathcal{V}' \to \mathcal{V}$ over S, such that we have a commutative diagram as follows:

$$\begin{tikzcd} \pi_S^* \mathcal{V}' \arrow[r, "\phi'"] \arrow[d, "\pi_S^*\beta"'] \arrow[dr, phantom, "\curvearrowright"] & \mathcal{E}' \arrow[d, "\alpha"] \\ \pi_S^* \mathcal{V} \arrow[r, "\phi"'] & \mathcal{E}. \end{tikzcd}$$

For every family of coherent systems $(\mathcal{E}, \mathcal{V})$ of type (n, d, k) parametrized by S and for every morphism of schemes $f : S' \to S$, the pullback via f is defined as

$$(f', f)^*(\mathcal{E}, \mathcal{V}) = \big(f'^*\mathcal{E}, f^*\mathcal{V}\big), \tag{1.2}$$

where f' is defined as the pullback

$$\begin{tikzcd} C \times S' \arrow[r, "f'"] \arrow[d, "\pi_{S'}"'] \arrow[dr, phantom, "\square"] & C \times S \arrow[d, "\pi_S"] \\ S' \arrow[r, "f"'] & S. \end{tikzcd} \tag{1.3}$$

It easy to see that (1.2) is a family of coherent systems of type (n, d, k) on C, parametrized by S'. If we use the definition of family as triple $(\mathcal{E}, \mathcal{V}, \phi)$, then the pullback of such a family by f is the triple

$$(f', f)^*(\mathcal{E}, \mathcal{V}, \phi) := (f'^*\mathcal{E}, f^*\mathcal{V}, \widetilde{\phi}),$$

where $\widetilde{\phi}$ is defined as the composition

$$\widetilde{\phi}:\pi_{S'}^*f^*\mathcal{V}\xrightarrow{\sim}f'^*\pi_S^*\mathcal{V}\xrightarrow{f'^*\phi}f'^*\mathcal{E},$$

where the first map is the canonical isomorphism induced by diagram (1.3).

Given any family $(\mathcal{E},\mathcal{V},\phi)$ of type (n,d,k) parametrized by a scheme S and any locally free $\mathcal{O}_S$-module $\mathcal{M}$, we define

$$(\mathcal{E},\mathcal{V},\phi)\otimes_S\mathcal{M}:=(\mathcal{E}\otimes_{C\times S}\pi_S^*\mathcal{M},\mathcal{V}\otimes_S\mathcal{M},\phi')$$

where $\phi':=\phi\otimes_{C\times S}\mathrm{id}_{\pi_S^*\mathcal{M}}$. This is a again a family of coherent systems parametrized by S.

Remark 1.0.3. If $\mathcal{M}$ is only a coherent or quasi-coherent $\mathcal{O}_S$-module, then the tensor product $(\mathcal{E},\mathcal{V},\phi)\otimes_S\mathcal{M}$ is in general only a family of weak coherent systems. To be more precise, it is an algebraic system on $C\times S/S$ in the sense of [He]. One should also need to consider such objects in order to define the functors Ext^i's (see below), but we will not need to deal explicitly with such objects in the present work, so we will only consider tensor products of families of coherent systems by locally free sheaves.

If $k\geq 1$, then by applying the α-semistability condition for (E,V) to the subsystem $(E,0)$ one obtains that a necessary condition for the existence of α-semistable coherent systems is that $\alpha\geq 0$, so one can simply restrict to that range for the parameter α.

Theorem 1.0.2. *([KN, theorem 1]) For every parameter $\alpha\in\mathbb{R}_{\geq 0}$ and for every type (n,d,k) there exist schemes $G(\alpha;n,d,k)$ and $\widetilde{G}(\alpha;n,d,k)$ which are coarse moduli spaces for families of α-stable (respectively α-semistable) coherent systems of type (n,d,k). The closed points of $G(\alpha;n,d,k)$ are in bijection with isomorphism classes of α-stable coherent systems. The closed points of $\widetilde{G}(\alpha;n,d,k)$ are in bijection with S-equivalence classes of α-semistable coherent systems. $\widetilde{G}(\alpha;n,d,k)$ is a projective variety and it contains $G(\alpha;n,d,k)$ as an open subscheme.*

Remark 1.0.4. For each (n,d,k) and $\alpha\geq 0$, the proof of this theorem follows from the usual GIT construction: there exist a projective scheme R and an action of $PGL(N)$ on R (both R and N depend on (n,d,k)), together with a linearization of that action depending on α. Then if we denote by R^s and R^{ss} the stable and the semistable loci, we get that the moduli space $G(\alpha;n,d,k)$ is obtained as the quotient $R^s/PGL(N)$ and analogously for the moduli space $\widetilde{G}(\alpha;n,d,k)$. In particular, there exist families $(\mathcal{Q}^{\mathrm{s}},\mathcal{W}^{\mathrm{s}})$ and $(\mathcal{Q}^{\mathrm{ss}},\mathcal{W}^{\mathrm{ss}})$ over R^{s}, respectively over R^{ss}, that have the local universal property (see [KN, §3.5]).

Definition 1.0.9. ([BGMN, definition 2.4]) A parameter $\alpha>0$ is a *virtual critical value* for a triple (n,d,k) if it is numerically possible to have a coherent system (E,V) of type (n,d,k) together with a proper coherent subsystem (E',V') such that $\frac{k'}{n'}\neq\frac{k}{n}$ but $\mu_\alpha(E',V')=\mu_\alpha(E,V)$. We also regard 0 as a virtual critical value. If there is a pair $(E,V),(E',V')$ such that this actually holds, we say that α is an *actual critical value*. Having fixed (n,d,k), all the non-zero virtual critical values lie in the set:

$$\left\{\frac{nd'-n'd}{n'k-nk'}\text{ s.t. }0\leq k'\leq k,\,0<n'<n,\,n'k\neq nk'\right\}\cap[0,\infty[;$$

we say that α is *generic* for (n, d, k) if it is not critical.

If $\mathrm{GCD}(n, d, k) = 1$ and α is generic, then α-semistability is equivalent to α-stability, so $\widetilde{G}(\alpha; n, d, k) = G(\alpha; n, d, k)$. The actual critical values are only a finite number. If we label them by α_i starting with $\alpha_0 = 0$, we get a partition of $\mathbb{R}_{\geq 0}$. For numerical reasons, within the interval $]\alpha_i, \alpha_{i+1}[$ the stability condition is independent of α, so $G(\alpha; n, d, k) \simeq G(\alpha'; n, d, k)$ for all $\alpha, \alpha' \in]\alpha_i, \alpha_{i+1}[$. Therefore we write $G_i(n, d, k) := G(\alpha; n, d, k)$ for every $\alpha \in]\alpha_i, \alpha_{i+1}[$. Analogously, we denote by $\widetilde{G}_i(n, d, k)$ the moduli space of α-semistable coherent systems for every α in that interval.

For every (n, d, k) the moduli spaces $G_0(n, d, k)$ and $\widetilde{G}_0(n, d, k)$ have strong connections with the Brill-Noether loci of stable and of semistable bundles defined by

$$B(n, d, k) := \{E \in M(n, d) \text{ s.t. } \dim \mathrm{H}^0(E) \geq k\},$$
$$\widetilde{B}(n, d, k) := \{[E] \in \widetilde{M}(n, d) \text{ s.t. } \dim \mathrm{H}^0(\mathrm{gr}(E)) \geq k\},$$

where $M(n, d)$ and $\widetilde{M}(n, d)$ denote the moduli spaces of stable, respectively semistable, vector bundles on C of rank n and degree d and $[E]$ denotes the S-equivalence class of any semistable vector bundle. In particular, the relationships between these loci and the moduli spaces of coherent systems are accounted for by the following proposition.

Proposition 1.0.3. *([BGMN, proposition 2.5]) Let $0 < \alpha < \alpha_1$; then*

(i) (E, V) α-stable implies E semistable;

(ii) E stable implies (E, V) α-stable for all $V \subseteq H^0(E)$.

Lemma 1.0.4. *([He, proposition 2.2]) Let us suppose that (E, V) and (E', V') are α-semistable for some value of α. Then:*

- *if $\mu_\alpha(E', V') > \mu_\alpha(E, V)$, then $Hom((E', V'), (E, V)) = 0$;*
- *if $\mu_\alpha(E', V') = \mu_\alpha(E, V)$ and both objects are α-stable, then $Hom((E', V'), (E, V))$ is $\mathbb{C}$ if the 2 objects are isomorphic, zero otherwise.*

Moreover, let us suppose that (E', V') is α_c-stable, that (E, V) is α_c-semistabe and let α be a non-zero morphism $(E', V') \to (E, V)$. Then the image of α is isomorphic to (E', V').

The following definition is taken from [He, §1.2]. In that paper the definition is given for families of algebraic systems, we state only the definition for the case of families of coherent systems.

Definition 1.0.10. Let S be any scheme, let $(\mathcal{E}, \mathcal{V})$ and $(\mathcal{E}', \mathcal{V}')$ be two families of coherent systems parametrized by S and let us denote by π_S the projection $C \times S \to S$. Then we define a sheaf of $\mathcal{O}_S$-modules

$$\mathcal{F} = \mathcal{H}om_{\pi_S}((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V}))$$

on S as follows: for every open set $U \subset S$ we set

$$\mathcal{F}(U) := \mathrm{Hom}((\mathcal{E}', \mathcal{V}')|_U, (\mathcal{E}, \mathcal{V})|_U) = \mathrm{Hom}\left((\mathcal{E}'|_{\pi_S^{-1}U}, \mathcal{V}'|_U), (\mathcal{E}|_{\pi_S^{-1}U}, \mathcal{V}|_U)\right).$$

This is actually a sheaf and the functor $\mathcal{H}om_{\pi_S}((\mathcal{E}', \mathcal{V}'), -)$ is left exact. We denote by

$$\mathcal{E}xt^i_{\pi_S}((\mathcal{E}', \mathcal{V}'), -) \quad \forall\, i \geq 1$$

its right derived functors. If $S = \mathrm{Spec}(\mathbb{C})$, then a family $(\mathcal{E}', \mathcal{V}')$ parametrized by S is simply a coherent system (E', V') and the previous functors are the derived functors of the functor $\mathrm{Hom}((E', V'), -)$, so we denote them by $\mathrm{Ext}^i((E', V'), -)$.

By using remark 1.0.2 and [He, corollaire 1.20] for the projection $\pi_S : C \times S \to S$ we have the following useful result of semicontinuity.

Proposition 1.0.5. *Let $(\mathcal{E}, \mathcal{V})$ and $(\mathcal{E}', \mathcal{V}')$ be two families of coherent systems (not necessarily of the same type), parametrized by the same scheme S. Then for all $i \geq 0$ the function*

$$t^i(s) := \mathit{dim}\ \mathit{Ext}^i((\mathcal{E}', \mathcal{V}')_s, (\mathcal{E}, \mathcal{V})_s)$$

is upper semicontinuous on S. If S is integral and for a certain i the function $t^i(s)$ is constant on S, then the sheaf $\mathcal{E}xt^i_{\pi_S}((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V}'))$ is locally free on S.

We recall also a result from [BGMN], stated here with slightly different notations.

Lemma 1.0.6. *([BGMN, lemma 6.3]) Let (E, V) be a coherent system of type (n, d, k) and let α_c be a critical value for (n, d, k).*

(i) Let us suppose that (E, V) is α_c^+-stable but α_c^--unstable. Then (E, V) appears as the middle term in a non-trivial extension:

$$0 \to (E_1, V_1) \xrightarrow{\alpha} (E, V) \xrightarrow{\beta} (E_2, V_2) \to 0 \tag{1.4}$$

in which

(a) (E_1, V_1) and (E_2, V_2) are both α_c^+-stable with

$$\mu_{\alpha_c^+}(E_1, V_1) < \mu_{\alpha_c^+}(E, V) < \mu_{\alpha_c^+}(E_2, V_2);$$

(b) (E_1, V_1) and (E_2, V_2) are both α_c-semistable with

$$\mu_{\alpha_c}(E_1, V_1) = \mu_{\alpha_c}(E, V) = \mu_{\alpha_c}(E_2, V_2);$$

(c) k_1/n_1 is a maximum among all proper subsystems $(E_1, V_1) \subset (E, V)$ which satisfy (b);

(d) n_1 is a minimum among all proper subsystems (E_1, V_1) that satisfy (c).

(ii) Similarly, if (E, V) is α_c^--stable but α_c^+-unstable, then (E, V) appears as the middle term in a non-trivial extension (1.4) in which

(a) (E_1, V_1) and (E_2, V_2) are both α_c^--stable with

$$\mu_{\alpha_c^-}(E_1, V_1) < \mu_{\alpha_c^-}(E, V) < \mu_{\alpha_c^-}(E_2, V_2);$$

(b) (E_1, V_1) and (E_2, V_2) are both α_c-semistable with

$$\mu_{\alpha_c}(E_1, V_1) = \mu_{\alpha_c}(E, V) = \mu_{\alpha_c}(E_2, V_2);$$

(c) k_1/n_1 is a minimum among all proper subsystems $(E_1, V_1) \subset (E, V)$ which satisfy (b);

(d) n_1 is a minimum among all proper subsystems (E_1, V_1) that satisfy (c).

Remark 1.0.5. In the original lemma it is not written explicitly that the sequence (1.4) is non-split, but actually this is an easy computation. Indeed, if that sequence is split, then we have $(E, V) \simeq (E_1, V_1) \oplus (E_2, V_2)$, which cannot be α-stable for any α.

Remark 1.0.6. Using condition (i-b), condition (i-a) can be restated saying that $\frac{k_1}{n_1} < \frac{k}{n} < \frac{k_2}{n_2}$ and analogously condition (ii-a) can be restated saying that $\frac{k_1}{n_1} > \frac{k}{n} > \frac{k_2}{n_2}$.

Remark 1.0.7. Obviously also the extensions obtained by (1.4) by multiplication by scalars in $\mathbb{C}^*$ do satisfy properties (a)-(d), so the extension with representative (1.4) can at most be unique only up to an action of $\mathbb{C}^*$. Moreover, as we will see in the following chapters, in general such an extension will not be unique also after quotienting by $\mathbb{C}^*$. First of all, this is because the numerical conditions (a)-(d) in general are not sufficient to get uniqueness of (E_1, V_1). Moreover, we can have problems if the automorphism groups of (E_1, V_1) or of (E_2, V_2) are larger than $\mathbb{C}^*$: in this case even if the objects (E_i, V_i)'s are unique, the class of the extension (1.4) is not unique. Finally, we have problems if (E, V) contains a subobject of the form $(E_1, V_1) \oplus (E_1, V_1)$; in this case even if (E_1, V_1) is unique with $\mathbb{C}^*$ as automorphism group, the morphism α is not uniquely determined up to scalars. An analogous problem can occur if there is a quotient $(E, V) \to (E_2, V_2) \oplus (E_2, V_2)$.

Proposition 1.0.7. *[BGMN, proposition 3.2] Let (E_1, V_1) and (E_2, V_2) be two coherent systems on C of types (n_1, d_1, k_1) and (n_2, d_2, k_2) respectively. Let*

$$\mathbb{H}^0_{21} := Hom((E_2, V_2), (E_1, V_1)) \quad and \quad \mathbb{H}^2_{21} := Ext^2((E_2, V_2), (E_1, V_1)).$$

Then:

$$dim\ Ext^1((E_2, V_2), (E_1, V_1)) = C_{21} + dim\ \mathbb{H}^0_{21} + dim\ \mathbb{H}^2_{21},$$

where

$$C_{21} := n_1 n_2 (g-1) - d_1 n_2 + d_2 n_1 + k_2 d_1 - k_2 n_1 (g-1) - k_1 k_2.$$

Chapter 2

Filtrations for semistable coherent systems

In this chapter we state some general properties of Jordan-Hölder filtrations and we also introduce a slightly different notion of filtration that will be useful in the next chapter. In general, we will be only interested in filtrations for coherent systems of type (n, d, k) that are (strictly) semistable at a value α_c that is critical for (n, d, k). This is the only interesting case, otherwise we end up with the trivial filtration.

2.1 Canonical filtrations

Since Jordan-Hölder filtrations in general are not unique, we look for some unique filtration. The basic idea is the same of [GM]. In order to find such a unique filtration, we first state this preliminary lemma.

Lemma 2.1.1. *Let us fix any α_c-semistable coherent system (E, V) with α_c-slope μ and let us consider the set:*

$$\mathcal{S} = \mathcal{S}(E, V) := \{\textit{all non-trivial coherent subsystems of } (E, V) \\ \textit{which are direct sums of } \alpha_c\textit{-stable coherent systems with } \alpha_c\textit{-slope } \mu\}.$$

Then this set is non-empty and it admits a unique maximal *element with respect to inclusions.*

Proof. $\mathcal{S}$ is nonempty (because it contains the first term of every α_c-JHF of (E, V)) and it is partially ordered by inclusions. By proposition 1.0.1 we are in a noetherian category, so we get that there exists at least a maximal element, that we denote by (E_1, V_1). Now by contradiction, let us suppose that we have also another $(\widetilde{E}_1, \widetilde{V}_1)$ which is a maximal object of $\mathcal{S}$. If we have that $E_1 \cap \widetilde{E}_1$ is the zero sheaf, then the direct sum of (E_1, V_1) and $(\widetilde{E}_1, \widetilde{V}_1)$ contradicts the maximality of both. Hence we can suppose that $E_1 \cap \widetilde{E}_1$ is not the trivial sheaf.

Then let us consider the morphism of coherent systems:

$$\varphi : (E_1, V_1) \hookrightarrow (E, V) \twoheadrightarrow (E, V)/(\widetilde{E}_1, \widetilde{V}_1).$$

The last object is again an α_c-semistable coherent system with α_c-slope μ by proposition 1.0.1. φ is not the zero morphism since $(E_1, V_1) \neq (\widetilde{E}_1, \widetilde{V}_1)$. Again by proposition 1.0.1 the category of α_c-semistable coherent systems of α_c-slope μ is an abelian category, hence there exists a kernel for φ in that category. Let us write $(\overline{E}, \overline{V}) := \mathrm{Ker}(\varphi)$; then we have an exact sequence of the form:

$$o \to (\overline{E}, \overline{V}) \xrightarrow{\alpha} (E_1, V_1) \xrightarrow{\varphi} (E, V)/(\widetilde{E}_1, \widetilde{V}_1) = (E/\widetilde{E}_1, V/\widetilde{V}_1). \tag{2.1}$$

Then by definition of exact sequence of coherent systems we get an exact sequence of vector bundles:

$$0 \to \overline{E} \to E_1 \to E/\widetilde{E}_1;$$

hence $\overline{E} = E_1 \cap \widetilde{E}_1$ (the part on the right a priori is just a sheaf, but since it is equal to $\overline{E}$, then it is a vector bundle). Moreover, we get exact sequences of vector spaces:

$$\begin{array}{ccccccc} 0 & \to & H^0(\overline{E}) & \to & H^0(E_1) & \to & H^0(E/\widetilde{E}_1) \\ & & \cup & & \cup & & \cup \\ 0 & \to & \overline{V} & \to & V_1 & \to & V/\widetilde{V}_1, \end{array} \tag{2.2}$$

where the second line is exact by (2.1). Hence we get that $\overline{V} = V_1 \cap \widetilde{V}_1$ (where the intersection is made in $H^0(E)$ that contains both V_1 and $\widetilde{V}_1$). Then we can consider the exact sequence induced by α:

$$0 \to (E_1 \cap \widetilde{E}_1, V_1 \cap \widetilde{V}_1) \xrightarrow{\alpha} (E_1, V_1) \xrightarrow{\beta} (E_1, V_1)/(E_1 \cap \widetilde{E}_1, V_1 \cap \widetilde{V}_1) \to 0. \tag{2.3}$$

Since $(E_1, V_1) \neq (\widetilde{E}_1, \widetilde{V}_1)$, then $(E_1 \cap \widetilde{E}_1, V_1 \cap \widetilde{V}_1)$ is strictly contained in (E_1, V_1); by construction (E_1, V_1) is the sum of α_c-stable coherent systems of α_c-slope μ, so there exists an object $(E', V') \subset (E_1, V_1)$ that is α_c-stable and that is not completely contained in the image of α. Now let us denote by (E'', V'') the image of (E', V') under β; by exactness of (2.3), we have that $\beta|_{(E', V')}$ is non-zero. Then by lemma 1.0.4 the image (E'', V'') of (E', V') is isomorphic to (E', V'), so it is α_c-stable with α_c-slope μ.

If do the same construction for all the (E', V')'s that are α_c-stable factors of (E_1, V_1) not contained in $(E_1 \cap \widetilde{E}_1, V_1 \cap \widetilde{V}_1)$, we get a split γ of (2.3), so that:

$$\begin{aligned} (E_1, V_1) &= (E_1 \cap \widetilde{E}_1, V_1 \cap \widetilde{V}_1) \oplus \mathrm{Im}(\gamma) \simeq \\ &\simeq (E_1 \cap \widetilde{E}_1, V_1 \cap \widetilde{V}_1) \oplus (E_1, V_1)/(E_1 \cap \widetilde{E}_1, V_1 \cap \widetilde{V}_1). \end{aligned} \tag{2.4}$$

Analogously, we can consider the morphism

$$\widetilde{\varphi} : (\widetilde{E}_1, \widetilde{V}_1) \hookrightarrow (E, V) \twoheadrightarrow (E, V)/(E_1, V_1)$$

and its kernel. Then the same explicit description given before for $\mathrm{Ker}(\varphi)$ proves that $\mathrm{Ker}(\widetilde{\varphi})$ is equal to $(\widetilde{E}_1 \cap E_1, \widetilde{V}_1 \cap V_1) = (E_1 \cap \widetilde{E}_1, V_1 \cap \widetilde{V}_1) = \mathrm{Ker}(\varphi)$. Hence by proceeding as before we get an exact sequence:

$$0 \to (E_1 \cap \widetilde{E}_1, V_1 \cap \widetilde{V}_1) \stackrel{\widetilde{\alpha}}{\to} (\widetilde{E}_1, \widetilde{V}_1) \stackrel{\widetilde{\beta}}{\to} (\widetilde{E}_1, \widetilde{V}_1)/(E_1 \cap \widetilde{E}_1, V_1 \cap \widetilde{V}_1) \to 0$$

and a split $\widetilde{\gamma}$ of it such that:

$$\begin{aligned} (\widetilde{E}_1, \widetilde{V}_1) = (E_1 \cap \widetilde{E}_1, V_1 \cap \widetilde{V}_1) \oplus \mathrm{Im}(\widetilde{\gamma}) \simeq \\ \simeq (E_1 \cap \widetilde{E}_1, V_1 \cap \widetilde{V}_1) \oplus (\widetilde{E}_1, \widetilde{V}_1)/(E_1 \cap \widetilde{E}_1, V_1 \cap \widetilde{V}_1). \end{aligned} \tag{2.5}$$

Note that by construction all the terms of the splittings (2.4) and (2.5) are non-trivial and consist of direct sums of α_c-stable coherent systems with α_c-slope μ. Now by considering only the part related to vector bundles, we get that $\mathrm{Im}(\gamma) \cap \mathrm{Im}(\widetilde{\gamma}) = 0$ in E. Hence it makes sense to consider the coherent subsystem of (E, V) given by:

$$(E_1 \cap \widetilde{E}_1, V_1 \cap \widetilde{V}_1) \oplus \mathrm{Im}(\gamma) \oplus \mathrm{Im}(\widetilde{\gamma}).$$

By construction, this is again a direct sum of α_c-stable coherent systems with α_c-slope μ. Since $\mathrm{Im}(\gamma)$ and $\mathrm{Im}(\widetilde{\gamma})$ are both non-trivial, the new coherent system strictly contains both (E_1, V_1) and $(\widetilde{E}_1, \widetilde{V}_1)$, and so it contradicts their maximality in $\mathcal{S}$. Hence (E_1, V_1) is unique. □

Lemma 2.1.2. *Let us fix any α_c-semistable coherent system (E, V) with α_c-slope μ. Then there exists a* unique *filtration*

$$0 = (E_0, V_0) \subset (E_1, V_1) \subset \cdots \subset (E_s, V_s) = (E, V) \tag{2.6}$$

such that:

(i) for all $i = 1, \cdots, s$ the quotients $(E_i, V_i)/(E_{i-1}, V_{i-1})$ are direct sums *of α_c-stable coherent sheaves with α_c-slope μ;*

(ii) for all $i = 1, \cdots, s$ the coherent systems $(E, V)/(E_{i-1}, V_{i-1})$ don't contain any coherent subsystem which is the direct sum of $(E_i, V_i)/(E_{i-1}, V_{i-1})$ with an α_c-stable coherent system with α_c-slope μ;

(iii) the graded associated to this filtration coincides with the graded associated to an α_c-JHF of (E, V);

(iv) if the α_c-JHF of (E, V) is unique, it coincides with (2.6);

(v) if (2.6) is an α_c-Jordan-Hölder filtration, then it is the unique α_c-Jordan-Hölder filtration of (E, V).

Definition 2.1.1. We will say that (2.6) is the *α_c-canonical filtration associated to* (E,V). We will say that such a filtration is of type $(t_1,\cdots,t_s)$ if for each $i=1,\cdots,s$ we have that the α_c-semistable coherent system $(E_i,V_i)/(E_{i-1},V_{i-1})$ is the direct sum of t_i α_c-stable coherent systems. In particular, we will have that $t_1+\cdots+t_s$ coincides with the length r of any α_c-Jordan-Hölder filtration of (E,V). Moreover, (E,V) has unique α_c-Jordan-Hölder filtration if and only if its α_c-canonical filtration is of type $(1,\cdots,1)$ (r times).

Proof. If (E,V) is α_c-stable, there's nothing to prove. Otherwise, properties (i) and (ii) force the first term of (2.6) to be equal to the maximal element (E_1,V_1) of $\mathcal{S}(E,V)$. Actually, if we choose that coherent subsystem, we get that it obviously satisfies both properties (i) and (ii) for $i=1$. Then we can consider $(E,V)/(E_1,V_1)$: by proposition 1.0.1, this is again α_c-semistable with α_c-slope μ, so we can apply the same procedure to the set $\mathcal{S}((E,V)/(E_1,V_1))$ to get the term (E_2,V_2) of (2.6), and so on. Since the rank of the coherent system is strictly decreasing at each step, after finitely many steps we get a filtration (2.6) that satisfies properties (i) and (ii) for all $i=1,\cdots,s$. Using lemma 2.1.1 at each step, we get that each term of the filtration is unique, hence (2.6) is unique.

Now let us consider the first term (E_1,V_1) of the filtration (2.6): since it α_c-polystable, then it is α_c-semistable, hence it appears in some α_c-JHF of (E,V). So the graded of (E,V) associated to that filtration is equal to the direct sum of the graded of (E_1,V_1) and of the graded of $(E,V)/(E_1,V_1)$. Since (E_1,V_1) is the sum of α_c stable coherent systems, it is obvious that a graded for it coincides with $(E_1,V_1)=(E_1,V_1)/(E_0,V_0)$. Then we have that:

$$\mathrm{gr}_{\alpha_c}(E,V)=(E_1,V_1)\oplus\mathrm{gr}_{\alpha_c}\Big((E,V)/(E_1,V_1)\Big).$$

Then we can apply the same procedure to $(E,V)/(E_1,V_1)$; by induction we conclude that there exists an α_c-JHF of (E,V) that completes (2.6) and such that:

$$\mathrm{gr}_{\alpha_c}(E,V)=\oplus_{i=1,\cdots,s}(E_i,V_i)/(E_{i-1},V_{i-1}).$$

So this proves property (iii). Now let us prove also (iv): by construction of α_c-JHF, we get that the first term of any α_c-JHF of (E,V) is given as a *minimal* element of $\mathcal{S}(E,V)$, i.e. an α_c-stable coherent subsystem of (E,V). If the α_c-JHF is unique, this implies that there are no direct sums of 2 or more coherent systems as subobject of (E,V) with α_c-slope μ, so the set $\mathcal{S}(E,V)$ consists of a unique element, hence its maximal object coincides with the minimal one. By construction, the maximal object is the first term of (2.6), hence we have proved that the first term of the unique α_c-JHF of (E,V) coincides with the first term of (2.6). Then in order to conclude, it suffices to consider the coherent system $(E,V)/(E_1,V_1)$ and to apply induction on it.

Conversely, if (2.6) is an α_c-Jordan-Hölder filtration, then this means that (E_1,V_1) is α_c-stable (and not only α_c-polystable). By construction we know that (E_1,V_1) is the maximal object of $\mathcal{S}(E,V)$; therefore $\mathcal{S}(E,V)$ consists of a single element. By definition of α_c-Jordan-Hölder filtration, the term (E_1,V_1) of an α_c-Jordan-Hölder filtration is any minimal object of

$\mathcal{S}(E,V)$. Therefore, any α_c-Jordan-Hölder filtration of (E,V) has the same first term, namely (E_1,V_1). Then we consider the coherent system $(E,V)/(E_1,V_1)$ and we apply induction on it. □

Now we need a way to characterize canonical filtrations. This is taken into account by the following proposition.

Proposition 2.1.3. *Let $(E_i,V_i)_{i=1,\cdots,t}$ be a filtration of a coherent system (E,V) such that the coherent systems $(E_i,V_i)/(E_{i-1},V_{i-1})$ are all α_c-polystable with the same α_c-slope μ for all $i=1,\cdots,t$. Then the following facts are equivalent*

(a) $(E_i,V_i)_{i=1,\cdots,t}$ is the α_c-canonical filtration of (E,V);

(b) for every $i=1,\cdots,t-1$, for every α_c-stable coherent system (Q,W) with α_c-slope equal to μ and for every morphism

$$\gamma:(Q,W)\to(E,V)/(E_{i-1},V_{i-1}),$$

we have $\beta_i\circ\gamma=0$, where β_i is the morphism appearing in the induced exact sequence

$$0\to(E_i,V_i)/(E_{i-1},V_{i-1})\xrightarrow{\alpha_i}(E,V)/(E_{i-1},V_{i-1})\xrightarrow{\beta_i}(E,V)/(E_i,V_i)\to 0;$$

(c) same statement of (b) but restricted for every fixed $i=1,\cdots,t-1$ only to those (Q,W)'s that appear in the graded of $(E,V)/(E_{i-1},V_{i-1})$ at α_c and to $\gamma\neq 0$.

Proof. First of all, we will prove that (a) and (b) are equivalent.

Let us suppose that (b) is not verified and let i be the smallest index in $\{1,\cdots,t-1\}$ such that there exists an α_c-stable coherent system (Q,W) with $\mu_{\alpha_c}(Q,W)=\mu$ and a morphism $\gamma:(Q,W)\to(E,V)/(E_{i-1},V_{i-1})$ such that $\beta_i\circ\gamma\neq 0$. In particular, $\gamma\neq 0$; since (Q,W) is α_c-stable, then by lemma 1.0.4 it is isomorphic to $\mathrm{Im}(\gamma)$. Moreover, since $\beta_i\circ\gamma\neq 0$, then we get that $\mathrm{Im}(\gamma)\nsubseteq\mathrm{Im}(\alpha_i)$, so we can form the coherent subsystem

$$(E_i,V_i)/(E_{i-1},V_{i-1})\oplus\mathrm{Im}(\gamma)\subset(E,V). \tag{2.7}$$

By hypothesis $(E_i,V_i)/(E_{i-1},V_{i-1})$ is α_c-polystable with α_c-slope equal to μ; since (Q,W) is α_c-stable with the same α_c-slope, we get that (2.7) is the sum of α_c-stable coherent systems with α_c-slope μ and it is contained in $(E,V)/(E_{i-1},V_{i-1})$. So by definition of α_c-canonical filtration the system (E_i,V_i) cannot be part of the α_c-canonical filtration of (E,V), so (a) is not verified.

Conversely, let us suppose that (b) is satisfied and let $\{(E'_i,V'_i)\}_{i=1,\cdots,s}$ be the α_c-canonical filtration of (E,V). By construction of the α_c-canonical filtration, we have that (E'_1,V'_1) is the unique maximal element of the set $\mathcal{S}(E,V)$, that is the set of all subsystems of (E,V) that are α_c-polystable with α_c-slope equal to μ. By hypothesis $(E_1,V_1)\in\mathcal{S}(E,V)$, so we get that $(E_1,V_1)\subseteq(E_1',V_1')$. By contradiction, let us suppose that (E_1,V_1) is strictly contained in (E_1',V_1'); since both coherent systems are α_c-polystable, this implies that there exists an α_c-stable coherent system (Q,W) with $\mu_{\alpha_c}(Q,W)=\mu$, such that $(E_1,V_1)\oplus(Q,W)\subseteq(E',V')$. Then let us define a morphism γ as the composition:

$$\gamma:\,(Q,W)\hookrightarrow(E_1,V_1)\oplus(Q,W)\hookrightarrow(E_1',V_1')\hookrightarrow(E,V)=(E,V)/(E_0,V_0).$$

Let us also consider the exact sequence

$$0\to(E_1,V_1)\xrightarrow{\alpha_1}(E,V)\xrightarrow{\beta_1}(E,V)/(E_1,V_1)\to 0.$$

Since we are assuming (b), we get that $\beta_1\circ\gamma=0$, then by exactness of this sequence we get that there exists $\gamma':(Q,W)\to(E_1,V_1)$ such that $\gamma=\alpha_1\circ\gamma'$, but this is impossible by definition of (Q,W) and of γ. Therefore we get that necessarily $(E_1,V_1)=(E_1',V_1')$. Then we consider the coherent systems $(E_2,V_2)/(E_1,V_1)$ and $(E_2',V_2')/(E_1,V_1)$ and we use the same argument; we conclude by induction on the length t of the filtration $\{(E_i,V_i)\}_i$.

Now obviously (b) implies (c). Conversely, let us fix any α_c-stable coherent system (Q,W) with $\mu_{\alpha_c}(E,V)=\mu$, together with a morphism γ to $(E,V)/(E_{i-1},V_{i-1})$. If $\gamma=0$, then $\beta_i\circ\gamma=0$. Otherwise, by lemma 1.0.4 γ maps isomorphically (Q,W) into a coherent subsystem (Q',W') of $(E,V)/(E_{i-1},V_{i-1})$ with $\mu_{\alpha_c}(Q',W')=\mu$. Then $(Q,W)\simeq(Q',W')$ is contained in the graded of $(E,V)/(E_{i-1},V_{i-1})$. □

2.2 Jordan-Hölder filtrations

Having fixed any α_c-semistable coherent system (E,V), we want to give necessary and sufficient conditions so that its α_c-Jordan-Hölder filtration is unique. By lemma 2.1.2 we have that an α_c-Jordan-Hölder filtration of (E,V) is its unique α_c-Jordan-Hölder filtration if and only if it coincides with the α_c-canonical filtration of (E,V). We want to restate the conditions that characterize the canonical filtration in the case when it coincides with a Jordan-Hölder filtration. Then we get the following result.

Proposition 2.2.1. *Let us fix any α_c-semistable coherent system (E,V) with $\mu_{\alpha_c}(E,V)=:\mu$ and any α_c-Jordan-Hölder filtration $\{(E_i,V_i)\}_{i=1,\cdots,r}$ for it, with graded at α_c given by $\oplus_{i=1}^r(Q_i,W_i)$. Then the following facts are equivalent:*

(a) the filtration $\{(E_i,V_i)\}_{i=1,\cdots,r}$ is the unique α_c-Jordan-Hölder filtration of (E,V);

(b) all the sequences

$$0\to(Q_i,W_i)\xrightarrow{\alpha_i'}(E_{i+1},V_{i+1})/(E_{i-1},V_{i-1})\xrightarrow{\beta_i'}(Q_{i+1},W_{i+1})\to 0 \tag{2.8}$$

are non-split for $i=1,\cdots,r-1$.

Proof. Let us suppose that the α_c-Jordan-Hölder filtration is unique and let us prove (b). Let us fix any index $i = 1, \cdots, r-1$ and any non-zero morphism

$$\gamma_i : (Q_{i+1}, W_{i+1}) \longrightarrow (E_{i+1}, V_{i+1})/(E_{i-1}, V_{i-1}).$$

Then let us consider the following commutative diagram:

$$\begin{tikzcd}
(E_{i+1}, V_{i+1})/(E_{i-1}, V_{i-1}) \arrow[r, "\beta_i'"] \arrow[d, hook, "\zeta_i'"] \arrow[dr, phantom, "\curvearrowright"] & (E_{i+1}, V_{i+1})/(E_i, V_i) \arrow[d, hook, "\zeta_i"] \\
(E, V)/(E_{i-1}, V_{i-1}) \arrow[r, "\beta_i"] & (E, V)/(E_i, V_i).
\end{tikzcd}$$

Let us write $\gamma := \zeta_i' \circ \gamma_i : (Q_{i+1}, W_{i+1}) \to (E, V)/(E_{i-1}, V_{i-1})$. Since the α_c-Jordan-Hölder filtration is unique, it coincides with the α_c-canonical filtration by lemma 2.1.2. So we can use part (b) of proposition 2.1.3, hence $\beta_i \circ \gamma = 0$. By commutativity of the previous diagram, this gives $\zeta_i \circ \beta_i' \circ \gamma_i = 0$. Since ζ_i is injective, we get $\beta_i' \circ \gamma_i = 0$. So we have proved that for every $i = 1, \cdots, r-1$ there are no splittings for (2.8).

Conversely, let us suppose that (2.8) is non-split for all $i = 1, \cdots, r-1$. We claim that $\{(E_i, V_i)\}_{i=1,\cdots,r}$ is the α_c-canonical filtration of (E, V). In order to do that, it suffices to prove condition (b) of proposition 2.1.3. So let us fix any index $i \in \{1, \cdots, r-1\}$, any α_c-stable coherent system (Q, W) with α_c-slope μ and any non-zero morphism

$$\gamma : (Q, W) \to (E, V)/(E_{i-1}, V_{i-1}).$$

By contradiction, let us suppose that $\beta_i \circ \gamma \neq 0$. Since γ has values in $(E, V)/(E_{i-1}, V_{i-1})$, there exists a coherent system (E_i', V_i') containing (E_{i-1}, V_{i-1}) and contained in (E, V), such that $\mathrm{Im}(\gamma) = (E_i', V_i')/(E_{i-1}, V_{i-1})$. Since $\beta_i \circ \gamma \neq 0$, we get that $(E_i', V_i') \neq (E_i, V_i)$. Since (E_i', V_i') is contained in $(E, V) = (E_r, V_r)$, there exists a unique index $j \in \{i+1, \cdots, r\}$ such that

$$(E_i', V_i') \subset (E_j, V_j) \quad \text{and} \quad (E_i', V_i') \nsubseteq (E_{j-1}, V_{j-1}). \tag{2.9}$$

Then let us consider the composition

$$\varphi : (E_i', V_i')/(E_{i-1}, V_{i-1}) \hookrightarrow (E_j, V_j)/(E_{i-1}, V_{i-1}) \twoheadrightarrow$$
$$\twoheadrightarrow (E_j, V_j)/(E_{j-1}, V_{j-1}) = (Q_j, W_j).$$

This morphism is non-zero by (2.9). Moreover, $(E_i', V_i')/(E_{i-1}, V_{i-1}) = \gamma(Q, W)$. Since (Q, W) is α_c-stable, then by lemma 1.0.4 also $(E_i', V_i')/(E_{i-1}, V_{i-1})$ is α_c-stable. Also (Q_j, W_j) is α_c-stable by definition of Jordan-Hölder filtration. Then φ is an isomorphism by lemma 1.0.4. Now let us consider the embedding η defined as follows:

$$\eta:\ (Q_j,W_j)\xrightarrow{\varphi^{-1}}(E_i',V_i')/(E_{i-1},V_{i-1})\overset{\delta}{\hookrightarrow}(E_j,V_j)/(E_{i-1},V_{j-1}).$$

Let us consider the short exact sequence

$$0\to\frac{(E_{j-1},V_{j-1})}{(E_{i-1},V_{i-1})}\overset{\alpha}{\to}\frac{(E_j,V_j)}{(E_{i-1},V_{i-1})}\overset{\beta}{\to}\frac{(E_j,V_j)}{(E_{j-1},V_{j-1})}=(Q_j,W_j)\to 0, \tag{2.10}$$

where the morphisms α and β are easy to guess. Now

$$\beta\circ\eta=\beta\circ\delta\circ\varphi^{-1}.$$

Let us suppose that $\beta\circ\eta=0$; then $\beta\circ\delta=0$, so by exactness of (2.10) we have that δ induces an embedding of (E_i',V_i') in (E_{j-1},V_{j-1}), but this is impossible by (2.9). Therefore $\beta\circ\eta\neq 0$. Since (Q_j,W_j) is α_c-stable, we get that $\beta\circ\eta=\lambda\cdot\mathrm{id}_{(Q_j,W_j)}$ for some $\lambda\in\mathbb{C}^*$. Now let us consider the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \frac{(E_{j-1},V_{j-1})}{(E_{i-1},V_{i-1})} \arrow[r, "\alpha"'] \arrow[d, two heads] \arrow[dr, phantom, "\curvearrowright"] & \frac{(E_j,V_j)}{(E_{i-1},V_{i-1})} \arrow[r, "\beta"'] \arrow[d, two heads, "\pi"] \arrow[dr, phantom, "\curvearrowright"] & \frac{(E_j,V_j)}{(E_{j-1},V_{j-1})} \arrow[r] \arrow[l, bend right, hook', "\eta"'] \arrow[d, equal] & 0 \\
0 \arrow[r] & \frac{(E_{j-1},V_{j-1})}{(E_{j-2},V_{j-2})} \arrow[r, "\alpha'_{j-1}"] & \frac{(E_j,V_j)}{(E_{j-2},V_{j-2})} \arrow[r, "\beta'_{j-1}"] & \frac{(E_j,V_j)}{(E_{j-1},V_{j-1})} \arrow[r] \arrow[l, bend left, hook', "\eta'_{j-1}"] & 0,
\end{tikzcd}$$

where we define $\eta'_{j-1}:=\pi\circ\eta$. By commutativity, it is easy to see that η'_{j-1} makes the second line split. This contradicts the hypothesis of (b) for $j-1$, so we conclude. $\square$

Chapter 3

Technical lemmas

In this chapter we state some results on pullbacks of families of extensions of coherent systems and tensor products by line bundles. Moreover, we define non-degenerate extensions and we describe necessary and sufficient conditions for having such kind of extensions.

3.1 Extensions of coherent systems

Definition 3.1.1. Let us fix any scheme S and any pair of families $(\mathcal{E}, \mathcal{V}), (\mathcal{E}', \mathcal{V}')$ (of coherent systems over C) parametrized by S. Then let us define the vector space

$$\mathrm{Hom}_S((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V}))$$

as the set of all morphisms (α, β) as described in definition 1.0.8. The functor $\mathrm{Hom}_S((\mathcal{E}', \mathcal{V}'), -)$ is right exact. We denote by $\mathrm{Ext}^i_S((\mathcal{E}', \mathcal{V}'), -)$ its right derived functors. If $S = \mathrm{Spec}\ \mathbb{C}$, we denote by $\mathrm{Ext}^i((E', V'), -)$ the corresponding derived functors. For the relationship between the functors $\mathrm{Ext}^i_S(-,-)$'s and $\mathcal{E}xt^i_{\pi_S}(-,-)$'s, see [BGMMN, proposition A.9].

The construction of the functors $\mathrm{Ext}^i_S(-,-)$ follows from [He], where the families of coherent systems parametrized by S are embedded in a larger category of algebraic systems on $C \times S/S$. This larger category is an abelian category with enough injectives, so derived functors are defined for every right exact functor. Given any pair of families as before, the sets

$$\mathrm{Ext}^i_S((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V}))$$

are all vector spaces.

The following lemma is a direct consequence of the definition of the functors $\mathrm{Ext}^1_S(-,-)$'s and it is already implicit in the proof of [BGMN, proposition A.9]. The lemma is also stated explicitly in [BGMN, proposition 3.1] in the particular case when $S = \mathrm{Spec}(\mathbb{C})$.

Lemma 3.1.1. *For all schemes S and for all pairs of families $(\mathcal{E}, \mathcal{V})$, $(\mathcal{E}', \mathcal{V}')$ parametrized by S, there is a canonical bijection from $Ext^1_S((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V}))$ to the set of all short exact sequences*

$$0 \longrightarrow (\mathcal{E}, \mathcal{V}) \longrightarrow (\mathcal{F}, \mathcal{W}) \longrightarrow (\mathcal{E}', \mathcal{V}') \longrightarrow 0 \tag{3.1}$$

modulo equivalences.

Here an extension (3.1) is equivalent to an extension

$$0 \longrightarrow (\mathcal{E}, \mathcal{V}) \longrightarrow (\mathcal{G}, \mathcal{Z}) \longrightarrow (\mathcal{E}', \mathcal{V}') \longrightarrow 0$$

if and only if there is an isomorphism $(\mathcal{F}, \mathcal{W}) \xrightarrow{\sim} (\mathcal{G}, \mathcal{Z})$ making the following diagram commute

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{E}, \mathcal{V}) \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\curvearrowright"] & (\mathcal{F}, \mathcal{W}) \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "\curvearrowright"] & (\mathcal{E}', \mathcal{V}') \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\mathcal{E}, \mathcal{V}) \arrow[r] & (\mathcal{G}, \mathcal{Z}) \arrow[r] & (\mathcal{E}', \mathcal{V}') \arrow[r] & 0.
\end{tikzcd}$$

Proof. This is a standard fact for an abelian category with enough injectives. It is therefore sufficient to observe that, given a short exact sequence in the category of algebraic systems on $C \times S/S$ for which the left and right hand members are families of coherent systems parametrized by S, then the whole sequence belongs to the category of coherent systems parametrized by S. □

3.2 Pullbacks of extensions of families and tensors of extensions of families by line bundles

Lemma 3.2.1. *Let S be any scheme and let*

$$0 \longrightarrow (\mathcal{E}_1, \mathcal{V}_1) \longrightarrow (\mathcal{E}, \mathcal{V}) \longrightarrow (\mathcal{E}_2, \mathcal{V}_2) \longrightarrow 0 \tag{3.2}$$

be any short exact sequence of families of coherent systems on C parametrized by S. Let $f : S' \to S$ be any morphism of schemes. Then the pullback of (3.2) via f is an exact sequence of families of coherent systems parametrized by S'.

Proof. Given the definition of pullbacks of coherent systems, it is enough to remark that pullbacks of short exact sequences of vector bundles remain short exact. □

Lemma 3.2.2. *Let us fix any scheme S and any short exact sequence (3.2) of families of coherent systems on C parametrized by S. Let us fix also also any line bundle $\mathcal{L}$ on S. Then the sequence obtained by tensoring (3.2) by $\mathcal{L}$ is again a short exact sequence.*

Proof. Tensoring (3.2) by $\mathcal{L}$ amounts to tensoring by $\pi_S^*\mathcal{L}$ an exact sequence of vector bundles on $C \times S$ and by $\mathcal{L}$ an exact sequence on S, so we get again 2 exact sequences of vector bundles on $C \times S$ and S respectively. If we put together these 2 sequences we get the desired result. □

3.3 Non-degenerate extensions

Definition 3.3.1. Let us fix any short exact sequence

$$0 \longrightarrow (E', V') \xrightarrow{\alpha} (E, V) \xrightarrow{\beta} (E'', V'') \longrightarrow 0 \tag{3.3}$$

of coherent systems with α_c-slope μ. Let us suppose that

$$(E', V') \simeq (Q_1, W_1)^{\oplus t_1} \oplus \cdots \oplus (Q_r, W_r)^{\oplus t_r},$$

where $t_1, \cdots, t_r \geq 1$ and $(Q_i, W_i) \not\simeq (Q_j, W_j)$ for all $i \neq j$. If ξ is the class of (3.3), then ξ is associated to a sequence

$$(\xi_1^1, \cdots, \xi_1^{t_1}, \cdots, \xi_r^1, \cdots, \xi_r^{t_1})$$

where every ξ_i^j belongs to $H_i := \mathrm{Ext}^1((E'', V''), (Q_i, W_i))$. Then we say that the sequence (3.3) is *non-degenerate on the left* (of rank $(t_1, \cdots, t_r)$) if for all $i = 1, \cdots, r$ we have that $\xi_i^1, \cdots, \xi_i^{t_i}$ are linearly independent in H_i.

Definition 3.3.2. Let us fix any short exact sequence (3.3) of coherent systems with α_c-slope μ. Let us suppose that

$$(E'', V'') \simeq (Q_1, W_1)^{\oplus t_1} \oplus \cdots \oplus (Q_r, W_r)^{\oplus t_r},$$

where $t_1, \cdots, t_r \geq 1$ and $(Q_i, W_i) \not\simeq (Q_j, W_j)$ for all $i \neq j$. If ξ is the class of (3.3), then ξ is associated to a sequence

$$(\xi_1^1, \cdots, \xi_1^{t_1}, \cdots, \xi_r^1, \cdots, \xi_r^{t_r}),$$

where every ξ_i^j belongs to $H_i := \mathrm{Ext}^1((Q_i, W_i), (E', V'))$. Then we say that the sequence (3.3) is *non-degenerate on the right* (of rank $(t_1, \cdots, t_r)$) if for all $i = 1, \cdots, r$ we have that $\xi_i^1, \cdots, \xi_i^{t_i}$ are linearly independent in H_i.

Lemma 3.3.1. *Let us fix any sequence (3.3) of coherent systems with α_c-slope μ, associated to a sequence $(\xi_1^1, \cdots, \xi_1^{t_1}, \cdots, \xi_r^1, \cdots, \xi_r^{t_r})$ as in definition (3.3.1) and let us suppose that (Q_i, W_i) is α_c-stable for every $i = 1, \cdots, r$. Then (3.3) is non-degenerate on the left if and only if for all $i = 1, \cdots, r$ and for all quotients $\zeta_i : (E, V) \twoheadrightarrow (Q_i, W_i)$ we have that $\zeta_i \circ \alpha = 0$.*

Proof. Let us suppose that there is any quotient ζ_i such that $\zeta_i \circ \alpha \neq 0$. Up to reordering the (Q_i, W_i)'s, we can assume that $i = 1$; we write (Q', W') for the direct sum of all the objects of (E', V') not isomorphic to (Q_1, W_1). Since $(Q_1, W_1) \not\simeq (Q_i, W_i)$ for all $i \neq 1$, then there exists $(a_1, \cdots, a_{t_1}) \in \mathbb{C}^{t_1} \smallsetminus \{0\}$ such that

$$0 \neq \zeta_1 \circ \alpha = (a_1, \cdots, a_{t_1}, 0) : (Q_1, W_1)^{\oplus t_1} \oplus (Q', W') \longrightarrow (Q_1, W_1).$$

Up to reordering, we can assume that $a_1 \neq 0$. If $t_1 = 1$, then this implies easily that $\xi_1^1 = 0$, so we get that (3.3) is degenerate on the left; so we only consider the case when $t_1 > 1$.

Since the target of $\zeta_1 \circ \alpha$ is α_c-stable, then such a morphism is surjective and we can complete to an exact sequence

$$\begin{aligned} 0 \to (Q_1, W_1)^{t_1 - 1} \oplus (Q', W') &\overset{f_{a_1,\cdots,a_{t_1}} \oplus \mathrm{id}_{(Q',W')}}{\longrightarrow} \\ \overset{f_{a_1,\cdots,a_{t_1}} \oplus \mathrm{id}_{(Q',W')}}{\longrightarrow} (Q_1, W_1)^{\oplus t_1} \oplus (Q', W') &\overset{\zeta_1 \circ \alpha = (a_1, \cdots, a_{t_1}, 0)}{\longrightarrow} (Q_1, W_1) \to 0, \end{aligned} \tag{3.4}$$

where

$$f_{a_1,\cdots,a_{t_1}} := \begin{pmatrix} -a_2/a_1 & \cdots & -a_{t_1}/a_1 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}.$$

Let ξ be the class of the sequence (3.3) in

$$\mathrm{Ext}^1\left((E'', V''), (Q_1, W_1)^{\oplus t_1} \oplus (Q', W')\right)$$

and let $\xi' := \overline{(a_1, \cdots, a_{t_1}, 0)}(\xi)$; so we have a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & (Q_1, W_1)^{\oplus t_1} \oplus (Q', W') \arrow[r, "\alpha"] \arrow[d, "{(a_1,\cdots,a_{t_1},0)}"'] \arrow[dr, phantom, "\curvearrowright"] & (E, V) \arrow[r, "\beta"] \arrow[d, "\eta"] \arrow[dr, phantom, "\curvearrowright"] & (E'', V'') \arrow[r] \arrow[d, equal] & 0 & \xi \arrow[d, "\overline{(a_1,\cdots,a_{t_1},0)}"] \\ 0 \arrow[r] & (Q_1, W_1) \arrow[r, "\alpha'"] & (\widetilde{E}, \widetilde{V}) \arrow[r, "\beta'"] & (E'', V'') \arrow[r] & 0 & \xi'. \end{tikzcd} \tag{3.5}$$

By the snake lemma and (3.4) we have:

$$\mathrm{Ker}\ \eta = \mathrm{Ker}\ (a_1, \cdots, a_{t_1}, 0) = (Q_1, W_1)^{\oplus t_1 - 1} \oplus (Q', W') = \mathrm{Im}\ \alpha \circ (f_{a_1,\cdots,a_{t_1}} \oplus \mathrm{id}_{(Q',W')}).$$

Now by definition of $(a_1, \cdots, a_{t_1})$, we have that

$$\zeta_1 \circ \alpha \circ (f_{a_1,\cdots,a_{t_1}} \oplus \mathrm{id}_{(Q',W')}) = (a_1, \cdots, a_{t_1}, 0) \circ (f_{a_1,\cdots,a_{t_1}} \oplus \mathrm{id}_{(Q',W')}) = 0.$$

Therefore ζ_1 is zero on the kernel of η. So we get that ζ_1 induces a non-zero morphisms ζ_1' from $(\widetilde{E}, \widetilde{V}) = (E, V)/\mathrm{Ker}\ \eta$ to (Q_1, W_1), such that $\zeta_1' \circ \eta = \zeta_1$. By commutativity of the previous diagram, we get that

$$\zeta_1' \circ \alpha' \circ (a_1, \cdots, a_{t_1}, 0) = \zeta_1' \circ \eta \circ \alpha = \zeta_1 \circ \alpha \neq 0.$$

In particular, we get that $\zeta_1' \circ \alpha' \neq 0$, so such a morphism belongs to $\mathrm{Aut}(Q_1, W_1) = \mathbb{C}^*$. Therefore ζ_1' gives a splitting of the second line of (3.5), so $\overline{(a_1, \cdots, a_{t_1}, 0)}(\xi) = \xi' = 0$. So we get that

$$a_1 \cdot \xi_1^1 + \cdots + a_{t_1} \cdot \xi_1^{t_1} = 0, \tag{3.6}$$

so $\xi_1^1, \cdots, \xi_1^{t_1}$ are linearly dependent.

Conversely, if $\xi_1^1, \cdots, \xi_1^{t_1}$ are linearly dependent, then there exists a sequence $(a_1, \cdots, a_{t_1}) \in \mathbb{C}^{t_1} \smallsetminus \{0\}$ such that (3.6) holds. Then we have that

$$\overline{(a_1, \cdots, a_{t_1}, 0)}(\xi) = 0,$$

so we have a diagram of the form (3.5) with the second line that is split. So we get a quotient:

$$\zeta_1 : (E, V) \twoheadrightarrow (\widetilde{E}, \widetilde{V}) \simeq (Q_1, W_1) \oplus (E'', V'') \twoheadrightarrow (Q_1, W_1),$$

A direct check proves that $\alpha \circ \zeta_1 = (a_1, \cdots, a_{t_1}, 0) \neq 0$, so this is enough to conclude. □

Lemma 3.3.2. *Let us fix any sequence (3.3) of coherent systems with α_c-slope μ, associated to a sequence $(\xi_1^1, \cdots, \xi_1^{t_1}, \cdots, \xi_r^1, \cdots \xi_r^{t_r})$ as in definition (3.3.2) and let us suppose that (Q_i, W_i) is α_c-stable for every $i = 1, \cdots, r$. Then (3.3) is non-degenerate on the right if and only if for all $i = 1, \cdots, r$ and for all morphisms $\gamma_i : (Q_i, W_i) \hookrightarrow (E, V)$ we have that $\beta \circ \gamma_i = 0$.*

Proof. This is the dual of the proof of lemma 3.3.1. □

Chapter 4

Universal families of extensions

We want to prove a series of statements analogous to those in [L] for families of extensions of coherent systems instead of families of extensions of coherent sheaves. The statements of [L] are true for every projective morphism $f : X \to Y$. In our case, we have to restrict to the case when f is the projection $\pi_S : X \times S \to S$ for any projective scheme X and for any noetherian scheme S because we have to use [He, proposition 1.13], that is proved only in this case. It seems possible to prove results analogous to those of [L] in full generality; anyway for this work the version we will prove below will be sufficient.

Almost all the results of this chapter hold under the hypothesis that X is any projective scheme, we don't require that it is a smooth irreducible projective curve C. Only in the last section we will restrict to that particular case.

Note that as in [L], we need a flatness hypothesis on the families we will use. Such an hypothesis is implicit in the definition of families of coherent systems, see remark 1.0.2.

4.1 Cohomology and base change for families of coherent systems

First of all, we will need to write a statement of cohomology and base change, so we need a result analogous to [L, corollary 1.2]. In this section we will have to consider every family of coherent systems as a family of weak coherent systems as in definition 1.0.7. Let us first state the following preliminary result.

Proposition 4.1.1. *Let X be any projective scheme and let $(\mathcal{E}, \mathcal{V}, \phi)$ and $(\mathcal{E}', \mathcal{V}', \phi')$ be two families of coherent systems over X, parametrized by a noetherian scheme S. Moreover, let us fix also any noetherian S-scheme $u : S' \to S$. Then there exists a resolution $\Delta_\bullet$ of $(\mathcal{E}', \mathcal{V}', \phi')$ such that:*

(i) $\Delta_0 = (\mathcal{P}_0, 0, 0) \oplus (\pi_S^* \mathcal{V}', \mathcal{V}', id_{\pi_S^* \mathcal{V}'})$

(ii) $\Delta_j = (\mathcal{P}_j, 0, 0)$ *for all* $j \geq 1$;

(iii) $\mathcal{P}_j$ *is locally free on* $X \times S$ *for all* $j \geq 0$*;*

(iv) for all locally free $\mathcal{O}_{S'}$*-modules* $\mathcal{M}$*, for all* $j \geq 0$ *and for all* $i \geq 1$ *we have* $\mathcal{E}xt^i_{\pi_{S'}}((u',u)^*\Delta_j, (u',u)^*(\mathcal{E},\mathcal{V},\phi) \otimes_{S'} \mathcal{M}) = 0$*;*

(v) for all $j \geq 0$ *the sheaf* $\mathcal{L}^j := \mathcal{H}om_{\pi_S}(\Delta_j, (\mathcal{E},\mathcal{V},\phi))$ *is locally free on* S*.*

Proof. The proof consists simply in combining the proof of [He, proposition 1.13] with the proof of [L, lemma 1.1]. Actually, point (iv) holds for every quasi-coherent $\mathcal{O}_{S'}$-module $\mathcal{M}$, once we suitably enlarge the category of coherent systems to take into account also algebraic systems (see remark 1.0.3). □

Definition 4.1.1. In the notation of [He], a *very negative resolution of* $(\mathcal{E}',\mathcal{V}',\phi')$ *with respect to* $(\mathcal{E},\mathcal{V},\phi)$ is any resolution $\Delta_\bullet$ of $(\mathcal{E}',\mathcal{V}',\phi')$ with properties (i), (ii), (iii) and:

(iv) $\mathcal{E}xt^i_{\pi_S}(\Delta_j, (\mathcal{E},\mathcal{V},\phi)) = 0$ for all $j \geq 0$ and for all $i \geq 1$.

Remark 4.1.1. The previous proposition proves that if we fix any morphism $u : S' \to S$ and any pair of families $(\mathcal{E},\mathcal{V},\phi)$, $(\mathcal{E}',\mathcal{V}',\phi')$ parametrized by S, then the resolution $(u',u)^*\Delta_\bullet$ is a very negative resolution of $(u',u)^*(\mathcal{E}',\mathcal{V}',\phi')$ with respect to $(u',u)^*(\mathcal{E},\mathcal{V},\phi) \otimes_{S'} \mathcal{M}$ for all locally free $\mathcal{O}_{S'}$-modules $\mathcal{M}$. In particular, if we choose $u = \mathrm{id}_S$ and $\mathcal{M} = \mathcal{O}_S$, we get a very negative resolution of $(\mathcal{E}',\mathcal{V}',\phi')$ with respect to $(\mathcal{E},\mathcal{V},\phi)$.

We recall the following result, obtained from [He, remarque 1.15] together with remark 1.0.2.

Lemma 4.1.2. *For every noetherian scheme* S*, for every pair of families* $(\mathcal{E},\mathcal{V},\phi)$*,* $(\mathcal{E}',\mathcal{V}',\phi')$ *of coherent systems parametrized by* S*, for every negative resolution* $\Delta_\bullet$ *of* $(\mathcal{E}',\mathcal{V}',\phi')$ *with respect to* $(\mathcal{E},\mathcal{V},\phi)$ *and for every* $i \geq 0$ *we have a canonical isomorphism of sheaves over* S*:*

$$\mathcal{E}xt^i_{\pi_S}\left((\mathcal{E}',\mathcal{V}',\phi'),(\mathcal{E},\mathcal{V},\phi)\right) = \mathcal{H}^i\left(\mathcal{H}om_{\pi_S}\left(\Delta_\bullet,(\mathcal{E},\mathcal{V},\phi)\right)\right).$$

Now let us fix any $u : S' \to S$ and any 2 families parametrized by S as before; let $\Delta_\bullet$ and $\mathcal{L}^\bullet$ be as in proposition 4.1.1. Then we have an analogue of [L, corollary 1.2 (ii)] as follows.

Lemma 4.1.3. *For all* S*-schemes* $u : S' \to S$*, for all locally free* $\mathcal{O}_{S'}$*-modules* $\mathcal{M}$ *and for all* $j \geq 0$ *there is a canonical isomorphism of sheaves on* S'*:*

$$\mathcal{H}om_{\pi_{S'}}\Big((u',u)^*\Delta_j,(u',u)^*(\mathcal{E},\mathcal{V},\phi) \otimes_{S'} \mathcal{M}\Big) \simeq u^*\mathcal{L}^j \otimes_{S'} \mathcal{M}. \tag{4.1}$$

Proof. We have to consider two different cases depending on j.

Case (i) Let us suppose that $j \geq 1$. Then for all V open in S' we have:

$$
\begin{aligned}
&\mathcal{H}om_{\pi_{S'}}\Big((u',u)^*\Delta_j,(u',u)^*(\mathcal{E},\mathcal{V},\phi)\otimes_{S'}\mathcal{M}\Big)(V)=\\
&=\mathrm{Hom}_V\Big((u'^*\mathcal{P}_j,0,0)|_V,(u'^*\mathcal{E}\otimes_{X\times S'}\pi_{S'}^*\mathcal{M},u^*\mathcal{V}\otimes_{S'}\mathcal{M},\widetilde{\phi})|_V\Big)=\\
&=\mathrm{Hom}_V\Big((u'^*\mathcal{P}_j|_{\pi_{S'}^{-1}(V)},0,0),(u'^*\mathcal{E}\otimes_{X\times S'}\pi_{S'}^*\mathcal{M}|_{\pi_{S'}^{-1}V},u^*\mathcal{V}\otimes_{S'}\mathcal{M}|_V,\widetilde{\phi}|_V)\Big)=\\
&=\mathcal{H}om_{\pi_{S'}}\Big(u'^*\mathcal{P}_j,u'^*\mathcal{E}\otimes_{X\times S'}\pi_{S'}^*\mathcal{M}\Big)(V).
\end{aligned}
$$

Therefore, we have:

$$
\begin{aligned}
&\mathcal{H}om_{\pi_{S'}}\left((u',u)^*\Delta_j,(u',u)^*(\mathcal{E},\mathcal{V},\phi)\otimes_{S'}\mathcal{M}\right)=\mathcal{H}om_{\pi_{S'}}(u'^*\mathcal{P}_j,u'^*\mathcal{E}\otimes_{X\times S'}\pi_{S'}^*\mathcal{M})=\\
&=(\pi_{S'})_*\mathcal{H}om_{\mathcal{O}_{X\times S'}}(u'^*\mathcal{P}_j,u'^*\mathcal{E}\otimes_{X\times S'}\pi_{S'}^*\mathcal{M})=(\pi_{S'})_*(u'^*\mathcal{P}_j^\vee\otimes_{X\times S'}u'^*\mathcal{E}\otimes_{X\times S'}\pi_{S'}^*\mathcal{M})=\\
&=(\pi_{S'})_*\Big(u'^*(\mathcal{P}_j^\vee\otimes_{X\times S}\mathcal{E})\otimes_{X\times S'}\pi_{S'}^*\mathcal{M}\Big)=(\pi_{S'})_*\Big(u'^*(\mathcal{P}_j^\vee\otimes_{X\times S}\mathcal{E})\Big)\otimes_{S'}\mathcal{M}. \qquad (4.2)
\end{aligned}
$$

Here the third equality is proved using the fact that $u'^*\mathcal{P}_j$ is locally free because $\mathcal{P}_j$ is so. Analogous computations (with u replaced by id_S and $\mathcal{M}$ by $\mathcal{O}_S$) prove that for all $j\geq 1$:

$$\mathcal{L}^j=\mathcal{H}om_{\pi_S}\left(\Delta_j,(\mathcal{E},\mathcal{V},\phi)\right)=(\pi_S)_*\left(\mathcal{P}_j^\vee\otimes_{X\times S}\mathcal{E}\right). \qquad (4.3)$$

Now by proposition 4.1.1 (v) we have that $\mathcal{L}^j$ is locally free on S; therefore by base change ([Ha, III, prop. 12.11 and prop. 12.5]) we have:

$$(\pi_{S'})_*u'^*(\mathcal{P}_j^\vee\otimes_{X\times S}\mathcal{E})=u^*\pi_{S*}(\mathcal{P}_j^\vee\otimes_{X\times S}\mathcal{E})=u^*\mathcal{L}^j.$$

Therefore, we have that (4.2) is equal to $u^*\mathcal{L}^j\otimes_{S'}\mathcal{M}$. So we have proved that (4.1) is true for all $j\geq 1$.

Case (ii) Let us suppose that $j=0$; then we have that $\Delta_0=(\mathcal{P}_0,0,0)\oplus(\pi_S^*\mathcal{V}',\mathcal{V}',\mathrm{id})$. By the same idea used in the previous case, we have a canonical isomorphism:

$$
\begin{aligned}
&\mathcal{H}om_{\pi_{S'}}\Big((u',u)^*(\mathcal{P}_0,0,0),(u',u)^*(\mathcal{E},\mathcal{V},\phi)\otimes_{S'}\mathcal{M}\Big)\simeq\\
&\simeq u^*\mathcal{H}om_{\pi_S}\Big((\mathcal{P}_0,0,0),(\mathcal{E},\mathcal{V},\phi)\Big)\otimes_{S'}\mathcal{M}.
\end{aligned}
$$

Therefore, in order to prove that (4.1) is still valid for $j=0$, it suffices to prove that there is a canonical isomorphism:

$$
\begin{aligned}
&\mathcal{H}om_{\pi_{S'}}\Big((u',u)^*(\pi_S^*\mathcal{V}',\mathcal{V}',\mathrm{id}),(u',u)^*(\mathcal{E},\mathcal{V},\phi)\otimes_{S'}\mathcal{M}\Big)\overset{?}{\simeq}\\
&\overset{?}{\simeq}u^*\mathcal{H}om_{\pi_S}\Big((\pi_S^*\mathcal{V}',\mathcal{V}',\mathrm{id}),(\mathcal{E},\mathcal{V},\phi)\Big)\otimes_{S'}\mathcal{M}. \qquad (4.4)
\end{aligned}
$$

Now for every open set V in S' we have:

$$\begin{aligned}\mathcal{H}om_{\pi_{S'}}\Big((u',u)^*(\pi_S^*\mathcal{V}',\mathcal{V}',\mathrm{id}),(u',u)^*(\mathcal{E},\mathcal{V},\phi)\otimes_{S'}\mathcal{M}\Big)(V)=\\ =\mathrm{Hom}_V\Big((u'^*\pi_S^*\mathcal{V}',u^*\mathcal{V}',\widetilde{\mathrm{id}})|_V,(u'^*\mathcal{E}\otimes_{X\times S'}\pi_{S'}^*\mathcal{M},u^*\mathcal{V}\otimes_{S'}\mathcal{M},\widetilde{\phi})|_V\Big)\end{aligned}\tag{4.5}$$

where $\widetilde{\mathrm{id}}$ is given by the composition:

$$\widetilde{\mathrm{id}}:\pi_{S'}^*u^*\mathcal{V}'\xrightarrow{\eta}u'^*\pi_S^*\mathcal{V}'\xrightarrow{u'^*(\mathrm{id})}u'^*\pi_S^*\mathcal{V}'$$

and η is the canonical isomorphism induced by $\pi_S\circ u'=u\circ\pi_{S'}$. Therefore, $\widetilde{\mathrm{id}}=\eta$ is an isomorphism. Hence (4.5) is the set of all pairs (α,β) of the form:

$$\begin{gathered}\alpha:u'^*\pi_S^*\mathcal{V}'|_{\pi_{S'}^{-1}(V)}\longrightarrow(u'^*\mathcal{E}\otimes_{X\times S'}\pi_{S'}^*\mathcal{M})|_{\pi_{S'}^{-1}(V)},\\ \beta:u^*\mathcal{V}'|_V\longrightarrow(u^*\mathcal{V}\otimes_{S'}\mathcal{M})|_V\end{gathered}$$

such that they make this diagram commute:

$$\begin{tikzcd}
\pi_{S'}^*u^*\mathcal{V}'|_{\pi_{S'}^{-1}(V)} \arrow[rr, "\eta|_{\pi_{S'}^{-1}(V)}"] \arrow[d, "\pi_{S'}^*\beta"'] & & u'^*\pi_S^*\mathcal{V}'|_{\pi_{S'}-1(V)} \arrow[d, "\alpha"] \\
\pi_{S'}^*(u^*\mathcal{V}\otimes_{S'}\mathcal{M})|_{\pi_{S'}^{-1}V} \arrow[rr, "\widetilde{\phi}|_{\pi_{S'}^{-1}(V)}"'] \arrow[urr, phantom, "\curvearrowright"] & & (u'^*\mathcal{E}\otimes_{X\times S'}\pi_{S'}^*\mathcal{M})|_{\pi_{S'}^{-1}(V)}.
\end{tikzcd}$$

Therefore, α is completely determined as

$$\alpha=\widetilde{\phi}|_{\pi_{S'}^{-1}(V)}\circ(\pi_{S'}^*\beta)\circ\left(\eta|_{\pi_{S'}^{-1}(V)}\right)^{-1}.$$

So, having fixed $(\pi_S^*\mathcal{V}',\mathcal{V}',\mathrm{id})$, $(\mathcal{E},\mathcal{V},\phi)$, $u:S'\to S$ and $\mathcal{M}$, we have that (4.5) is naturally identified with the set of all morphisms β as before, i.e. with the set

$$\mathrm{Hom}_V(u^*\mathcal{V}'|_V,u^*\mathcal{V}\otimes_{S'}\mathcal{M}|_V)=\mathcal{H}om_{\mathcal{O}_{S'}}(u^*\mathcal{V}',u^*\mathcal{V}\otimes_{S'}\mathcal{M})(V).$$

Therefore, the left hand side of (4.4) is given by:

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_{S'}}(u^*\mathcal{V}',u^*\mathcal{V}\otimes_{S'}\mathcal{M})=u^*\mathcal{V}'^\vee\otimes_{S'}u^*\mathcal{V}\otimes_{S'}\mathcal{M}=\\ =u^*(\mathcal{V}'^\vee\otimes_S\mathcal{V})\otimes_{S'}\mathcal{M}=u^*\mathcal{H}om_{\mathcal{O}_S}(\mathcal{V}',\mathcal{V})\otimes_{S'}\mathcal{M}.\end{aligned}$$

Here we used several times the fact that $\mathcal{V}'$ is locally free on S. By using exactly the same technique, we can prove that also the right hand side of (4.4) is given by the same expression, so we conclude. □

With the same ideas we can also prove the following result; we omit the proof since it is quite similar to the previous one.

Lemma 4.1.4. *For all S-schemes $u : S' \to S$, for all locally free $\mathcal{O}_{S'}$-modules $\mathcal{M}$ and for all $j \geq 0$ there is a canonical isomorphism of sheaves on S':*

$$\mathcal{H}om_{\pi_{S'}}\Big((u',u)^*\Delta_j \otimes_{S'} \mathcal{M}, (u',u)^*(\mathcal{E},\mathcal{V},\phi)\Big) \simeq u^*\mathcal{L}^j \otimes_{S'} \mathcal{M}^\vee.$$

Lemma 4.1.5. *For every pair of families as before parametrized by S, for every morphism of noetherian schemes $u : S' \to S$, for every locally free $\mathcal{O}_{S'}$-module $\mathcal{M}$ and for every $i \geq 0$ we have:*

$$\begin{aligned}&\mathcal{E}xt^i_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}',\phi'), (u',u)^*(\mathcal{E},\mathcal{V},\phi) \otimes_{S'} \mathcal{M}\Big) = \\ &= \mathcal{H}^i\Big(\mathcal{H}om_{\pi_{S'}}\Big((u',u)^*\Delta_\bullet, (u',u)^*(\mathcal{E},\mathcal{V},\phi) \otimes_{S'} \mathcal{M}\Big)\Big)\end{aligned}$$

where $\Delta_\bullet$ is as in proposition 4.1.1.

Proof. Let us consider the pullback $(u',u)^*\Delta_\bullet$: by remark 4.1.1 we get that it is a very negative resolution of $(u',u)^*(\mathcal{E}',\mathcal{V}',\phi')$ with respect to $(u',u)^*(\mathcal{E},\mathcal{V},\phi) \otimes_{S'}\mathcal{M}$ for all locally free $\mathcal{O}_{S'}$-modules $\mathcal{M}$. Therefore we can use lemma 4.1.2 for the families $(u',u)^*(\mathcal{E}',\mathcal{V}',\phi')$ and $(u',u)^*(\mathcal{E},\mathcal{V},\phi) \otimes_{S'} \mathcal{M}$ over S' and we conclude. □

Now if we combine lemma 4.1.5 with the canonical isomorphism of lemma 4.1.3, we get the following statement, that is analogous to [L, cor. 1.2.iii].

Lemma 4.1.6. *For every $i \geq 0$, for every morphism $u : S' \to S$ and for every locally free $\mathcal{O}_{S'}$-module $\mathcal{M}$ we have a canonical isomorphism of sheaves over S':*

$$\mathcal{E}xt^i_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}',\phi'), (u',u)^*(\mathcal{E},\mathcal{V},\phi) \otimes_{S'} \mathcal{M}\Big) = \mathcal{H}^i(u^*\mathcal{L}^\bullet \otimes_{S'} \mathcal{M})$$

where $\mathcal{L}^\bullet$ is as in proposition 4.1.1. Since $\mathcal{L}^\bullet$ is a complex of locally free sheaves on S by that proposition, this implies that the sheaf on the left is coherent on S'.

Now for every locally free $\mathcal{O}_S$-module $\mathcal{M}$ and for every $i \geq 0$, we define

$$\mathcal{T}^i(\mathcal{M}) := \mathcal{H}^i(\mathcal{L}^\bullet \otimes_S \mathcal{M}) = \mathcal{E}xt^i_{\pi_S}\Big((\mathcal{E}',\mathcal{V}',\phi'), (\mathcal{E},\mathcal{V},\phi) \otimes_S \mathcal{M}\Big),$$

where the last equality is given by the previous lemma with $u = id_S$. By [Ha, III, proposition 12.5] we get natural homomorphisms for every $i \geq 0$:

$$\varphi(i,\mathcal{M}) : \mathcal{T}^i(\mathcal{O}_S) \otimes_S \mathcal{M} \to \mathcal{T}^i(\mathcal{M}).$$

Moreover, for every morphism $u : S' \to S$ by using the same computation as [Ha, III, proposition 9.3 and remark 9.3.1] we get the base change homomorphism:

$$\tau^i(u) : u^*\mathcal{E}xt^i_{\pi_S}\Big((\mathcal{E}',\mathcal{V}',\phi'), (\mathcal{E},\mathcal{V},\phi)\Big) \longrightarrow \mathcal{E}xt^i_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}',\phi'), (u',u)^*(\mathcal{E},\mathcal{V},\phi)\Big).$$

In addition, by using again a resolution $\Delta_\bullet$ as in proposition 4.1.1 together with [Ha, III, proposition 9.3], we get the following result.

Proposition 4.1.7. *For every* flat *morphism $u : S' \to S$ of noetherian schemes and for every $i \geq 0$, the base change homomorphism $\tau^i(u)$ is an isomorphism.*

This is exactly [He, théorème 1.16 (i)], but with a more explicit construction of such an isomorphism, that was not described in that work. Moreover, by proceeding as in [Ha, III.12] we get the following result.

Proposition 4.1.8. (cohomology and base change for families of coherent systems) *Let X be a projective scheme, S any noetherian scheme and let $(\mathcal{E}, \mathcal{V}, \phi)$ and $(\mathcal{E}', \mathcal{V}', \phi')$ be two families of coherent systems on X, parametrized by S. Let s be any point in S and let us assume that the base change homomorphism*

$$\tau^i(s) : \mathcal{E}xt^i_{\pi_S}\Big((\mathcal{E}', \mathcal{V}', \phi'), (\mathcal{E}, \mathcal{V}, \phi)\Big) \otimes k(s) \to Ext^i\Big((\mathcal{E}', \mathcal{V}', \phi')_s, (\mathcal{E}, \mathcal{V}, \phi)_s\Big)$$

is surjective. Then:

(i) there is an open neighborhood U of s in S such that $\tau^i(s')$ is an isomorphism for all s' in U;

(ii) $\tau^{i-1}(s)$ is surjective if and only if $\mathcal{E}xt^i_{\pi_S}\Big((\mathcal{E}', \mathcal{V}', \phi'), (\mathcal{E}, \mathcal{V}, \phi)\Big)$ is locally free in an open neighborhood of s in S.

According to the usual definitions for coherent sheaves, if $\tau^i(s)$ is an isomorphism for all s in S, then we will say that $\mathcal{E}xt^i_{\pi_S}\Big((\mathcal{E}', \mathcal{V}', \phi'), (\mathcal{E}, \mathcal{V}, \phi)\Big)$ *commutes with base change.* If this is the case, then $\tau^i(u)$ is an isomorphism for all morphisms $u : S' \to S$ of noetherian schemes.

Remark 4.1.2. From now on, we will not need to refer explicitly to the maps of the form ϕ, so in the following lemmas and propositions we will use the notation of definition 1.0.6.

Exactly as in [L, lemma 4.1], we can prove the following consequence of lemma 4.1.3.

Lemma 4.1.9. *For every scheme S, for every pair of families $(\mathcal{E}, \mathcal{V})$, $(\mathcal{E}', \mathcal{V}')$ parametrized by S, for every locally free $\mathcal{O}_S$-module $\mathcal{M}$ and for every $i \geq 0$, there are canonical isomorphisms*

$$\mathcal{E}xt^i_{\pi_S}\Big((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V})\Big) \otimes_S \mathcal{M} \simeq \mathcal{E}xt^i_{\pi_S}\Big((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V}) \otimes_S \mathcal{M}\Big) \simeq$$
$$\simeq \mathcal{E}xt^i_{\pi_S}\Big((\mathcal{E}', \mathcal{V}') \otimes_S \mathcal{M}^\vee, (\mathcal{E}, \mathcal{V})\Big).$$

Lemma 4.1.10. *Let us suppose that $\mathcal{E}xt^1_{\pi_S}\Big((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V})\Big)$ commutes with base change and that $Hom\Big((\mathcal{E}', \mathcal{V}')_s, (\mathcal{E}, \mathcal{V})_s\Big) = 0$ for all s in S. Then for every S-scheme $u : S' \to S$ and for every locally free $\mathcal{O}_{S'}$-module $\mathcal{M}$, we have a canonical isomorphism:*

$$\mu : Ext^1_{S'}\Big((u', u)^*(\mathcal{E}', \mathcal{V}'), (u', u)^*(\mathcal{E}, \mathcal{V}) \otimes_{S'} \mathcal{M}\Big) \xrightarrow{\sim}$$
$$\xrightarrow{\sim} H^0\Big(S', \mathcal{E}xt^1_{\pi_{S'}}\Big((u', u)^*(\mathcal{E}', \mathcal{V}'), (u', u)^*(\mathcal{E}, \mathcal{V})\Big) \otimes_{S'} \mathcal{M}\Big).$$

The same conclusion holds if we assume that S is affine.

Proof. We recall that by [BGMMN, proposition A.9], there is a spectral sequence

$$H^p\Big(S', \mathcal{E}xt^q_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'), (u',u)^*(\mathcal{E},\mathcal{V})\otimes_{S'}\mathcal{M}\Big)\Big) \Rightarrow$$
$$\Rightarrow \mathrm{Ext}^{p+q}_{S'}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'), (u',u)^*(\mathcal{E},\mathcal{V})\otimes_{S'}\mathcal{M}\Big).$$

This induces a long exact sequence:

$$\begin{aligned}
0 \to H^1\Big(S', \mathcal{H}om_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'), (u',u)^*(\mathcal{E},\mathcal{V})\otimes_{S'}\mathcal{M}\Big)\Big) \to \\
\to \mathrm{Ext}^1_{S'}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'), (u',u)^*(\mathcal{E},\mathcal{V})\otimes_{S'}\mathcal{M}\Big) \xrightarrow{\mu} \\
\xrightarrow{\mu} H^0\Big(S', \mathcal{E}xt^1_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'), (u',u)^*(\mathcal{E},\mathcal{V})\otimes_{S'}\mathcal{M}\Big)\Big) \to \\
\to H^2\Big(S', \mathcal{H}om_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'), (u',u)^*(\mathcal{E},\mathcal{V})\otimes_{S'}\mathcal{M}\Big)\Big) \to \cdots
\end{aligned} \tag{4.6}$$

Now let us assume the first hypothesis. If $\mathrm{Hom}\Big((\mathcal{E}',\mathcal{V}')_s, (\mathcal{E},\mathcal{V})_s\Big) = 0$ for all $s \in S$, then this implies that the base change morphisms $\tau^0(s)$ are surjective for all s in S. Therefore, by base change $\mathcal{H}om_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'),\ (u',u)^*(\mathcal{E},\mathcal{V})\Big) = 0$; moreover, by lemma 4.1.9 (over S' instead of S) and base change we have that

$$\mathcal{H}om_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'), (u',u)^*(\mathcal{E},\mathcal{V})\otimes_{S'}\mathcal{M}\Big) =$$
$$= \mathcal{H}om_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'), (u',u)^*(\mathcal{E},\mathcal{V})\Big)\otimes_{S'}\mathcal{M} = u^*\mathcal{H}om_{\pi_S}\Big((\mathcal{E}',\mathcal{V}'), (\mathcal{E},\mathcal{V})\Big)\otimes_{S'}\mathcal{M} = 0.$$

So by substituting in the previous long exact sequence we get that there is an isomorphism

$$\mu : \mathrm{Ext}^1_{S'}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'), (u',u)^*(\mathcal{E},\mathcal{V})\otimes_{S'}\mathcal{M}\Big) \xrightarrow{\sim}$$
$$\xrightarrow{\sim} H^0\Big(S', \mathcal{E}xt^1_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'), (u',u)^*(\mathcal{E},\mathcal{V})\otimes_{S'}\mathcal{M}\Big)\Big).$$

Then we can apply again lemma 4.1.9 (over S' instead of S) for $i = 1$ and we get the result. If we assume that S is affine, then both the first and the last term of the previous long exact sequence are zero, so we conclude as before. □

4.2 Families of (classes of) extensions

Let us consider any scheme S and any pair of families parametrized by S as before. Moreover, let us consider any extension of $(\mathcal{E}',\mathcal{V}')$ by $(\mathcal{E},\mathcal{V})$:

$$0 \longrightarrow (\mathcal{E},\mathcal{V}) \longrightarrow (\mathcal{F},\mathcal{W}) \longrightarrow (\mathcal{E}',\mathcal{V}') \longrightarrow 0;$$

according to lemma 3.1.1, we can consider this as a representative of an object in $\mathrm{Ext}^1_S((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V}))$. Then for every point s in S the pullback of such an exact sequence to $X_s = X \times \{s\}$ gives rise to an extension:

$$0 \longrightarrow (\mathcal{E}, \mathcal{V})_s \longrightarrow (\mathcal{F}, \mathcal{W})_s \longrightarrow (\mathcal{E}', \mathcal{V}')_s \to 0.$$

Therefore, by lemma 3.1.1 we get a well defined map:

$$\Phi_s : \mathrm{Ext}^1_S\Big((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V})\Big) \to \mathrm{Ext}^1\Big((\mathcal{E}', \mathcal{V}')_s, (\mathcal{E}, \mathcal{V})_s\Big).$$

As in [L], we give the definition of family of extensions as follows:

Definition 4.2.1. A *family of (classes of) extensions of* $(\mathcal{E}', \mathcal{V}')$ *by* $(\mathcal{E}, \mathcal{V})$ *over* S is any family

$$\Big\{e_s \in \mathrm{Ext}^1\Big((\mathcal{E}', \mathcal{V}')_s, (\mathcal{E}, \mathcal{V})_s\Big)\Big\}_{s \in S}$$

such that there is an open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of S and for each $i \in I$ there is an element σ_i in $\mathrm{Ext}^1_{U_i}\Big((\mathcal{E}', \mathcal{V}')|_{U_i}, (\mathcal{E}, \mathcal{V})|_{U_i}\Big)$ such that $e_s = \Phi_{i,s}(\sigma_i)$ for every s in S and for every $i \in I$ such that $s \in U_i$. Here $\Phi_{i,s}$ denotes the canonical map

$$\Phi_{i,s} : \mathrm{Ext}^1_{U_i}\Big((\mathcal{E}', \mathcal{V}')|_{U_i}, (\mathcal{E}, \mathcal{V})|_{U_i}\Big) \to \mathrm{Ext}^1\Big((\mathcal{E}', \mathcal{V}')_s, (\mathcal{E}, \mathcal{V})_s\Big).$$

A family of extensions is called *globally defined* if the covering $\mathfrak{U}$ can be taken to be S itself.

For every s in S, let us define the canonical homomorphism

$$\iota_s : \mathcal{E}xt^1_{\pi_S}\Big((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V})\Big) \to \mathcal{E}xt^1_{\pi_S}\Big((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V})\Big) \otimes k(s).$$

Then we get a result analogous to that of [L, lemma 2.1].

Lemma 4.2.1. *For every s in S, the map Φ_s coincides with the composition:*

$$\begin{aligned} Ext^1_S\Big((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V})\Big) &\xrightarrow{\mu} H^0\Big(S, \mathcal{E}xt^1_{\pi_S}\Big((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V})\Big)\Big) \xrightarrow{H^0(\iota_s)} \\ &\xrightarrow{H^0(\iota_s)} H^0\Big(S, \mathcal{E}xt^1_{\pi_S}\Big((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V})\Big) \otimes k(s)\Big) = \\ &= \mathcal{E}xt^1_{\pi_S}\Big((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V})\Big) \otimes k(s) \xrightarrow{\tau^1(s)} Ext^1\Big((\mathcal{E}', \mathcal{V}')_s, (\mathcal{E}, \mathcal{V})_s\Big), \end{aligned}$$

where $\tau^1(s)$ is the base change homomorphism induced by the inclusion of s in S and μ is the map described in (4.6) with $u = id_S$ and $\mathcal{M} = \mathcal{O}_S$ (μ is not necessarily an isomorphism in this case).

Having fixed $(\mathcal{E},\mathcal{V})$ and $(\mathcal{E}',\mathcal{V}')$, we define

$$\mathrm{EXT}((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))$$

as the set of all the families of extensions between these two families of coherent systems. It is clear that such a set has a natural structure of vector space, so we would like to describe an isomorphism of it with some known vector space. First of all, we will give the description of the subvector space

$$\mathrm{EXT}_{glob}((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))$$

consisting of those families of extensions that are globally defined. By definition, every globally defined family is induced by an element of $\mathrm{Ext}^1_S((\mathcal{E}',\mathcal{V}'),\ (\mathcal{E},\mathcal{V}))$, but two elements of such a space can sometimes define the same family. The following proposition tells exactly when this happens.

Proposition 4.2.2. *Let us suppose that S is* reduced *and that $\mathcal{E}xt^1_{\pi_S}((\mathcal{E}',\mathcal{V}'),\ (\mathcal{E},\mathcal{V}))$ commutes with base change. Then there is a canonical isomorphism between the set $EXT_{glob}((\mathcal{E}',\mathcal{V}'),$ $(\mathcal{E},\mathcal{V}))$ and*

$$Ext^1_S((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))/H^1(S,\mathcal{H}om_{\pi_S}((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))) \subseteq$$
$$\subseteq H^0(S,\mathcal{E}xt^1_{\pi_S}((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))).$$

Proof. For every class of extensions $\sigma\in\mathrm{Ext}^1_S((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))$, by lemma 4.2.1 the family

$$\{\Phi_s(\sigma)=(\tau^1(s)\circ H^0(\iota_s)\circ\mu)(\sigma)\}_{s\in S}$$

is a globally defined family of extensions of $(\mathcal{E}',\mathcal{V}')$ by $(\mathcal{E},\mathcal{V})$ over S. Let us consider the exact sequence (4.6) of lemma 4.1.10 (with $u=\mathrm{id}_S$ and $\mathcal{M}=\mathcal{O}_S$) and let us denote by $\overline{\mu}$ the morphism induced by (4.6):

$$H:=\mathrm{Ext}^1_S((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))/H^1(S,\mathcal{H}om_{\pi_S}((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))) \xrightarrow{\overline{\mu}}$$
$$\xrightarrow{\overline{\mu}} H^0(S,\mathcal{E}xt^1_{\pi_S}((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))).$$

Now let us consider the set map f defined from H to EXT_{glob} as follows: for every class $[\sigma]$ in H we associate to it the family

$$f([\sigma]):=\{(\tau^1(s)\circ H^0(\iota_s)\circ\overline{\mu})([\sigma])\}_{s\in S}=\{\tau^1(s)\circ H^0(\iota_s)\circ\mu(\sigma)\}_{s\in S}.$$

Now $\overline{\mu}$ is injective by construction and by (4.6). Moreover the family $\{\iota_s\}_{s\in S}$ is injective by using Nakayama's lemma and the fact that S is reduced by hypothesis. So also the family $\{H^0(\iota_s)\}_{s\in S}$ is injective. In addition, every $\tau^1(s)$ is an isomorphism by hypothesis (base change for $i=1$), so in particular it is injective. Therefore the set map f is injective. Moreover, f is surjective by definition of globally defined family and by lemma 4.2.1. Finally, this map is clearly linear, so we get the desired isomorphism. $\square$

Proposition 4.2.3. *Let us assume the same hypotheses as for proposition 4.2.2. Then there is a canonical isomorphism between the set $EXT((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))$ and $H^0(S,\mathcal{E}xt^1_{\pi_S}((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})))$.*

Proof. Let us fix any $\sigma \in H^0(S,\mathcal{E}xt^1_{\pi_S}((\mathcal{E}',\mathcal{V}'),\ (\mathcal{E},\mathcal{V})))$, let $\mathfrak{U} = \{U_i\}_{i\in I}$ be any open *affine* covering of S and let $\sigma_i := \sigma|_{U_i}$. By the second part of lemma 4.1.10 for $u : U_i \hookrightarrow S$ and $\mathcal{M} = \mathcal{O}_S$, for all $i \in I$ we have an isomorphism

$$\mu_i : \mathrm{Ext}^1_{U_i}((\mathcal{E}',\mathcal{V}')|_{U_i},(\mathcal{E},\mathcal{V})|_{U_i}) \xrightarrow{\sim} H^0(U_i,\mathcal{E}xt^1_{\pi_S}((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))|_{U_i}). \tag{4.7}$$

For every point $s \in U_i$, we define $e_s := \Phi_{i,s}(\mu_i^{-1}(\sigma_i))$; a direct check proves that such an extension is well defined, i.e. it depends only on s and not on i. So the family $\{e_s\}_{s\in S}$ is a family of extensions of $(\mathcal{E}',\mathcal{V}')$ by $(\mathcal{E},\mathcal{V})$ over S. Since σ is a global section of $\mathcal{E}xt^1_{\pi_S}((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))$, a direct computation shows that such a family does not depend on the choice of the affine covering $\mathfrak{U}$. So we get a well defined set map

$$H^0(\mathcal{E}xt^1_{\pi_S}((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))) \longrightarrow \mathrm{EXT}((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})). \tag{4.8}$$

We explicitly describe an inverse for such a map. Let $\{e_s\}_{s\in S}$ be any family in the set $\mathrm{EXT}(-,-)$. By definition of family of extensions, there is an open covering $\mathfrak{U} = \{U_i\}_{i\in I}$ of S and for every i there is an object

$$\widetilde{\sigma_i} \in \mathrm{Ext}^1_{U_i}((\mathcal{E}',\mathcal{V}')|_{U_i},(\mathcal{E},\mathcal{V})|_{U_i})$$

such that $e_s = \Phi_{i,s}(\widetilde{\sigma_i})$ for all $s \in U_i$. Without loss of generality, we can assume that $\mathfrak{U}$ is an affine covering. Therefore we can use (4.7) and we define

$$\sigma_i := \mu_i(\widetilde{\sigma_i}) \in H^0(U_i,\mathcal{E}xt^1_{\pi_S}((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))).$$

As in lemma 4.2.1 on U_i instead of S, we get that for every $i \in I$ and for every s in U_i, the morphism $\Phi_{i,s}$ coincides with the composition:

$$\begin{aligned}\mathrm{Ext}^1_{U_i}\Big((\mathcal{E}',\mathcal{V}')|_{U_i},(\mathcal{E},\mathcal{V})|_{U_i}\Big) \xrightarrow{\mu_i} H^0\Big(U_i,\mathcal{E}xt^1_{\pi_S}\Big((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\Big)\Big) &\xrightarrow{H^0(U_i,\iota_s)} \\ \xrightarrow{H^0(U_i,\iota_s)} H^0\Big(U_i,\mathcal{E}xt^1_{\pi_S}\Big((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\Big)\otimes k(s)\Big) &= \\ = \mathcal{E}xt^1_{\pi_S}\Big((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\Big)\otimes k(s) \xrightarrow{\tau^1(s)} \mathrm{Ext}^1\Big((\mathcal{E}',\mathcal{V}')_s,(\mathcal{E},\mathcal{V})_s\Big)&.\end{aligned}$$

So for every $s \in U_i$ we have:

$$\Phi_{i,s}(\widetilde{\sigma_i}) = \tau^1(s)\circ H^0(U_i,\iota_s)\circ\mu_i(\widetilde{\sigma_i}) = \tau^1(s)\circ H^0(U_i,\iota_s)(\sigma_i) = \tau^1(s)(\sigma_i(s)).$$

Analogously, for every $s \in U_j$ we have $\Phi_{j,s}(\widetilde{\sigma_j}) = \tau^1(s)(\sigma_j(s))$. So if $s \in U_i \cap U_j$, then we have

$$\tau^1(s)(\sigma_i(s)) = \tau^1(s)(\sigma_j(s)).$$

By hypothesis, $\tau^1(s)$ is an isomorphism for all s in S, so we conclude that for all pairs i, j in I and for all $s \in U_i \cap U_j$ we have $\sigma_i(s) = \sigma_j(s)$. Since S is reduced, we conclude that σ_i coincides with σ_j over $U_i \cap U_j$. So there exists a unique

$$\sigma \in H^0(S, \mathcal{E}xt^1_{\pi_S}((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V})))$$

such that $\sigma|_{U_i} = \sigma_i$ for all $i \in I$. A direct computation shows that σ does not depend on the choice of the covering $\mathfrak{U}$ nor on the choice of the family $\{\widetilde{\sigma_i}\}_{i \in I}$, so we get a well defined map

$$\mathrm{EXT}((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V})) \to H^0(\mathcal{E}xt^1_{\pi_S}((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V}))). \tag{4.9}$$

Now it is easy to see that the map in (4.9) is the inverse of (4.8), so we conclude. □

4.3 Universal families of extensions

Now let us suppose that $\mathcal{E}xt^i_{\pi_S}\Big((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V})\Big)$ *commutes with base change for* $i = 0, 1$. Then let us define a contravariant functor E from the category of noetherian S-schemes to the category of sets. For every morphism $u : S' \to S$, let us consider the pullback diagram:

$$\begin{tikzcd} X \times S' \arrow[r, "u'"] \arrow[d, "\pi_{S'}"'] \arrow[dr, phantom, "\square"] & X \times S \arrow[d, "\pi_S"] \\ S' \arrow[r, "u"'] & S \end{tikzcd} \tag{4.10}$$

and let us define:

$$E(S') := H^0\left(S', \mathcal{E}xt^1_{\pi_{S'}}\left((u', u)^*(\mathcal{E}', \mathcal{V}'), (u', u)^*(\mathcal{E}, \mathcal{V})\right)\right).$$

We want to make E into a contravariant functor, so for every morphism $v : S'' \to S'$ of noetherian S-schemes we define $E(v) : E(S') \to E(S'')$ as the composition:

$$\begin{aligned} &H^0\left(S', \mathcal{E}xt^1_{\pi_{S'}}\left((u', u)^*(\mathcal{E}', \mathcal{V}'), (u', u)^*(\mathcal{E}, \mathcal{V})\right)\right) \longrightarrow \\ &\longrightarrow H^0\left(S'', v^*\mathcal{E}xt^1_{\pi_{S'}}\left((u', u)^*(\mathcal{E}', \mathcal{V}'), (u', u)^*(\mathcal{E}, \mathcal{V})\right)\right) \overset{H^0(\tau^1(v))}{\longrightarrow} \\ &\overset{H^0(\tau^1(v))}{\longrightarrow} H^0\left(S'', \mathcal{E}xt^1_{\pi_{S''}}\left((u' \circ v', u \circ v)^*(\mathcal{E}', \mathcal{V}'), (u' \circ v', u \circ v)^*(\mathcal{E}, \mathcal{V})\right)\right). \end{aligned} \tag{4.11}$$

Since we are assuming that $\mathcal{E}xt^1_{\pi_S}((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V}))$ commutes with base change, so does $\mathcal{E}xt^1_{\pi_{S''}}((u' \circ v', u \circ v)^*(\mathcal{E}', \mathcal{V}'), (u' \circ v', u \circ v)^*(\mathcal{E}, \mathcal{V}))$. Therefore, E is a contravariant functor from the category of noetherian S-schemes to the category of sets. For the moment we have not used the fact that $\mathcal{H}om_{\pi_S}((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V}))$ commutes with base change. We need also this fact in order to prove that E is representable.

Proposition 4.3.1. *Let us suppose that* $\mathcal{E}xt^i_{\pi_S}\Big((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\Big)$ *commutes with base change for* $i=0,1$. *Then the functor* E *is representable by the vector bundle*

$$V := \mathbb{V}\left(\mathcal{E}xt^1_{\pi_S}\left((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\right)^\vee\right) \xrightarrow{\pi} S$$

associated to the locally free sheaf $\mathcal{E}xt^1_{\pi_S}\left((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\right)^\vee$.

Proof. By hypothesis and base change for $i=1$, the sheaf $\hat{E} := \mathcal{E}xt^1_{\pi_S}\left((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\right)$ commutes with base change, so for every S-scheme $u: S' \to S$ we have that

$$E(S') = H^0\left(S', u^*\mathcal{E}xt^1_{\pi_S}\left((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\right)\right) = H^0(S', u^*\hat{E}).$$

Moreover, using base change for $i=0,1$, we get that $\hat{E}$ is a locally free sheaf. Therefore, the functor E is representable by the vector bundle V associated to $\hat{E}^\vee$ by the universal property of that object. Note that by assumption $\hat{E}$ is locally free, so $\hat{E}^{\vee\vee} = \hat{E}$. □

Remark 4.3.1. The universal element of $E(V)$ is constructed in the following way. Let us consider the inclusion of sheaves on S given by $\hat{E}^\vee \hookrightarrow \pi_*\mathcal{O}_V$ and the induced canonical inclusion

$$\begin{aligned} H^0(S, \mathrm{End}\hat{E}) = H^0(S, \hat{E}\otimes\hat{E}^\vee) \hookrightarrow H^0(S, \hat{E}\otimes\pi_*\mathcal{O}_V) = \\ = H^0(S, \pi_*\pi^*\hat{E}) = H^0(V, \pi^*\hat{E}) = E(V). \end{aligned}$$

Then we consider the image of the identity of $\hat{E}$ under this series of maps and we get that this is the universal object for the functor E.

By combining this proposition with proposition 4.2.3 we get the following corollary.

Corollary 4.3.2. *Let us suppose that* S *is* reduced *and that* $\mathcal{E}xt^i_{\pi_S}((\mathcal{E}',\mathcal{V}'),\,(\mathcal{E},\mathcal{V}))$ *commutes with base change for* $i=0,1$. *Let us denote by* π' *the morphism* $X\times V \to X\times S$ *induced by* π. *Then there is a family of extensions* $\{e_v\}_{v\in V}$ *of* $(\pi',\pi)^*(\mathcal{E}',\mathcal{V}')$ *by* $(\pi',\pi)^*(\mathcal{E},\mathcal{V})$ *over the vector bundle* $\pi: V\to S$. *Such a family is universal over the category of* reduced *noetherian* S*-schemes.*

Here "universal" means the following: given any reduced S-scheme $u: S'\to S$ and any family of extensions $\{e_{s'}\}_{s'\in S'}$ of $(u',u)^*(\mathcal{E}',\mathcal{V}')$ by $(u',u)^*(\mathcal{E},\mathcal{V})$ over S', there is exactly one morphism $\psi: S'\to V$ of S-schemes such that $\{e_{s'}\}_{s'\in S'}$ is the pullback of $\{e_v\}_{v\in V}$ via (ψ',ψ), where ψ' is given as follows:

$$\begin{tikzcd}
X \times S' \arrow[r, "\psi'"] \arrow[d, "\pi_{S'}"] \arrow[rr, bend left, "u'"] \arrow[dr, phantom, "\square"] & X \times V \arrow[r, "\pi'"] \arrow[d, "\pi_V"] \arrow[dr, phantom, "\square"] & X \times S \arrow[d, "\pi_S"] \\
S' \arrow[r, "\psi"] \arrow[rr, bend right, "u"'] & V \arrow[r, "\pi"] & S.
\end{tikzcd} \tag{4.12}$$

Corollary 4.3.3. *Let us suppose that $Hom((\mathcal{E}',\mathcal{V}')_s,(\mathcal{E},\mathcal{V})_s) = 0$ for all $s \in S$ and that $\mathcal{E}xt^1_{\pi_S}((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))$ commutes with base change. Let us denote by π' the morphism $X \times V \to X \times S$ induced by π. Then there is an extension*

$$0 \to (\pi',\pi)^*(\mathcal{E},\mathcal{V}) \to (\mathcal{E}_V,\mathcal{V}_V) \to (\pi',\pi)^*(\mathcal{E}',\mathcal{V}') \to 0 \tag{4.13}$$

over V that is universal *on the category of noetherian S-schemes.*

Here "universal" means the following: let us fix any noetherian S-scheme $u : S' \to S$ and any extension

$$0 \to (u',u)^*(\mathcal{E},\mathcal{V}) \to (\mathcal{E}_{S'},\mathcal{V}_{S'}) \to (u',u)^*(\mathcal{E}',\mathcal{V}') \to 0 \tag{4.14}$$

over S'. Then there is a unique morphism $\psi : S' \to V$ of S-schemes such that (4.14) is the pullback of (4.13) via (ψ',ψ) where ψ' is as in (4.12).

Proof. If we assume the hypotheses, then by lemma 4.1.10 for all morphisms $u : S' \to S$ we get a canonical isomorphism

$$\mu : \mathrm{Ext}^1_{S'}((u',u)^*(\mathcal{E}',\mathcal{V}'),(u',u)^*(\mathcal{E},\mathcal{V})) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} H^0(S',\mathcal{E}xt^1_{\pi_{S'}}((u',u)^*(\mathcal{E}',\mathcal{V}'),(u',u)^*(\mathcal{E},\mathcal{V}))).$$

If we use proposition 4.2.2 and the hypothesis, then this coincides also with

$$\mathrm{EXT}_{glob}((u',u)^*(\mathcal{E}',\mathcal{V}'),(u',u)^*(\mathcal{E},\mathcal{V})).$$

So for every S-scheme S' as before we can consider the set $E(S')$ as the set of all extensions of $(u',u)^*(\mathcal{E}',\mathcal{V}')$ by $(u',u)^*(\mathcal{E},\mathcal{V})$ over S'. In particular, the universal object of the functor E corresponds to an extension (4.13). The universal property of such an object (together with the fact that μ is canonical) then proves the claim. $\square$

4.4 Universal families of classes of non-split extensions

Let us suppose again that $\mathcal{E}xt^i_{\pi_S}\Big((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\Big)$ *commutes with base change for* $i=0,1$. Then let us define a contravariant functor F from the category of noetherian S-schemes to the category of sets. For every morphism $u: S'\to S$, let us consider the pullback diagram (4.10) and let us define:

$$F(S') := \Big\{\text{invertible quotients of } \mathcal{E}xt^1_{\pi_{S'}}\left((u',u)^*(\mathcal{E}',\mathcal{V}'),(u',u)^*(\mathcal{E},\mathcal{V})\right)^\vee\Big\}.$$

We want to make F into a contravariant functor, so let us fix any morphism $v: S''\to S'$ of noetherian S-schemes and any object of $F(S')$, i.e. any invertible quotient:

$$\mathcal{E}xt^1_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'),(u',u)^*(\mathcal{E},\mathcal{V})\Big)^\vee \longrightarrow \mathcal{L}\longrightarrow 0.$$

Then by pullback via v, we get an exact sequence:

$$v^*\mathcal{E}xt^1_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'),(u',u)^*(\mathcal{E},\mathcal{V})\Big)^\vee \longrightarrow v^*\mathcal{L}\longrightarrow 0. \tag{4.15}$$

Using base change for $i=1$ we get:

$$v^*\mathcal{E}xt^1_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'),(u',u)^*(\mathcal{E},\mathcal{V})\Big)^\vee = \Big(v^*\mathcal{E}xt^1_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'),(u',u)^*(\mathcal{E},\mathcal{V})\Big)\Big)^\vee \simeq$$
$$\simeq \mathcal{E}xt^1_{\pi_{S''}}\Big((u'\circ v',u\circ v)^*(\mathcal{E}',\mathcal{V}'),(u'\circ v',u\circ v)^*(\mathcal{E},\mathcal{V})\Big)\Big)^\vee.$$

Therefore, (4.15) gives an element of $F(S'')$, so we get a set map $F(v): F(S')\to F(S'')$. Using again base change for $i=1$, it is immediate to prove that this gives rise to a contravariant functor F on the category of noetherian S-schemes. Actually, in order to define the functor F we don't need base change for $i=0$; we need also that hypothesis in order to prove that F is representable.

Proposition 4.4.1. *Let us suppose that* $\mathcal{E}xt^i_{\pi_S}\Big((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\Big)$ *commutes with base change for* $i=0,1$. *Then the functor* F *is representable by the projective bundle*

$$P := \mathbb{P}\Big(\mathcal{E}xt^1_{\pi_S}\Big((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\Big)^\vee\Big) \xrightarrow{\varphi} S$$

associated to the locally free sheaf $\mathcal{E}xt^1_{\pi_S}\Big((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\Big)^\vee$ *on* S.

Proof. By base change for $i=0,1$, the sheaf $\hat{E}:=\mathcal{E}xt^1_{\pi_S}\left((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\right)$ commutes with base change and is locally free. Therefore, for every noetherian S-scheme $u: S'\to S$, $F(S')$ is equal to the set of invertible quotients of $u^*\hat{E}^\vee$. Moreover, since $\hat{E}$ is locally free, it makes sense to consider the projective bundle $\varphi: P=\mathbb{P}(\hat{E}^\vee)\to S$. Now F is representable by that projective bundle by the universal property of the grassmannian functor associated to every quasi-coherent $\mathcal{O}_S$-module. Note that since $\hat{E}$ is locally free, then $\hat{E}^{\vee\vee}=\hat{E}$. $\square$

Remark 4.4.1. The universal element of $F(P)$ is constructed in the following way. We consider the canonical isomorphisms:

$$H^0(S, \mathrm{End}\hat{E}) = H^0(S, \hat{E} \otimes \hat{E}^\vee) = H^0(S, \hat{E} \otimes \varphi_* \mathcal{O}_P(1)) =$$
$$= H^0\Big(S, \varphi_*\Big(\varphi^* \hat{E} \otimes \mathcal{O}_P(1)\Big)\Big) = H^0(P, \varphi^* \hat{E} \otimes_P \mathcal{O}_P(1)).$$

Then we consider the image of the identity of $\hat{E}$ under this series of isomorphisms and we get that this is a non-vanishing section of $\varphi^* \hat{E} \otimes_P \mathcal{O}_P(1)$. Using base change for $i = 1$, this gives a non-vanishing section of

$$\mathcal{E}xt^1_{\pi_P}((\varphi', \varphi)^*(\mathcal{E}', \mathcal{V}'), (\varphi', \varphi)^*(\mathcal{E}, \mathcal{V})) \otimes_P \mathcal{O}_P(1),$$

so it defines a quotient:

$$\mathcal{E}xt^1_{\pi_P}((\varphi', \varphi)^*(\mathcal{E}', \mathcal{V}'), (\varphi', \varphi)^*(\mathcal{E}, \mathcal{V}))^\vee \longrightarrow \mathcal{O}_P(1) \longrightarrow 0.$$

This will be the universal object of the functor F.

Definition 4.4.1. Given any scheme S' and any exact sequence of families of coherent systems parametrized by S' we call it *non-splitting* if its restriction to every fiber $X_{s'} = X \times \{s'\}$ over any point s' of S' is non-splitting. Analogously, we call non-splitting any family $\{e_{s'}\}_{s' \in S'}$ of extensions such that each $e_{s'}$ is non-splitting.

Lemma 4.4.2. *Let us assume the same hypotheses as for the previous proposition. Then for every S-scheme $u : S' \to S$ we have that $F(S')$ is the set of all the families of classes of non-splitting extensions of $(u', u)^*(\mathcal{E}', \mathcal{V}')$ by $(u', u)^*(\mathcal{E}, \mathcal{V}) \otimes_{S'} \mathcal{L}$ with arbitrary $\mathcal{L}$ in Pic(S'), modulo the canonical operation of $H^0(S', \mathcal{O}^*_{S'})$.*

Proof. By construction, $F(S')$ is equal to the set of all nowhere vanishing global sections of every sheaf on S' of the form

$$\mathcal{E}xt^1_{\pi_{S'}}\Big((u', u)^*(\mathcal{E}', \mathcal{V}'), (u', u)^*(\mathcal{E}, \mathcal{V})\Big) \otimes_{S'} \mathcal{L}$$

with arbitrary $\mathcal{L} \in \mathrm{Pic}(S')$, modulo the canonical operation of $H^0(S', \mathcal{O}^*_{S'})$. Since every such $\mathcal{L}$ is in particular locally free, we can use lemma 4.1.9 and we conclude by proposition 4.2.3. $\square$

The proofs of the following two corollaries are modeled on the proofs of corollaries 4.3.2 and 4.3.3 together with lemma 4.4.2 and proposition 4.4.1, so we omit the details.

Corollary 4.4.3. *Let us suppose that S is* reduced *and that $\mathcal{E}xt^i_{\pi_S}((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V}))$ commutes with base change for $i = 0, 1$. Let us denote by φ' the morphism $X \times P \to X \times S$ induced by φ. Then there is a family of non-splitting extensions $\{e_p\}_{p \in P}$ of $(\varphi', \varphi)^*(\mathcal{E}', \mathcal{V}')$ by $(\varphi', \varphi)^*(\mathcal{E}, \mathcal{V}) \otimes_P \mathcal{O}_P(1)$ over P which is universal on the category of* reduced *noetherian S-schemes in the following sense: given any reduced noetherian S-scheme $u : S' \to S$, any $\mathcal{L} \in$ Pic(S') and any family $\{e_{s'}\}_{s' \in S'}$ of non-splitting extensions of $(u', u)^*(\mathcal{E}', \mathcal{V}')$ by*

$(u',u)^*(\mathcal{E},\mathcal{V})\otimes_{S'}\mathcal{L}$ over S', then there is a unique morphism of S-schemes $\psi : S' \to P$ such that the family $\{e_{s'}\}_{s'\in S'}$ is the pullback (modulo the canonical operation of $H^0(S',\mathcal{O}^*_{S'})$) of $\{e_p\}_{p\in P}$ via (ψ',ψ), where ψ' is given as follows

$$\begin{tikzcd}
X\times S' \arrow[r, "\psi'"] \arrow[d, "\pi_{S'}"] \arrow[rr, bend left, "u'"] \arrow[dr, phantom, "\square"] & X\times P \arrow[r, "\varphi'"] \arrow[d, "\pi_P"] \arrow[dr, phantom, "\square"] & X\times S \arrow[d, "\pi_S"] \\
S' \arrow[r, "\psi"] \arrow[rr, bend right, "u"'] & P \arrow[r, "\varphi"] & S.
\end{tikzcd} \tag{4.16}$$

Corollary 4.4.4. *Let us suppose that $Hom((\mathcal{E}',\mathcal{V}')_s,(\mathcal{E},\ \mathcal{V})_s) = 0$ for all $s \in S$ and that $\mathcal{E}xt^1_{\pi_S}((\mathcal{E}',\ \mathcal{V}'),(\mathcal{E},\mathcal{V}))$ commutes with base change. Let us denote by φ' the morphism $X\times P \to X\times S$ induced by φ. Then there is a family $(\mathcal{E}_P,\mathcal{V}_P)$ parametrized by P and a non-splitting extension:*

$$0 \to (\varphi',\varphi)^*(\mathcal{E},\mathcal{V})\otimes_P \mathcal{O}_P(1) \to (\mathcal{E}_P,\mathcal{V}_P) \to (\varphi',\varphi)^*(\mathcal{E}',\mathcal{V}') \to 0 \tag{4.17}$$

parametrized by P. This extension is universal *on the category of noetherian S-schemes in the following sense: let us fix any morphism $u : S' \to S$, any line bundle $\mathcal{L} \in Pic(S')$ and any non-splitting extension*

$$0 \to (u',u)^*(\mathcal{E},\mathcal{V})\otimes_{S'}\mathcal{L} \to (\mathcal{E}_S,\mathcal{V}_S) \to (u',u)^*(\mathcal{E}',\mathcal{V}') \to 0. \tag{4.18}$$

Then there is a unique morphism of S-schemes $\psi : S' \to P$ such that (4.18) is the pullback (modulo the canonical operation of $H^0(S',\mathcal{O}^*_{S'})$) of (4.17) via (ψ',ψ), where ψ' is as in (4.16).

4.5 Universal families of non-degenerate extensions

In the following chapters we will have also to exhibit universal families of non-degenerate extensions (either on the left or on the right, see definitions 3.3.1 and 3.3.2). Actually, we have already described a particular case of non-degenerate extensions, namely the non-split extensions that we studied in the previous section. The constructions in the present section generalize the previous ones by allowing the object on the left or on the right of any extension to be the sum of $t \geq 2$ copies of a fixed stable coherent system. We will not consider here the most general case described in definitions 3.3.1 and 3.3.2 because the results of this section will be sufficient for the computations of the next chapters.

In the first part of this section we consider the case of non-degenerate extensions on the left (of rank $t \geq 2$).

Let us fix any integer $t \geq 2$ and let us suppose again that $\mathcal{E}xt^i_{\pi_S}\Big((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\Big)$ *commutes with base change for* $i=0,1$. Then let us define a contravariant functor G_t from the category of noetherian S-schemes to the category of sets. For every morphism $u: S' \to S$, let us consider the pullback diagram (4.10) and let us define:

$$G_t(S') := \Big\{\text{locally free quotients of rank } t \text{ of } \mathcal{E}xt^1_{\pi_{S'}}\left((u',u)^*(\mathcal{E}',\mathcal{V}'),(u',u)^*(\mathcal{E},\mathcal{V})\right)^\vee\Big\}.$$

We want to make G_t into a contravariant functor, so let us fix any morphism $v: S'' \to S'$ of noetherian S-schemes and any object of $G_t(S')$, i.e. any locally free quotient of rank t:

$$\mathcal{E}xt^1_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'),(u',u)^*(\mathcal{E},\mathcal{V})\Big)^\vee \longrightarrow \mathcal{M} \longrightarrow 0.$$

Then by pullback via v, we get an exact sequence:

$$v^*\mathcal{E}xt^1_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'),(u',u)^*(\mathcal{E},\mathcal{V})\Big)^\vee \longrightarrow v^*\mathcal{M} \longrightarrow 0. \tag{4.19}$$

As in the previous section, by base change we get that:

$$v^*\mathcal{E}xt^1_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'),(u',u)^*(\mathcal{E},\mathcal{V})\Big)^\vee \simeq \mathcal{E}xt^1_{\pi_{S''}}\Big((u'\circ v',u\circ v)^*(\mathcal{E}',\mathcal{V}'),(u'\circ v',u\circ v)^*(\mathcal{E},\mathcal{V})\Big)\Big)^\vee.$$

Therefore, (4.19) gives an element of $G_t(S'')$, so we get a set map $G_t(v): G_t(S') \to G_t(S'')$. Using base change for $i=1$, this gives rise to a contravariant functor G_t on the category of noetherian S-schemes. Actually, in order to define the functor G_t we don't need base change for $i=0$; we need also that hypothesis in order to prove that G_t is representable.

Proposition 4.5.1. *Let us suppose that* $\mathcal{E}xt^i_{\pi_S}\Big((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\Big)$ *commutes with base change for* $i=0,1$. *Then for every* $t \geq 2$ *the functor* G_t *is representable by the relative grassmannian of rank* t

$$Q_t := Grass\Big(t, \mathcal{E}xt^1_{\pi_S}\Big((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\Big)^\vee\Big) \xrightarrow{\theta_t} S$$

associated to the locally free sheaf $\mathcal{E}xt^1_{\pi_S}\Big((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\Big)^\vee$ *on* S.

Proof. By hypothesis and base change for $i=0,1$, the sheaf $\hat{E} := \mathcal{E}xt^1_{\pi_S}\left((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V})\right)$ commutes with base change and is locally free. Therefore, for every noetherian S-scheme $u: S' \to S$, $G_t(S')$ is equal to the set of locally free quotients of rank t of $u^*\hat{E}^\vee$. Now G_t is represented by the grassmannian bundle $\theta_t : \mathrm{Grass}(t,\hat{E}^\vee) \to S$ by the universal property of the grassmannian functor associated to every quasi-coherent $\mathcal{O}_S$-module. Note that since $\hat{E}$ is locally free, then $\hat{E}^{\vee\vee} = \hat{E}$. ☐

Remark 4.5.1. In this case we don't know how to explicitly describe the universal object of the functor G_t. We only know that it will be something of the form

$$\mathcal{E}xt^1_{\pi_G}\Big((\theta'_t,\theta_t)^*(\mathcal{E}',\mathcal{V}'),(\theta'_t,\theta_t)^*(\mathcal{E},\mathcal{V})\Big)^\vee \xrightarrow{q} \overline{\mathcal{M}}_t \longrightarrow 0$$

for some locally free sheaf $\overline{\mathcal{M}}_t$ on Q_t of rank t (it is reasonable that $\overline{\mathcal{M}}_t$ is the very ample sheaf on Q_t that gives the Plücker embedding of the relative Grassmannian Q_t into a projective space, but we don't have a proof of this fact). Note that in particular

$$
\begin{aligned}
q \in Hom_{Q_t}\Big(\mathcal{E}xt^1_{\pi_{G_t}}\Big((\theta'_t,\theta_t)^*(\mathcal{E}',\mathcal{V}'),(\theta'_t,\theta_t)^*(\mathcal{E},\mathcal{V})\Big)^{\vee},\overline{\mathcal{M}}_t\Big) = \\
= H^0\Big(Q_t,\mathcal{E}xt^1_{\pi_{Q_t}}\Big((\theta'_t,\theta_t)^*(\mathcal{E}',\mathcal{V}'),(\theta'_t,\theta_t)^*(\mathcal{E},\mathcal{V})\Big)\otimes_{Q_t}\overline{\mathcal{M}}_t\Big) = \\
= H^0\Big(Q_t,\mathcal{E}xt^1_{\pi_{Q_t}}\Big((\theta'_t,\theta_t)^*(\mathcal{E}',\mathcal{V}'),(\theta'_t,\theta_t)^*(\mathcal{E},\mathcal{V})\otimes_{Q_t}\overline{\mathcal{M}}_t\Big)\Big),
\end{aligned}
$$

where the last identity comes from lemma 4.1.9 since $\overline{\mathcal{M}}_t$ is locally free.

Definition 4.5.1. Let us fix any scheme S', any locally free sheaf of rank t $\mathcal{M}$ on S and any exact sequence of families of coherent systems of the form

$$0 \to (\mathcal{E}_{S'},\mathcal{V}_{S'})\otimes_{S'}\mathcal{M} \to (\mathcal{F}_{S'},\mathcal{Z}_{S'}) \to (\mathcal{E}'_{S'},\mathcal{V}'_{S'}) \to 0. \tag{4.20}$$

By restriction to any fiber $X_{s'} = X\times\{s'\}$ over any point s' of S', we get a sequence that is a representative for an object

$$
\begin{aligned}
\xi_{s'} \in \mathrm{Ext}^1((\mathcal{E}'_{S',s'},\mathcal{V}'_{S',s'}),(\mathcal{E}_{S',s'},\mathcal{V}_{S',s'})\otimes_{s'}\mathcal{M}_{s'}) = \\
= \mathrm{Ext}^1((\mathcal{E}'_{S',s'},\mathcal{V}'_{S',s'}),(\mathcal{E}_{S',s'},\mathcal{V}_{S',s'})^{\oplus t}) =: H_{s'}^{\oplus t}.
\end{aligned}
$$

So we can write $\xi_{s'} = (\xi^1_{s'},\cdots,\xi^t_{s'})$. Then we say that (4.20) is *non-degenerate of rank t on the left* if for all points s' of S' the objects $\xi^i_{s'}$ for $i=1,\cdots,t$ are linearly independent in $H_{s'}$. Analogously, we call non-degenerate on the left any family $\{e_{s'}\}_{s'\in S'}$ of extensions of the same 2 objects on the left and on the right of (4.20) such that each $e_{s'}$ is non-degenerate. We can also give analogous definitions for non-degenerate extensions of rank t on the right.

Lemma 4.5.2. *Let us assume the same hypotheses as for the previous proposition. Then for every S-scheme $u : S' \to S$ we have that $G_t(S')$ is the set of all the families of classes of non-degenerate extensions of $(u',u)^*(\mathcal{E}',\mathcal{V}')$ by $(u',u)^*(\mathcal{E},\mathcal{V})\otimes_{S'}\mathcal{M}$ with arbitrary $\mathcal{M}$ locally free of rank t on S', modulo the canonical operation of $H^0(S',GL(t,\mathcal{O}_{S'}))$.*

Proof. By construction, $G_t(S')$ is equal to the set of all nowhere vanishing global sections of every sheaf on S' of the form

$$\mathcal{E}xt^1_{\pi_{S'}}\Big((u',u)^*(\mathcal{E}',\mathcal{V}'),(u',u)^*(\mathcal{E},\mathcal{V})\Big)\otimes_{S'}\mathcal{M}$$

with arbitrary $\mathcal{M}$ locally free of rank t on S', modulo the canonical operation of $H^0(S',GL(t,\mathcal{O}_{S'}))$. Since every such $\mathcal{M}$ is locally free, we can use lemma 4.1.9 and we conclude by proposition 4.2.3. □

The proofs of the following two corollaries are modeled on the proofs of corollaries 4.3.2 and 4.3.3 together with lemma 4.5.2 and proposition 4.5.1, so we omit the details.

Corollary 4.5.3. *Let us fix any $t \geq 2$, let us suppose that S is* reduced *and that $\mathcal{E}xt^i_{\pi_S}((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))$ commutes with base change for $i = 0, 1$. Let us denote by θ'_t the morphism $X \times Q_t \to X \times S$ induced by θ_t. Then there is a family of non-degenerate extensions of rank t on the left $\{e_q\}_{q \in Q_t}$ of $(\theta'_t, \theta_t)^*(\mathcal{E}',\mathcal{V}')$ by $(\theta'_t, \theta_t)^*(\mathcal{E},\mathcal{V}) \otimes_{Q_t} \overline{\mathcal{M}}_t$ over G_t which is universal on the category of* reduced *noetherian S-schemes in the following sense. Given any reduced noetherian S-scheme $u : S' \to S$, any locally free sheaf $\mathcal{M}$ of rank t on S' and any class of a family $\{e_{s'}\}_{s' \in S'}$ of non-degenerate extensions of rank t on the left of $(u', u)^*(\mathcal{E}',\mathcal{V}')$ by $(u', u)^*(\mathcal{E},\mathcal{V}) \otimes_{S'} \mathcal{M}$ over S', then there is a unique morphism of S-schemes $\psi : S' \to Q_t$, such that the class of the family $\{e_{s'}\}_{s' \in S'}$ is the pullback (modulo the canonical operation of $H^0(S', GL(t, \mathcal{O}_{S'}))$) of the class of $\{e_q\}_{q \in Q_t}$ via (ψ', ψ), where ψ' is given as follows*

$$\begin{tikzcd}
X \times S' \arrow[r, "\psi'"] \arrow[d, "\pi_{S'}"'] \arrow[rr, bend left, "u'"] \arrow[dr, phantom, "\square"] & X \times Q_t \arrow[r, "\theta'_t"] \arrow[d, "\pi_P"] \arrow[dr, phantom, "\square"] & X \times S \arrow[d, "\pi_S"] \\
S' \arrow[r, "\psi"] \arrow[rr, bend right, "u"'] & Q_t \arrow[r, "\theta_t"] & S.
\end{tikzcd} \tag{4.21}$$

Corollary 4.5.4. *Let us fix any $t \geq 2$, let us suppose that $Hom((\mathcal{E}',\mathcal{V}')_s, (\mathcal{E},\mathcal{V})_s) = 0$ for all $s \in S$ and that $\mathcal{E}xt^1_{\pi_S}((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))$ commutes with base change. Let us denote by θ'_t the morphism $X \times G_t \to X \times S$ induced by θ_t. Then there is a family $(\mathcal{E}_{Q_t}, \mathcal{V}_{Q_t})$ parametrized by Q_t and a non-degenerate extension of rank t on the left:*

$$0 \to (\theta'_t, \theta_t)^*(\mathcal{E},\mathcal{V}) \otimes_{Q_t} \overline{\mathcal{M}}_t \to (\mathcal{E}_{Q_t}, \mathcal{V}_{Q_t}) \to (\theta'_t, \theta_t)^*(\mathcal{E}',\mathcal{V}') \to 0 \tag{4.22}$$

parametrized by Q_t. This extension is universal *on the category of noetherian S-schemes in the following sense: let us fix any morphism $u : S' \to S$, any locally free sheaf $\mathcal{M}$ of rank t on S' and any non-degenerate extension of rank t on the left:*

$$0 \to (u', u)^*(\mathcal{E},\mathcal{V}) \otimes_{S'} \mathcal{M} \to (\mathcal{E}_S, \mathcal{V}_S) \to (u', u)^*(\mathcal{E}',\mathcal{V}') \to 0. \tag{4.23}$$

Then there is a unique morphism of S-schemes $\psi : S' \to Q_t$ such that (4.23) is the pullback (modulo the canonical operation of $H^0(S', GL(t, \mathcal{O}_{S'}))$) of (4.22) via (ψ', ψ), where ψ' is as in (4.21).

Analogously, using the second part of lemma 4.1.9 we can prove the following results.

Corollary 4.5.5. *Let us fix any $t \geq 2$, let us suppose that S is* reduced *and that $\mathcal{E}xt^i_{\pi_S}((\mathcal{E}',\mathcal{V}'),(\mathcal{E},\mathcal{V}))$ commutes with base change for $i = 0, 1$. Let us denote by θ'_t the morphism $X \times Q_t \to X \times S$ induced by θ_t. Then there is a family of non-degenerate extensions of rank t on the right $\{e_q\}_{q \in Q_t}$ of $(\theta'_t, \theta_t)^*(\mathcal{E}',\mathcal{V}') \otimes_{Q_t} \overline{\mathcal{M}}_t^{\vee}$ by $(\theta'_t, \theta_t)^*(\mathcal{E},\mathcal{V})$ over G_t which is universal on the category* *of* reduced *noetherian S-schemes in the following sense. Given any reduced noetherian S-scheme $u : S' \to S$, any locally free sheaf $\mathcal{M}$ of rank t on S' and any class of a family $\{e_{s'}\}_{s' \in S'}$ of non-degenerate extensions of rank t on the right of $(u', u)^*(\mathcal{E}', \mathcal{V}') \otimes_{S'} \mathcal{M}$ by $(u', u)^*(\mathcal{E}, \mathcal{V})$ over S', then there is a unique morphism of S-schemes $\psi : S' \to Q_t$, such that the class of the family $\{e_{s'}\}_{s' \in S'}$ is the pullback (modulo the canonical operation of $H^0(S', GL(t, \mathcal{O}_{S'}))$) of the class of $\{e_q\}_{q \in Q_t}$ via (ψ', ψ), where ψ' is given as in (4.21)*

Corollary 4.5.6. *Let us fix any $t \geq 2$, let us suppose that $Hom((\mathcal{E}', \mathcal{V}')_s, (\mathcal{E}, \mathcal{V})_s) = 0$ for all $s \in S$ and that $\mathcal{E}xt^1_{\pi_S}((\mathcal{E}', \mathcal{V}'), (\mathcal{E}, \mathcal{V}))$ commutes with base change. Let us denote by θ'_t the morphism $X \times Q_t \to X \times S$ induced by θ_t. Then there is a family $(\mathcal{E}_{Q_t}, \mathcal{V}_{Q_t})$ parametrized by Q_t and a non-degenerate extension on the right of rank t:*

$$0 \to (\theta'_t, \theta_t)^*(\mathcal{E}, \mathcal{V}) \to (\mathcal{E}_{Q_t}, \mathcal{V}_{Q_t}) \to (\theta'_t, \theta_t)^*(\mathcal{E}', \mathcal{V}') \otimes_{Q_t} \overline{\mathcal{M}}_t^{\vee} \to 0 \tag{4.24}$$

parametrized by Q_t. This extension is universal *on the category of noetherian S-schemes in the following sense: let us fix any morphism $u : S' \to S$, any locally free sheaf $\mathcal{M}$ of rank t on S' and any non-degenerate extension on the right of rank t:*

$$0 \to (u', u)^*(\mathcal{E}, \mathcal{V}) \to (\mathcal{E}_S, \mathcal{V}_S) \to (u', u)^*(\mathcal{E}', \mathcal{V}') \otimes_{S'} \mathcal{M} \to 0. \tag{4.25}$$

Then there is a unique morphism of S-schemes $\psi : S' \to Q_t$ such that (4.23) is the pullback (modulo the canonical operation of $H^0(S', GL(t, \mathcal{O}_{S'}))$) of (4.22) via (ψ', ψ), where ψ' is as in (4.21).

4.6 Applications on curves

All the previous constructions work for every projective scheme X and for every noetherian scheme S (with the additional hypothesis of being reduced or affine in some cases). Now let us restrict to the case when X is a smooth projective irreducible curve C.

Lemma 4.6.1. *Let C be any smooth projective irreducible curve. Let us fix any noetherian scheme T and any pair of families of coherent systems $(\mathcal{E}, \mathcal{V})$, $(\mathcal{E}', \mathcal{V}')$ parametrized by T (of type (n, d, k) and (n', d', k') respectively). Let us suppose that dim $Hom((\mathcal{E}', \mathcal{V}')_t, (\mathcal{E}, \mathcal{V})_t) = c$ is constant for all $t \in T$. For every $\alpha \in \mathbb{N}_0$ let us define*

$$T_\alpha := \Big\{ t \in T \text{ s.t. } \dim Ext^1\Big((\mathcal{E}', \mathcal{V}')_t, (\mathcal{E}, \mathcal{V})_t\Big) = \alpha \Big\}.$$

Then only a finite number of T_α's is non-empty; moreover, each T_α is locally closed in T with the induced reduced structure and it has a covering $\{T_{\alpha,k}\}_k$ consisting of a finite set of disjoint locally closed reduced subschemes. On each $T_{\alpha,k}$ the sheaves

$$\mathcal{E}xt^i_{\pi_{T_{\alpha,k}}}\Big((\mathcal{E}', \mathcal{V}')|_{T_{\alpha,k}}, (\mathcal{E}, \mathcal{V})|_{T_{\alpha,k}}\Big) \quad \text{for } i = 0, 1, 2$$

are locally free and commute with every noetherian base change to $T_{\alpha,k}$. If T_α is integral for a certain α, then the set $\{T_{\alpha,k}\}_k$ coincides with T_α itself.

Proof. Let us fix any $\alpha \in \mathbb{N}_0$; then by proposition 1.0.5, the set T_α is locally closed in T with the induced reduced structure. By [BGMN, proposition 3.2 and lemma 3.3], the set of T_α's that are non-empty is finite. Now for each T_α, let us consider the set $\{T_\alpha^l\}_l$ of its irreducible components; since we are working in noetherian hypothesis, such a set is finite for each α. Moreover, each T_α^l is locally closed in T. In addition, by construction every T_α^l is reduced and irreducible, hence integral. Now for every pair (α, l) (such that $T_\alpha^l \neq \varnothing$), for every $i \geq 0$ and for every $t \in T_\alpha^l$, let us denote by $\tau^i(\alpha, l, t)$ the base change:

$$\tau^i(\alpha, l, t) : \mathcal{E}xt^i_{\pi_{T_\alpha^l}}\Big((\mathcal{E}', \mathcal{V}')|_{T_\alpha^l}, (\mathcal{E}, \mathcal{V})|_{T_\alpha^l}\Big) \otimes k(t) \longrightarrow \mathrm{Ext}^i\Big((\mathcal{E}', \mathcal{V}')_t, (\mathcal{E}, \mathcal{V})_t\Big).$$

Since C is a curve, for every point t in T we have that

$$\mathrm{Ext}^3\Big((\mathcal{E}', \mathcal{V}')_t, (\mathcal{E}, \mathcal{V})_t\Big) = 0.$$

Therefore, every $\tau^3(\alpha, l, t)$ is surjective and $\mathcal{E}xt^3_{\pi_{T_\alpha^l}}\Big((\mathcal{E}', \mathcal{V}')|_{T_\alpha^l}, (\mathcal{E}, \mathcal{V})|_{T_\alpha^l}\Big) = 0$, so in particular it is locally free. Now by using proposition 1.0.7 we get that that for every $t \in T_\alpha^l$:

$$\begin{aligned} &\dim\, \mathrm{Ext}^2((\mathcal{E}', \mathcal{V}')_t, (\mathcal{E}, \mathcal{V})_t) = \\ &= \dim\, \mathrm{Ext}^1((\mathcal{E}', \mathcal{V}')_t, (\mathcal{E}, \mathcal{V})_t) - c' - \dim\, \mathrm{Hom}((\mathcal{E}', \mathcal{V}')_t, (\mathcal{E}, \mathcal{V})_t) = \alpha - c' - c \end{aligned} \tag{4.26}$$

where c' is a constant that depends only on $(n, d, k), (n', d', k')$ and on the genus of C. So we get that for $i = 0, 1, 2$ the dimension of $\mathrm{Ext}^i((\mathcal{E}', \mathcal{V}')_t, (\mathcal{E}, \mathcal{V})_t)$ is constant on every T_α^l. Since every T_α^l is integral, then by proposition 1.0.5 we get that on each T_α^l the sheaves

$$\mathcal{E}xt^i_{\pi_{T_\alpha^l}}\Big((\mathcal{E}', \mathcal{V}')|_{T_\alpha^l}, (\mathcal{E}, \mathcal{V})|_{T_\alpha^l}\Big)$$

are locally free for $i = 0, 1, 2$. Then by descending induction and base change (proposition 4.1.8) we can prove that for every $i = 0, 1, 2$, for every pair (α, l) and for every t in T_α^l the base change $\tau^i(\alpha, l, t)$ is an isomorphism.

Now let us fix any α and let us denote by $L = \{l_1 < \cdots < l_r\}$ the corresponding set of indices. For each subset $\{l'_1 < \cdots < l'_s\} \subset L$ we denote by $\{l'_{s+1} < \cdots < l'_r\}$ its complement in L and we define

$$T_\alpha^{l'_1, \cdots, l'_s} := (T_\alpha^{l'_1} \cap \cdots \cap T_\alpha^{l'_s}) \smallsetminus (T_\alpha^{l'_{s+1}} \cup \cdots \cup T_\alpha^{l'_r}). \tag{4.27}$$

Each such scheme is locally closed in T and any two such schemes are disjoint if they are associated to different sets of indices; moreover each T_α is covered by such subschemes. Then we denote by k any set of indices $k := \{l'_1 < \cdots < l'_s\}$ and by $T_{\alpha,k}$ the corresponding scheme defined as in (4.27). For each α, the set of all such k is finite. Now for each such k, let us consider the inclusion $T_{\alpha,k} \hookrightarrow T_\alpha^{l'_1}$. By base change for $i = 0, 1, 2$ the sheaves

$$\mathcal{E}xt^i_{\pi_{T_{\alpha,k}}}\Big((\mathcal{E}', \mathcal{V}')|_{T_{\alpha,k}}, (\mathcal{E}, \mathcal{V})|_{T_{\alpha,k}}\Big) = \left(\mathcal{E}xt^i_{\pi_{T_\alpha^{l'_1}}}\Big((\mathcal{E}', \mathcal{V}')|_{T_\alpha^{l'_1}}, (\mathcal{E}, \mathcal{V})|_{T_\alpha^{l'_1}}\Big)\right)|_{T_{\alpha,k}}$$

are locally free for $i = 0, 1, 2$ and commute with base change, so we conclude. $\square$

In particular, we will apply these results in the cases when the constant c is equal to 0 or 1. By using lemma 4.6.1 together with the results of the previous 3 sections we get the following propositions.

Proposition 4.6.2. *Let us fix any noetherian scheme T and any pair of families of coherent systems $(\mathcal{E}, \mathcal{V})$, $(\mathcal{E}', \mathcal{V}')$ parametrized by T. Let us suppose that $Hom((\mathcal{E}', \mathcal{V}')_t, (\mathcal{E}, \mathcal{V})_t)$ has constant dimension (not necessarily zero) for all $t \in T$. Then there exists a finite covering of T by* disjoint reduced locally closed subschemes *$T_{\alpha,k}$ defined as in lemma 4.6.1, such that the conclusions of corollaries 4.3.2, 4.4.3, 4.5.3 and 4.5.5 hold for each $S = T_{\alpha,k}$ and for the pair of families $(\mathcal{E}', \mathcal{V}')|_{T_{\alpha,k}}$ and $(\mathcal{E}, \mathcal{V})|_{T_{\alpha,k}}$. If we denote by*

$$\pi_{\alpha,k} : V_{\alpha,k} \to T_{\alpha,k}, \quad \varphi_{\alpha,k} : P_{\alpha,k} \to T_{\alpha,k}, \quad \theta_{t,\alpha,k} : Q_{t,\alpha,k} \to T_{\alpha,k}$$

the vector bundles, the projective and the grassmannian fibrations obtained by those corollaries, then we get that the fibers of $\pi_{\alpha,k}$ are isomorphic to $\mathbb{C}^{\alpha}$, the fibers of $\varphi_{\alpha,k}$ are isomorphic to $\mathbb{P}^{\alpha-1}$ and the fibers of $\theta_{t,\alpha,k}$ are isomorphic to $Grass(t, \alpha)$.

If T_{α} is irreducible for some α, then the covering $\{T_{\alpha;k}\}_k$ coincides with $\{T_{\alpha}\}$.

Proposition 4.6.3. *Let us fix any noetherian scheme T and any pair of families of coherent systems $(\mathcal{E}, \mathcal{V})$, $(\mathcal{E}', \mathcal{V}')$ parametrized by T. Let us suppose that $Hom((\mathcal{E}', \mathcal{V}')_t, (\mathcal{E}, \mathcal{V})_t) = 0$ for all $t \in T$. Then there exists a finite covering of T by* disjoint reduced locally closed subschemes *$T_{\alpha,k}$ defined as in the previous lemma, such that the conclusions of corollaries 4.3.3, 4.4.4, 4.5.4 and 4.5.6 hold for each $S = T_{\alpha,k}$ and for the pair of families $(\mathcal{E}', \mathcal{V}')|_{T_{\alpha,k}}$ and $(\mathcal{E}, \mathcal{V})|_{T_{\alpha,k}}$. The description of the various fibrations that are obtained in this way is the same given in previous proposition.*

Also in this case if T_{α} is irreducible for some α, then the covering $\{T_{\alpha;k}\}_k$ coincides with $\{T_{\alpha}\}$.

Chapter 5

Fibrations associated to binary trees

We recall that for every (E, V) that is α_c-semistable, the length of any of its α_c-JHF is constant, so we have denoted that number by $r_{\alpha_c}(E, V)$ (or simply $r(E, V)$ if α_c is fixed). We need to parametrize any (E, V) with a given Jordan-Hölder graded $\oplus_{i=1}^{r}(Q_i, W_i)$ at α_c. Then the basic idea should be that of considering a binary tree with r leaves representing the various (Q_i, W_i)'s and internal nodes representing subsequent classes of extensions of their descendents on the left and on the right.

In particular, given any (E, V) in $G^+(\alpha_c; n, d, k)$ one should be interested in applying several times lemma 1.0.6. First of all, we should apply it on (E, V), so that we get a tree with root associated to (E, V) and 2 descendents associated to (E_1, V_1) and (E_2, V_2). Then we look at the 2 (E_i, V_i)'s separately: each of them is α_c^+-stable and α_c-semistable. If (E_i, V_i) is α_c-stable, then it is an object of the graded of (E, V) at α_c, and we stop the construction of the tree on that side. If it is strictly α_c-semistable, then it is necessarily α_c^--unstable, so it belongs to $G^+(\alpha_c; n_i, d_i, k_i)$, so we can apply again lemma 1.0.6 on such an object and we add 2 more nodes on the tree as descendents of (E_i, V_i). Every time we apply that lemma to some object, the 2 new nodes that are added represent objects that are α_c-semistable (possibly, α_c-stable); the new objects have both strictly smaller length of any Jordan-Hölder filtration at α_c. So after finitely many steps one gets a binary tree where the set of the leaves (i.e. those nodes without descendents) coincides with the set of the α_c-stable coherent systems in the graded of (E, V) at α_c.

Every pair of segments joining a node with its 2 descendents should represent a non-split extension similar to (1.4). As we said in remark 1.0.7, we have a natural action of $\mathbb{C}^*$, so actually every pair of segments like that should represent a class of equivalence of such an extension, modulo multiplication by invertible scalars.

As we said in that remark, in general one does not have uniqueness of the extension (1.4) even after quotienting by the action of $\mathbb{C}^*$. Anyway, sometimes this can actually happen, so one should be interested in having a good geometric description of such a situation. The idea is basically the following. Let us fix some data:

- any integer $r \geq 2$;
- any finite binary tree with r leaves, let us name them from 1 to r, from left to right;
- any triple (n,d,k) and any critical value α_c for it; we write $\mu := \mu_{\alpha_c}(n,d,k)$;
- any triple of invariants (n_i,d_i,k_i) for each leaf i, such that $\mu_{\alpha_c}(n_i,d_i,k_i) = \mu$ and such that $G(\alpha_c; n_i,d_i,k_i) \neq \varnothing$.

For each leaf i, we fix a stable coherent system (Q_i,W_i) in $G(\alpha_c; n_i,d_i,k_i)$. Then for each node N with 2 leaves named i and $i+1$ as descendents, we consider the projective space

$$\mathbb{P}(\mathrm{Ext}^1((Q_{i+1},W_{i+1}),(Q_i,W_i)))$$

and we fix any $[\sigma]$ in it. If σ is represented by an extension with middle term (E',V'), then this coherent systems will be associated to the node N. The class $[\sigma]$ will be associated to the pair of segments from N to its 2 descendents. By finite induction we can associate to every node of the tree a coherent system. The one associated to the root of the tree will be simply denoted by (E,V). We would like to globalize such kind of construction by letting vary the various (Q_i,W_i)'s in the corresponding moduli spaces any by letting vary also the various $[\sigma]$'s in the corresponding vector spaces.

Some caveat are necessary. Since our construction is bottom-up (starting from the various objects (Q_i,W_i)'s and obtaining objects of the form (E,V)), in general we can have 2 problems as follows.

- The (E,V)'s that one gets at the end are certainly α_c-semistable by proposition 1.0.1, but in general they can be not α_c^+-stable.
- A fixed (E,V) in $G^+(\alpha_c; n,d,k)$ can in general be associated to more than one tree (or, to the same tree but with different objects in the internal nodes) because of remark 1.0.7.

So in general the objects that we will obtain will not be exactly what one needs in order to describe $G^+(\alpha_c; n,d,k)$. Analogous considerations hold for $G^-(\alpha_c; n,d,k)$. We decided anyway to give such a description because we think that it is interesting in its own. Moreover, we will use it directly in the cases when the Jordan-Hölder filtration is unique and has length 3 (see next chapter). Note that by globalizing the previous description, in general it will be possible that some (E,V)'s obtained in the root of a fixed tree belong to $G^+(\alpha_c; n,d,k)$ or to $G^-(\alpha_c; n,d,k)$ and are counted exactly once, while some other do not satisfy those properties. Regrettably, it is not possible to say a priori which elements (E,V)'s obtained in that way are good and which are not interesting for our purposes. Indeed, in order to say something more one has to know precisely both the shape of the particular tree under consideration and the moduli spaces we are considering in the leaves of the tree.

Remark 5.0.1. So in this chapter we will only give a geometric description of certain sets of sequences of classes of extensions, but with no claim about the final objects (E,V)'s that one gets on the root of the tree.

Given any binary tree with $r \geq 2$ leaves, we denote the root of the tree by $(0,r)$. Starting from the 2 descendents of the root, for any node N we do the following: we denote by $r(N)$ the number of leaves under N (in particular $r(N)=1$ if and only if N is a leaf); if the node N is the left descendent of a node $M=(a,b)$, then we write $N=(a,a+r(N))$; if N is the right descendent of M, then we write $N=(b-r(N),b)$. In particular, if N is a leaf, then it is denoted by $(i-1,i)$ for some $i=1,\cdots,r$. By induction on the tree we can prove that every node (a,b) is such that $a<b$; if it is an internal node (i.e. not a leaf), then $b-a\geq 2$ and its 2 descendents are of the form (a,c) and (c,b) for some $c\in\{a+1,\cdots,b-1\}$.

Now let us fix any triple (n,d,k) and any critical value α_c for it; we write $\mu := \mu_{\alpha_c}(n,d,k)$. Then we consider any set of data as follows:

- any integer $r\in\{2,\cdots,n\}$;
- any binary tree such that it has exactly r leaves; this in particular fixes a finite set of internal nodes, i.e. those of the form $(j,i;-)$ with 2 descendents of the form (j,l) and (l,i) with $j<l<i$;
- for every leaf $(i-1,i)$ of the tree, a pair $(n_{i-1,i},k_{i-1,i}) := (n_i,k_i)\in\mathbb{N}\times\mathbb{N}_0$ such that the quantity

$$d_{i-1,i} = d_i := n_i\mu - \alpha_c k_i$$

  is an integer and such that the moduli space $G(\alpha_c;n_i,d_i,k_i)$ is not empty (the definition of d_i is such that $\mu_{\alpha_c}(n_i,d_i,k_i)=\mu$);
- for all the triples (j,l,i) as before, a non-negative integer $e_l = e_{j,l,i}$.

Any such set of data will be denoted by $\mathscr{D}$. If all the data but the last one are clear from the context, we will simply write $\mathscr{D} = \{e_l = e_{j,l,i}\}_{(j,i;-)}$.

For every node (j,i) of a tree, we define its height $h(j,i)$ as the maximum number of segments needed to reach any of the descendents of that node. In particular, if (j,i) is a leaf, that number will be considered equal to zero. We denote by H the height of the whole tree, i.e. $H=h(0,r)$. For every internal node (j,i), we define $\mathrm{desc}(j,i)$ as the unique integer l such that the descendents of that node are (j,l) and (l,i). Moreover, for every internal node (j,i) of the tree, we define $\underline{\mathrm{desc}}(j,i)$ as the set of all *internal* nodes of that tree descending from (j,i) (considered also). Then we fix also the following notation.

- For every leaf $(i-1,i)$ we set $\mathcal{G}(\mathscr{D},i-1,i) := G_i = G(\alpha_c;n_i,d_i,k_i)$. We will denote any object of this moduli space by (Q_i,W_i).

- For every *internal* node (a,b) of the tree, we denote by $\alpha_{a,b}$ any set of the form

$$\alpha_{a,b} := \Big\{[\sigma_{j,i}] \in \mathbb{P}(\mathrm{Ext}^1((E_{l,i},V_{l,i}),(E_{j,l},V_{j,l})))\Big\}_{(j,i)\in\underline{\mathrm{desc}}(a,b)}.$$

Here for every internal node (j,i) we are writing $l := \mathrm{desc}(j,i)$ and for every $\sigma_{j,i}$ we are considering a representative of it of the form:

$$0 \to (E_{j,l},V_{j,l}) \xrightarrow{\lambda_{j,l}} (E_{j,i},V_{j,i}) \xrightarrow{\gamma_{l,i}} (E_{l,i},V_{l,i}) \to 0. \tag{5.1}$$

Then for every internal node (a,b) we denote by $\mathcal{G}(\mathscr{D},a,b)$ the set of all those $\alpha_{a,b}$'s such that the following conditions are satisfied:

(1) every object of the form $(E_{i-1,i},V_{i-1,i})$ that appears in any sequence of the form (5.1) (either on the left or on the right) belongs to G_i. Hence it will be also denoted by (Q_i,W_i);

(2) $\dim \mathrm{Ext}^1\Big((E_{l,i},V_{l,i}),(E_{j,l},V_{j,l})\Big) = e_l$ for all nodes (j,i) in the set $\underline{\mathrm{desc}}(a,b)$, with $l = \mathrm{desc}(j,i)$;

(3) $\mathrm{Hom}\Big((E_{l,i},V_{l,i}),(E_{j,l},V_{j,l})\Big) = 0$ for all nodes (j,i) in $\underline{\mathrm{desc}}(a,b)$ and with $l = \mathrm{desc}(j,i)$.

In particular, the first condition induces a set map $\mathrm{gr}_{a,b} : \mathcal{G}(\mathscr{D},a,b) \to \prod_{i=a+1,\cdots,b} G_i$ that associates to every $\alpha_{a,b}$ as before the graded of $(E_{a,b},V_{a,b})$.

- For every internal node (a,b) with $\mathrm{desc}(a,b)=c$, we denote by $\mathcal{F}(\mathscr{D},a,b)$ the set of all pairs $(\alpha_{a,c},\alpha_{c,b})$ in $\mathcal{G}(\mathscr{D},a,c)\times\mathcal{G}(\mathscr{D},c,b)$ such that

(4) $\dim \mathrm{Ext}^1\Big((E_{c,b},V_{c,b}),(E_{a,c},V_{a,c})\Big) = e_c = e_{a,c,b}$;

(5) $\mathrm{Hom}\Big((E_{c,b},V_{c,b}),(E_{a,c},V_{a,c})\Big) = 0$.

Lemma 5.0.4. *For each node $(a,b;-)$, condition (3) (and therefore also condition (5)) is automatically satisfied (and therefore it can be omitted) in each of the following 2 cases:*

(i) $\mu_{\alpha_c^+}(E_{l,i},V_{l,i}) \neq \mu_{\alpha_c^+}(E_{j,l},V_{j,l})$ *(or the same inequality for $\mu_{\alpha_c^-}$) for all $(j,i;-)\in\underline{\mathrm{desc}}(a,b)$ with $l = \mathrm{desc}(j,i)$;*

(ii) $(Q_i,W_i)\not\simeq(Q_j,W_j)$ *for all $i\neq j\in\{a+1,\cdots,b\}$.*

Proof. Let us suppose that (i) holds. Since $\mu_{\alpha_c}(E_{l,i},V_{l,i}) = \mu = \mu_{\alpha_c}(E_{j,l},V_{j,l})$ by construction, then (i) implies that either

$$\mu_{\alpha_c^+}(E_{l,i},V_{l,i}) > \mu_{\alpha_c^+}(E_{j,l},V_{j,l})$$

or

$$\mu_{\alpha_c^-}(E_{l,i}, V_{l,i}) > \mu_{\alpha_c^-}(E_{j,l}, V_{j,l}).$$

In both cases, lemma 1.0.4 implies condition (3).

For the second case, we prove the result only for condition (5); the proof of condition (3) is analogous. So let us consider the vector space

$$\mathrm{Hom}\Big((E_{c,b}, V_{c,b}), (E_{a,c}, V_{a,c})\Big);$$

by contradiction, let us suppose that it contains a non-zero morphism η; then if $(a, c) = (a, a+1)$ this implies that we have a non-zero morphism from $(E_{c,b}, V_{c,b})$ to (Q_{a+1}, W_{a+1}). Otherwise, (a, c) is an internal node, hence $\alpha_{a,c}$ contains an object $[\sigma_{a,c}]$ with $\sigma_{a,c}$ represented by:

$$0 \to (E_{a,d}, V_{a,d}) \xrightarrow{\lambda} (E_{a,c}, V_{a,c}) \xrightarrow{\gamma} (E_{d,c}, V_{d,c}) \to 0$$

where $d = \mathrm{desc}(a, c)$. In that case if $\gamma \circ \eta = 0$, this implies that η has values in the image of λ, so it induces a non-zero morphism from $(E_{c,b}, V_{c,b})$ to $(E_{a,d}, V_{a,d})$. Otherwise, $\gamma \circ \eta \neq 0$, so we have a non-zero morphism from $(E_{c,b}, V_{c,b})$ to $(E_{d,c}, V_{d,c})$. Therefore, by applying induction on the length of the nodes, we get that η induces a non-zero morphism η' from $(E_{c,b}, V_{c,b})$ to some $(E_{i-1,i}, V_{i-1,i}) = (Q_i, W_i)$ for an index i in $\{a+1, \cdots, c\}$.

Now let us consider the node (c, b). If it is a leaf, we get that $(E_{c,b}, V_{c,b}) = (Q_b, W_b)$; otherwise it is an internal node with $f := \mathrm{desc}(c, b)$. In this case, $\alpha_{c,b}$ contains an object $[\sigma_{c,b}]$ with $\sigma_{c,b}$ represented by an exact sequence of the form:

$$0 \to (E_{c,f}, V_{c,f}) \xrightarrow{\lambda'} (E_{c,b}, V_{c,b}) \xrightarrow{\gamma'} (E_{f,b}, V_{f,b}) \to 0.$$

If $\eta' \circ \lambda' \neq 0$, this is a non-zero morphism from $(E_{c,f}, V_{c,f})$ to (Q_i, W_i); otherwise η' induces a non-zero morphism from $(E_{f,b}, V_{f,b}) \simeq (E_{c,b}, V_{c,b})/\mathrm{Im}(\lambda')$ to (Q_i, W_i). By applying induction on the length of the nodes also in this case, we end up with a non-zero morphism η'' from $(Q_j, W_j) = (E_{j-1,j}, V_{j-1,j})$ to (Q_i, W_i) for some index j in $\{c+1, \cdots, b\}$.

Now that morphism must be an isomorphism by lemma 1.0.4, but this is impossible because we are assuming condition (ii). Hence we conclude that $\mathrm{Hom}\Big((E_{c,b}, V_{c,b}), (E_{a,c}, V_{a,c})\Big)$ is zero for every $(\alpha_{a,c}, \alpha_{c,b}) \in \mathcal{G}(\mathscr{D}, a, c) \times \mathcal{G}(\mathscr{D}, c, b)$. $\square$

Now let us fix any index $i \in \{1, \cdots, r\}$. Then we denote by $R_i = R(\alpha_c; n_i, d_i, k_i)$ the Quot scheme used in the construction of $G(\alpha_c; n, d, k)$ and $\widetilde{G}(\alpha; n, d, k)$; we denote by $PGL(N_i)$ the group acting on such scheme (see remark 1.0.4). For every $i \in \{1, \cdots, r\}$ let us define a set $L_{i-1,i}$ consisting of a single abstract index $l_{i-1,i}$. We denote by

$$\hat{\mathcal{G}}(\mathscr{D}, l_{i-1,i}) = \hat{G}_i = R_i^s(\alpha_c; n_i, d_i, k_i)$$

the subschemes of R_i consisting of α_c-stable points. By remark 1.0.4 there exists a family $(\mathcal{Q}_i^{\mathrm{s}}, \mathcal{W}_i^{\mathrm{s}})$ of coherent systems over the stable locus with a local universal property; we denote such a family also by $(\hat{\mathcal{E}}_{l_{i-1,i}}, \hat{\mathcal{V}}_{l_{i-1,i}})$. In addition, for every node (a,b) let us define the group $G_{a,b} := PGL(N_{a+1}) \times \cdots \times PGL(N_b)$. Then we have the following result.

Proposition 5.0.5. *Let us fix any type (n,d,k), any critical value α_c for it and any data $\mathscr{D}$ as before. Let us suppose that for each internal node (j,i) with desc$(j,i) = l$ and for every $\alpha_{j,i} \in \mathcal{G}(\mathscr{D}, j, i)$ we have*

$$Hom\Big((E_{l,i}, V_{l,i}), (E_{j,l}, V_{j,l})\Big) = 0. \tag{5.2}$$

Then for every internal *node $(a,b;-)$ there exist the following objects:*

- *a finite set $L_{a,b}$ of indices together with a natural map $L_{a,b} \to L_{a,c} \times L_{c,b}$, where $c =$ desc(a,b); for each $l_{a,b} \in L_{a,b}$ its image will be denoted by $(l_{a,c}, l_{c,b})$;*
- *for each $l_{a,b} \in L_{a,b}$, 2 schemes $\hat{\mathcal{F}}(\mathscr{D}, l_{a,b})$ and $\hat{\mathcal{G}}(\mathscr{D}, l_{a,b})$;*
- *for each $l_{a,b} \in L_{a,b}$, a family $(\hat{\mathcal{E}}_{l_{a,b}}, \hat{\mathcal{V}}_{l_{a,b}})$ of coherent systems parametrized by $\hat{\mathcal{G}}(\mathscr{D}, l_{a,b})$,*

such that the following properties hold.

(i) For each $l_{a,b} \in L_{a,b}$, $\hat{\mathcal{F}}(\mathscr{D}, l_{a,b})$ is a locally closed subscheme of $\hat{\mathcal{G}}(\mathscr{D}, l_{a,c}) \times \hat{\mathcal{G}}(\mathscr{D}, l_{c,b})$; if we denote by $\hat{p}_{l_{a,b}}$, $\hat{q}_{l_{a,b}}$ and $\hat{\pi}_{l_{a,b}}$ the projections

$$\begin{gathered} \hat{\mathcal{G}}(\mathscr{D}, l_{a,c}) \xleftarrow{\hat{p}_{l_{a,b}}} \hat{\mathcal{F}}(\mathscr{D}, l_{a,b}) \xrightarrow{\hat{q}_{l_{a,b}}} \hat{\mathcal{G}}(\mathscr{D}, l_{c,b}), \\ \hat{\pi}_{l_{a,b}} : \hat{\mathcal{F}}(\mathscr{D}, l_{a,b}) \times C \to \hat{\mathcal{F}}(\mathscr{D}, l_{a,b}), \end{gathered} \tag{5.3}$$

then there exists a locally free sheaf

$$\hat{\mathcal{H}}_{l_{a,b}} := \mathcal{E}xt^1_{\hat{\pi}_{l_{a,b}}}\Big((\hat{q}'_{l_{a,b}}, \hat{q}_{l_{a,b}})^*(\hat{\mathcal{E}}_{l_{c,b}}, \hat{\mathcal{V}}_{l_{c,b}}), (\hat{p}'_{l_{a,b}}, \hat{p}_{l_{a,b}})^*(\hat{\mathcal{E}}_{l_{a,c}}, \hat{\mathcal{V}}_{l_{a,c}})\Big)^\vee$$

over $\hat{\mathcal{F}}(\mathscr{D}, l_{a,b})$, and $\hat{\mathcal{G}}(\mathscr{D}, l_{a,b})$ is equal to $\mathbb{P}(\hat{\mathcal{H}}_{l_{a,b}})$. In particular, this gives a projective fibration $\hat{\varphi}_{l_{a,b}} : \hat{\mathcal{G}}(\mathscr{D}, l_{a,b}) \to \hat{\mathcal{F}}(\mathscr{D}, l_{a,b})$ with fibers isomorphic to $\mathbb{P}^{e_c - 1}$; we denote by $\mathcal{O}_{l_{a,b}}(1)$ the tautological bundle of $\hat{\mathcal{G}}(\mathscr{D}, l_{a,b})$.

(ii) There exists a family of non-splitting extensions parametrized by $\hat{\mathcal{G}}(\mathscr{D}, l_{a,b})$, of the form

$$\begin{gathered} 0 \to (\hat{\varphi}'_{l_{a,b}}, \hat{\varphi}_{l_{a,b}})^*(\hat{p}'_{l_{a,b}}, \hat{p}_{l_{a,b}})^*(\hat{\mathcal{E}}_{l_{a,c}}, \hat{\mathcal{V}}_{l_{a,c}}) \otimes_{\hat{\mathcal{G}}(\mathscr{D}, l_{a,b})} \mathcal{O}_{l_{a,b}}(1) \to \\ \to (\hat{\mathcal{E}}_{l_{a,b}}, \hat{\mathcal{V}}_{l_{a,b}}) \to (\hat{\varphi}'_{l_{a,b}}, \hat{\varphi}_{l_{a,b}})^*(\hat{q}'_{l_{a,b}}, \hat{q}_{l_{a,b}})^*(\hat{\mathcal{E}}_{l_{c,b}}, \hat{\mathcal{V}}_{l_{c,b}}) \to 0. \end{gathered} \tag{5.4}$$

Such an extension is universal *in the following sense: let us suppose that we have fixed any morphism of noetherian schemes $u : T \to \hat{\mathcal{F}}(\mathscr{D}, l_{a,b})$, any line bundle $\mathcal{L} \in Pic(T)$ and any family of non-splitting extensions parametrized by T:*

$$
\begin{aligned}
0 \to (u',u)^*(\hat{p}'_{l_{a,b}}, \hat{p}_{l_{a,b}})^*(\hat{\mathcal{E}}_{l_{a,c}}, \hat{\mathcal{V}}_{l_{a,c}}) \otimes_T \mathcal{L} \to \\
\to (\mathcal{E}_T, \mathcal{V}_T) \to (u',u)^*(\hat{q}'_{l_{a,b}}, \hat{q}_{l_{a,b}})^*(\hat{\mathcal{E}}_{l_{c,b}}, \hat{\mathcal{V}}_{l_{c,b}}) \to 0.
\end{aligned} \tag{5.5}
$$

Then there exists a unique morphism $\psi : T \to \hat{\mathcal{G}}(\mathscr{D}, l_{a,b})$ *over* $\hat{\mathcal{F}}(\mathscr{D}, l_{a,b})$*, such that (5.5) is the pullback of (5.4) via* (ψ', ψ)*, modulo the action of* $H^0(T, \mathcal{O}_T^*)$*.*

(iii) *For each* $l_{a,b}$ *there are free actions of* $G_{a,b}$ *on* $\hat{\mathcal{F}}(\mathscr{D}, l_{a,b})$ *and on* $\hat{\mathcal{G}}(\mathscr{D}, l_{a,b})$*. For both schemes there exist good quotients, denoted by:*

$$
pr^{\mathcal{F}}_{l_{a,b}} : \hat{\mathcal{F}}(\mathscr{D}, l_{a,b}) \twoheadrightarrow \mathcal{F}(\mathscr{D}, l_{a,b}), \quad pr^{\mathcal{G}}_{l_{a,b}} : \hat{\mathcal{G}}(\mathscr{D}, l_{a,b}) \twoheadrightarrow \mathcal{G}(\mathscr{D}, l_{a,b}).
$$

(iv) *The family of schemes* $\{\mathcal{F}(\mathscr{D}, l_{a,b})\}_{l_{a,b} \in L_{a,b}}$ *gives a disjoint locally closed covering of the set* $\mathcal{F}(\mathscr{D}, a, b)$ *and analogously for* $\mathcal{G}(\mathscr{D}, a, b)$*.*

(v) *For each* $l_{a,b}$ *there is a projective fibration* $\varphi_{l_{a,b}}$ *making the following diagram commute:*

$$
\begin{tikzcd}
\hat{\mathcal{G}}(\mathscr{D}, l_{a,b}) \arrow[r, "\hat{\varphi}_{l_{a,b}}"] \arrow[d, two heads, "pr^{\mathcal{G}}_{l_{a,b}}"'] \arrow[dr, phantom, "\curvearrowright"] & \hat{\mathcal{F}}(\mathscr{D}, l_{a,b}) \arrow[d, two heads, "pr^{\mathcal{F}}_{l_{a,b}}"] \\
\mathcal{G}(\mathscr{D}, l_{a,b}) \arrow[r, "\varphi_{l_{a,b}}"'] & \mathcal{F}(\mathscr{D}, l_{a,b}).
\end{tikzcd} \tag{5.6}
$$

The fibers of $\varphi_{l_{a,b}}$ *are again isomorphic to* $\mathbb{P}^{e_c - 1}$*.*

(vi) *For every* $l_{a,b}$*, for every* $\alpha_{a,b} = \{[\sigma_{j,i}]\}_{(j,i)}$ *in* $\mathcal{G}(\mathscr{D}, l_{a,b})$ *and for every* $t \in (pr^{\mathcal{G}}_{l_{a,b}})^{-1}(\alpha_{a,b})$ *we have that the sequence (5.4) restricted to* t *gives a non-splitting exact sequence*

$$
0 \to (E_{a,c}, V_{a,c}) \to (E_{a,b}, V_{a,b}) \to (E_{c,b}, V_{c,b}) \to 0
$$

that is a representative of $\sigma_{a,b}$*.*

Proof. (modeled on the proof of [GM, proposition 6.5]) We proceed by induction on the height of the node (a, b) starting from nodes of height 1 (note that if the height is zero, we will have a local universal property as stated in [KN, §3.5]). If $h(a, b) = 1$, then $b = a + 2$ and the 2 descendents of $(a, a+2)$ are $(a, a+1)$ and $(a+1, a+2)$. So let us consider the 2 projections

$$
\begin{aligned}
\hat{p}_{a,a+2} : \hat{G}_{a+1} \times \hat{G}_{a+2} \to \hat{G}_{a+1} = \hat{\mathcal{G}}(\mathscr{D}, l_{a,a+1}), \\
\hat{q}_{a,a+2} : \hat{G}_{a+1} \times \hat{G}_{a+2} \to \hat{G}_{a+2} = \hat{\mathcal{G}}(\mathscr{D}, l_{a+1,a+2})
\end{aligned}
$$

and the projection $\hat{\pi}_{a,a+2} : \hat{G}_{a+1} \times \hat{G}_{a+2} \times C \to \hat{G}_{a+1} \times \hat{G}_{a+2}$. Let us denote by T the scheme $\hat{G}_{a+1} \times \hat{G}_{a+2}$ and by t any point $(t_1, t_2) \in T$; then we can define:

$$\begin{gathered}\hat{\mathcal{F}}(\mathscr{D}, a, a+2) := \Big\{ t \in T \text{ s.t. } \dim \operatorname{Ext}^1\big((\hat{q}'_{a,a+2}, \hat{q}_{a,a+2})^*(\hat{\mathcal{E}}_{l_{a+1,a+2}}, \hat{\mathcal{V}}_{l_{a+1,a+2}})_t, \\ (\hat{p}'_{a,a+2}, \hat{p}_{a,a+2})^*(\hat{\mathcal{E}}_{l_{a,a+1}}, \hat{\mathcal{V}}_{l_{a,a+1}})_t\big) = e_{a+1} \text{ and} \\ \operatorname{Hom}\big((\hat{q}'_{a,a+2}, \hat{q}_{a,a+2})^*(\hat{\mathcal{E}}_{l_{a+1,a+2}}, \hat{\mathcal{V}}_{l_{a+1,a+2}})_t, (\hat{p}'_{a,a+2}, \hat{p}_{a,a+2})^*(\hat{\mathcal{E}}_{l_{a,a+1}}, \hat{\mathcal{V}}_{l_{a,a+1}})_t\big) = 0 \Big\}. \quad (5.7)\end{gathered}$$

Using (5.2) and the previous definition of the families $(\hat{\mathcal{E}}_{l_{i-1,i}}, \hat{\mathcal{V}}_{l_{i-1,i}})$ for every $i = 1, \cdots, r$, the last condition can be dropped. Therefore, if we consider the families parametrized by T

$$(\hat{q}'_{a,a+2}, \hat{q}_{a,a+2})^*(\hat{\mathcal{E}}_{l_{a+1,a+2}}, \hat{\mathcal{V}}_{l_{a+1,a+2}}), \quad (\hat{p}'_{a,a+2}, \hat{p}_{a,a+2})^*(\hat{\mathcal{E}}_{l_{a,a+1}}, \hat{\mathcal{V}}_{l_{a,a+1}}),$$

then each set (5.7) coincides with a scheme of the form $T_{e_{a+1}}$ as described in lemma 4.6.1. Then we can apply proposition 4.6.3 on $T_{e_{a+1}}$ and we get that there is a finite set $L_{a,a+2}$ and a disjoint covering $\{\hat{\mathcal{F}}(\mathscr{D}, l_{a,a+2})\}_{l_{a,a+2} \in L_{a,a+2}}$ of $\hat{\mathcal{F}}(\mathscr{D}, a, a+2)$ by reduced locally closed subschemes. In this case the sets $L_{a,a+1}$ and $L_{a+1,a+2}$ consist of a single element, so the set map $L_{a,a+2} \to L_{a,a+1} \times L_{a+1,a+2}$ is easy to define.

Now for each index $l_{a,a+2}$ we define the morphisms $\hat{p}_{l_{a,a+2}}$, $\hat{q}_{l_{a,a+2}}$ and $\hat{\pi}_{l_{a,a+2}}$ as in (5.3). Again by proposition 4.6.3, we get that on each $\hat{\mathcal{F}}(\mathscr{D}, l_{a,a+2})$ the sheaf

$$\hat{\mathcal{H}}_{l_{a,a+2}} := \mathcal{E}xt^1_{\hat{\pi}_{l_{a,a+2}}}\Big((\hat{q}'_{l_{a,a+2}}, \hat{q}_{l_{a,a+2}})^*(\hat{\mathcal{E}}_{l_{a+1,a+2}}, \hat{\mathcal{V}}_{l_{a+1,a+2}}), (\hat{p}'_{l_{a,a+2}}, \hat{p}_{l_{a,a+2}})^*(\hat{\mathcal{E}}_{l_{a,a+1}}, \hat{\mathcal{V}}_{l_{a,a+1}})\Big)^{\vee}$$

is locally free of rank e_{a+1}, so it makes sense to define $\hat{\mathcal{G}}(\mathscr{D}, l_{a,a+2}) := \mathbb{P}(\hat{\mathcal{H}}_{l_{a,a+2}})$ and to consider the induced projective fibration

$$\hat{\varphi}_{l_{a,a+2}} : \hat{\mathcal{G}}(\mathscr{D}, l_{a,a+2}) \longrightarrow \hat{\mathcal{F}}(\mathscr{D}, l_{a,a+2})$$

with fibers isomorphic to $\mathbb{P}^{e_{a+1}-1}$. We denote by $\mathcal{O}_{l_{a,a+2}}(1)$ the tautological bundle of $\hat{\mathcal{G}}(\mathscr{D}, l_{a,a+2})$. Again the same proposition proves that there is a family $(\hat{\mathcal{E}}_{l_{a,a+2}}, \hat{\mathcal{V}}_{l_{a,a+2}})$ of coherent systems parametrized by $\hat{\mathcal{G}}(\mathscr{D}, l_{a,a+2})$ and a family of non-splitting extensions of the form:

$$\begin{gathered}0 \to (\hat{\varphi}'_{l_{a,a+2}}, \hat{\varphi}_{l_{a,a+2}})^*(\hat{p}'_{l_{a,a+2}}, \hat{p}_{l_{a,a+2}})^*(\hat{\mathcal{E}}_{l_{a,a+1}}, \hat{\mathcal{V}}_{l_{a,a+1}}) \otimes_{\hat{\mathcal{G}}(\mathscr{D}, l_{a,a+2})} \\ \otimes_{\hat{\mathcal{G}}(\mathscr{D}, l_{a,a+2})} \mathcal{O}_{l_{a,a+2}}(1) \to (\hat{\mathcal{E}}_{l_{a,a+2}}, \hat{\mathcal{V}}_{l_{a,a+2}}) \to \qquad (5.8) \\ \to (\hat{\varphi}'_{l_{a,a+2}}, \hat{\varphi}_{l_{a,a+2}})^*(\hat{q}'_{l_{a,a+2}}, \hat{q}_{l_{a,a+2}})^*(\hat{\mathcal{E}}_{l_{a+1,a+2}}, \hat{\mathcal{V}}_{l_{a+1,a+2}}) \to 0.\end{gathered}$$

This family is universal in the sense of corollary 4.4.4.

Now let us consider the free action of $G_{a,a+2} = PGL(N_{a+1}) \times PGL(N_{a+2})$ on $\hat{G}_{a+1} \times \hat{G}_{a+2}$, with quotient

$$pr_{a,a+2} = (pr^{\mathcal{G}}_{l_{a,a+1}}, pr^{\mathcal{G}}_{l_{a+1,a+2}}) : \hat{G}_{a+1} \times \hat{G}_{a+2} \twoheadrightarrow G_1 \times G_2.$$

This action restricts to an action of the same group on the subscheme $\hat{\mathcal{F}}(\mathscr{D}, l_{a,a+2})$ and we denote by $\mathcal{F}(\mathscr{D}, l_{a,a+2})$ the image of it under $pr_{a,a+2}$. Since $\hat{\mathcal{F}}(\mathscr{D}, l_{a,a+2})$ is locally closed in $\hat{G}_1 \times \hat{G}_2$ and saturated with respect to that action, then $\mathcal{F}(\mathscr{D}, l_{a,a+2})$ is locally closed in $G_1 \times G_2$. For every index $l_{a,a+2}$ we denote by

$$pr^{\mathcal{F}}_{l_{a,a+2}} : \hat{\mathcal{F}}(\mathscr{D}, l_{a,a+2}) \twoheadrightarrow \mathcal{F}(\mathscr{D}, l_{a,a+2})$$

the restriction of $pr_{a,a+2}$. Now the action of $G_{a,a+2}$ extends to an action on the projective bundle $\hat{\mathcal{G}}(\mathscr{D}, l_{a,a+2})$ over $\hat{\mathcal{F}}(\mathscr{D}, l_{a,a+2})$. Such a group action is free and the quotients are geometric, so there is a good quotient

$$pr^{\mathcal{G}}_{l_{a,a+2}} : \hat{\mathcal{G}}(\mathscr{D}, l_{a,a+2}) \twoheadrightarrow \mathcal{G}(\mathscr{D}, l_{a,a+2})$$

and an induced projective fibration $\varphi_{l_{a,a+2}} : \mathcal{G}(\mathscr{D}, l_{a,a+2}) \to \mathcal{F}(\mathscr{D}, l_{a,a+2})$ making diagram (5.6) commute for $(a,b) = (a, a+2)$. Set theoretically $pr_{a,a+2}(\hat{\mathcal{F}}(\mathscr{D}, a, a+2)) = \mathcal{F}(\mathscr{D}, a, a+2)$. Since

$$\{\hat{\mathcal{F}}(\mathscr{D}, l_{a,a+2})\}_{l_{a,a+2} \in L_{a,a+2}}$$

is a disjoint locally closed covering of $\hat{\mathcal{F}}(\mathscr{D}, a, a+2)$, then we get that the family of schemes $\{\mathcal{F}(\mathscr{D}, l_{a,a+2})\}_{l_{a,a+2} \in L_{a,a+2}}$ gives a disjoint covering of the set $\mathcal{F}(\mathscr{D}, a, a+2)$. Now let us fix any $l_{a,b}$, any point $((Q_{a+1}, W_{a+1}), (Q_{a+2}, W_{a+2}))$ in $\mathcal{F}(\mathscr{D}, l_{a,a+2})$ and let

$$(t_1, t_2) \in (pr^{\mathcal{F}}_{l_{a,a+2}})^{-1}((Q_{a+1}, W_{a+1}), (Q_{a+2}, W_{a+2})).$$

By construction the fiber of $\hat{\varphi}_{l_{a,a+2}}$ over (t_1, t_2) is given by

$$\mathbb{P}(\mathrm{Ext}^1((Q_{a+2}, W_{a+2}), (Q_{a+1}, W_{a+1}))).$$

By the commutativity of (5.6), this coincides also with the fiber of $\varphi_{l_{a,a+2}}$ over $((Q_{a+1}, W_{a+1}),$ $(Q_{a+2}, W_{a+2}))$. Therefore this gives a canonical identification of the set underlying the scheme $\mathcal{G}(\mathscr{D}, l_{a,a+2})$ with a subset of $\mathcal{G}(\mathscr{D}, a, a+2)$. Moreover, this also proves property (vi) for the node $(a, a+2)$. In addition, since $\{\mathcal{F}(\mathscr{D}, l_{a,a+2})\}_{l_{a,a+2} \in L_{a,a+2}}$ is a locally closed disjoint covering of $\mathcal{F}(\mathscr{D}, a, a+2)$, then we get that the family of schemes $\{\mathcal{G}(\mathscr{D}, l_{a,a+2})\}_{l_{a,a+2} \in L_{a,a+2}}$ gives a disjoint covering of the set $\mathcal{G}(\mathscr{D}, a, a+2)$.

Now for the inductive step, let us suppose that all the results from (i) to (vi) are verified for every internal node with height less or equal than $h-1$. Then let us fix any node (a,b) with $h(a,b) = h$ and let us write $c := \mathrm{desc}(a,b)$; let us fix also any pair of indices $l_{a,c} \in L_{a,c}$ and $l_{c,b} \in L_{c,b}$.

By inductive hypothesis, we have constructed 2 projective fibrations $\hat{\mathcal{G}}(\mathscr{D}, l_{a,c})$ over $\hat{\mathcal{F}}(\mathscr{D}, l_{a,c})$ and $\hat{\mathcal{G}}(\mathscr{D}, l_{c,b})$ over $\hat{\mathcal{F}}(\mathscr{D}, l_{c,b})$. Moreover, we have constructed families of coherent systems

$(\hat{\mathcal{E}}_{l_{a,c}}, \hat{\mathcal{V}}_{l_{a,c}})$ and $(\hat{\mathcal{E}}_{l_{c,b}}, \hat{\mathcal{V}}_{l_{c,b}})$ parametrized by $\hat{\mathcal{G}}(\mathscr{D}, l_{a,c})$ and $\hat{\mathcal{G}}(\mathscr{D}, l_{c,b})$ respectively. Now let us fix any point

$$t = (t_{a,c}, t_{c,b}) \in \hat{\mathcal{G}}(\mathscr{D}, l_{a,c}) \times \hat{\mathcal{G}}(\mathscr{D}, l_{c,b}) =: T$$

and let us denote by $(\alpha_{a,c}, \alpha_{c,b}) := (pr^{\mathcal{G}}_{l_{a,c}}, pr^{\mathcal{G}}_{l_{c,b}})(t_{a,c}, t_{c,b})$. For simplicity, let us suppose that $\alpha_{a,c} = \{[\sigma_{j,i}]\}_{(j,i)\in\underline{\mathrm{desc}}(a,c)}$ with every $\sigma_{j,i}$ represented by a sequence of the form:

$$0 \to (E_{j,l}, V_{j,l}) \to (E_{j,i}, V_{j,i}) \to (E_{l,i}, V_{l,i}) \to 0$$

and analogously for $\alpha_{c,b}$. Then let us consider the 2 projections:

$$\hat{\mathcal{G}}(\mathscr{D}, l_{a,c}) \overset{\hat{p}_{l_{a,c},l_{c,b}}}{\longleftarrow} \hat{\mathcal{G}}(\mathscr{D}, l_{a,c}) \times \hat{\mathcal{G}}(\mathscr{D}, l_{c,b}) \overset{\hat{q}_{l_{a,c},l_{c,b}}}{\longrightarrow} \hat{\mathcal{G}}(\mathscr{D}, l_{c,b}).$$

Now by property (vi) of the inductive step, for each $t = (t_{a,c}, t_{c,b}) \in T$ as before, we have:

$$\begin{aligned}(\hat{p}'_{l_{a,c},l_{c,b}}, \hat{p}_{l_{a,c},l_{c,b}})^*(\hat{\mathcal{E}}_{l_{a,c}}, \hat{\mathcal{V}}_{l_{a,c}})_t = (\hat{\mathcal{E}}_{l_{a,c}}, \hat{\mathcal{V}}_{l_{a,c}})_{t_{a,c}} \simeq (E_{a,c}, V_{a,c}),\\ (\hat{q}'_{l_{a,c},l_{c,b}}, \hat{q}_{l_{a,c},l_{c,b}})^*(\hat{\mathcal{E}}_{l_{c,b}}, \hat{\mathcal{V}}_{l_{c,b}})_t = (\hat{\mathcal{E}}_{l_{c,b}}, \hat{\mathcal{V}}_{l_{c,b}})_{t_{c,b}} \simeq (E_{c,b}, V_{c,b}).\end{aligned}$$

Now we define

$$\begin{aligned}\hat{\mathcal{F}}(\mathscr{D}, l_{a,c}, l_{a,b}) := \Big\{(t_{a,c}, t_{c,b}) \in T \text{ s.t. } \dim \mathrm{Ext}^1((\hat{q}'_{l_{a,c},l_{c,b}}, \hat{q}_{l_{a,c},l_{c,b}})^*(\hat{\mathcal{E}}_{l_{c,b}}, \hat{\mathcal{V}}_{l_{c,b}})_t,\\ (\hat{p}'_{l_{a,c},l_{c,b}}, \hat{p}_{l_{a,c},l_{c,b}})^*(\hat{\mathcal{E}}_{l_{a,c}}, \hat{\mathcal{V}}_{l_{a,c}})_t) = e_c\Big\}.\end{aligned} \tag{5.9}$$

By proposition 1.0.5, this set is locally closed in T. Then we can apply proposition 4.6.3 on such a scheme, so there exists a finite set $L = L(l_{a,c}, l_{c,b})$ and a disjoint covering $\{\hat{\mathcal{F}}(\mathscr{D}, l)\}_{l\in L}$ of (5.9). Each object of that covering is locally closed in T and in (5.9). We perform this construction for every pair of indices $(l_{a,c}, l_{c,b}) \in L_{a,c} \times L_{c,b}$. Then we define

$$L_{a,b} := \bigsqcup_{(l_{a,c},l_{c,b})} L(l_{a,c}, l_{c,b})$$

and we denote by $l_{a,b}$ any object of that set. By construction, we have an obvious morphism from this set to $L_{a,c} \times L_{c,b}$ sending every $l_{a,b}$ to the pair $(l_{a,c}, l_{c,b})$. Then we define set maps $\hat{p}_{l_{a,b}}, \hat{q}_{l_{a,b}}, \hat{\pi}_{l_{a,b}}$ as in (5.3). Again using proposition 4.6.3, we get that on each $\hat{\mathcal{F}}(\mathscr{D}, l_{a,b})$ the sheaf $\hat{\mathcal{H}}_{l_{a,b}}$ defined as

$$\mathcal{E}xt^1_{\hat{\pi}_{l_{a,b}}}\Big((\hat{q}'_{l_{a,b}}, \hat{q}_{l_{a,b}})^*(\hat{\mathcal{E}}_{l_{c,b}}, \hat{\mathcal{V}}_{l_{c,b}}), (\hat{p}'_{l_{a,b}}, \hat{p}_{l_{a,b}})^*(\hat{\mathcal{E}}_{l_{a,c}}, \hat{\mathcal{V}}_{l_{a,c}})\Big)^{\vee}$$

is a locally free sheaf of rank e_c, so it makes sense to define $\hat{\mathcal{G}}(\mathscr{D}, l_{a,b}) := \mathbb{P}(\hat{\mathcal{H}}_{l_{a,b}})$ and to consider the induced projective fibration:

$$\hat{\varphi}_{l_{a,b}} : \hat{\mathcal{G}}(\mathscr{D}, l_{a,b}) \longrightarrow \hat{\mathcal{F}}(\mathscr{D}, l_{a,b})$$

with fibers isomorphic to $\mathbb{P}^{e_c-1}$. We denote by $\mathcal{O}_{l_{a,b}}(1)$ the tautological bundle of $\hat{\mathcal{G}}(\mathscr{D}, l_{a,b})$. Again the same proposition proves that there is a family $(\hat{\mathcal{E}}_{l_{a,b}}, \hat{\mathcal{V}}_{l_{a,b}})$ of coherent systems parametrized by $\hat{\mathcal{G}}(\mathscr{D}, l_{a,b})$, together with a family of non-splitting extensions:

$$0 \to (\hat{\varphi}'_{l_{a,b}}, \hat{\varphi}_{l_{a,b}})^*(\hat{p}'_{l_{a,b}}, \hat{p}_{l_{a,b}})^*(\hat{\mathcal{E}}_{l_{a,c}}, \hat{\mathcal{V}}_{l_{a,c}}) \otimes_{\hat{\mathcal{G}}(\mathscr{D}, l_{a,b})} \mathcal{O}_{l_{a,b}}(1) \to$$
$$\to (\hat{\mathcal{E}}_{l_{a,b}}, \hat{\mathcal{V}}_{l_{a,b}}) \to (\hat{\varphi}'_{l_{a,b}}, \hat{\varphi}_{l_{a,b}})^*(\hat{q}'_{l_{a,b}}, \hat{q}_{l_{a,b}})^*(\hat{\mathcal{E}}_{l_{c,b}}, \hat{\mathcal{V}}_{l_{c,b}}) \to 0$$

that is universal in the sense of corollary 4.4.4. Now let us consider the action of $G_{a,b} = G_{a,c} \times G_{c,b}$ on $\hat{\mathcal{G}}(\mathscr{D}, l_{a,c}) \times \hat{\mathcal{G}}(\mathscr{D}, l_{c,b})$, with quotient

$$(pr^{\mathcal{G}}_{l_{a,c}} \times pr^{\mathcal{G}}_{l_{c,b}}) : \hat{\mathcal{G}}(\mathscr{D}, l_{a,c}) \times \hat{\mathcal{G}}(\mathscr{D}, l_{c,b}) \twoheadrightarrow \mathcal{G}(\mathscr{D}, l_{a,c}) \times \mathcal{G}(\mathscr{D}, l_{c,b}).$$

For each $l_{a,b}$, this action restricts to an action of the same group on the subscheme $\hat{\mathcal{F}}(\mathscr{D}, l_{a,b})$ and we denote by $\mathcal{F}(\mathscr{D}, l_{a,b})$ its image via $(pr^{\mathcal{G}}_{l_{a,c}} \times pr^{\mathcal{G}}_{l_{c,b}})$. Since $\hat{\mathcal{F}}(\mathscr{D}, l_{a,b})$ is locally closed in $\hat{\mathcal{G}}(\mathscr{D}, l_{a,c}) \times \hat{\mathcal{G}}(\mathscr{D}, l_{c,b})$ and saturated with respect to the action of $G_{a,b}$, then $\mathcal{F}(\mathscr{D}, l_{a,b})$ is locally closed in $\mathcal{G}(\mathscr{D}, l_{a,c}) \times \mathcal{G}(\mathscr{D}, l_{c,b})$. For every index $l_{a,b}$ we denote by

$$pr^{\mathcal{F}}_{l_{a,b}} : \hat{\mathcal{F}}(\mathscr{D}, l_{a,b}) \twoheadrightarrow \mathcal{F}(\mathscr{D}, l_{a,b})$$

the restriction of $(pr^{\mathcal{G}}_{l_{a,c}} \times pr^{\mathcal{G}}_{l_{c,b}})$. Now the action of $G_{a,b}$ extends to an action on the projective bundle $\hat{\mathcal{G}}(\mathscr{D}, l_{a,b})$ over $\hat{\mathcal{F}}(\mathscr{D}, l_{a,b})$. Such an action is free and there is a geometric quotient:

$$pr^{\mathcal{G}}_{l_{a,b}} : \hat{\mathcal{G}}(\mathscr{D}, l_{a,b}) \twoheadrightarrow \mathcal{G}(\mathscr{D}, l_{a,b})$$

and an induced projective fibration $\varphi_{l_{a,b}} : \mathcal{G}(\mathscr{D}, l_{a,b}) \to \mathcal{F}(\mathscr{D}, l_{a,b})$ making diagram (5.6) commute. Set theoretically $(pr^{\mathcal{G}}_{l_{a,c}} \times pr^{\mathcal{G}}_{l_{c,b}})\hat{\mathcal{F}}(\mathscr{D}, a, b) = \mathcal{F}(\mathscr{D}, a, b)$. Since

$$\{\hat{\mathcal{F}}(\mathscr{D}, l_{a,b})\}_{l_{a,b} \in L_{a,b}}$$

is a disjoint locally closed covering of $\hat{\mathcal{F}}(\mathscr{D}, a, b)$, then we get that the family of schemes $\{\mathcal{F}(\mathscr{D}, l_{a,b})\}_{l_{a,b} \in L_{a,b}}$ gives a disjoint covering of the set $\mathcal{F}(\mathscr{D}, a, b)$. Now let us fix any $l_{a,b}$, any point $(\alpha_{a,c}, \alpha_{c,b})$ in $\mathcal{F}(\mathscr{D}, l_{a,b})$ and let

$$(t_{a,c}, t_{c,b}) \in (pr^{\mathcal{F}}_{l_{a,b}})^{-1}(\alpha_{a,c}, \alpha_{c,b}).$$

Then by construction we get that the fiber of $\hat{\varphi}_{l_{a,c}}$ over $(t_{a,c}, t_{c,b})$ is given by

$$\mathbb{P}(\mathrm{Ext}^1((E_{c,b}, V_{c,b}), (E_{a,c}, V_{a,c}))).$$

By commutativity of (5.6), this coincides also with the fiber of $\varphi_{l_{a,b}}$ over $(\alpha_{a,c}, \alpha_{c,b})$. Therefore this gives a canonical identification of the set underlying the scheme $\mathcal{G}(\mathscr{D}, l_{a,b})$ with a subset of $\mathcal{G}(\mathscr{D}, a, b)$. Moreover, this proves also property (vi) for the node (a, b). In addition, since $\{\mathcal{F}(\mathscr{D}, l_{a,b})\}_{l_{a,b} \in L_{a,b}}$ is a disjoint covering of $\mathcal{F}(\mathscr{D}, a, b)$, then we get that the family $\{\mathcal{G}(\mathscr{D}, l_{a,b})\}_{l_{a,b} \in L_{a,b}}$ gives a disjoint covering of the set $\mathcal{G}(\mathscr{D}, a, b)$.

□

We recall that in [BGMMN, proposition A.8], it is proved that if $GCD(n_i, d_i, k_i) = 1$, then there exists a universal family of coherent systems over $G(i) = G(\alpha_c; n_i, d_i, k_i)$. Hence by repeating all the previous proof on the level of the moduli spaces $G(i)$ instead of $\hat{G}(i)$, we get:

Corollary 5.0.6. *If $GCD(n_i, d_i, k_i) = 1$ for all $i = 1, \cdots, r$, then all the previous results hold not only at the Quot scheme level, but also at the moduli space level. In particular, the families $(\hat{\mathcal{E}}_{l_{a,b}}, \hat{\mathcal{V}}_{l_{a,b}})$ can be defined also at the moduli space level.*

The following corollary is simply a consequence of proposition 1.0.5 applied at the level of the node (a, b) with the same 2 families that we used at that step of the previous proof.

Corollary 5.0.7. *Let us suppose that for a certain node (a, b) the condition*

$$Hom((E_{c,b}, V_{c,b}), (E_{a,c}, V_{a,c})) = 0$$

is not satisfied. Then the results of the previous proposition for the node (a, b) still hold if we restrict to the subscheme

$$\mathcal{F}'(\mathscr{D}, a, b); = \{(\alpha_{a,c}, \alpha_{c,b}) \in \mathcal{F}(\mathscr{D}, a, b) \ s.t. \ Hom((E_{c,b}, V_{c,b}), (E_{a,c}, V_{a,c})) = 0\}.$$

Such a subscheme is locally closed in $\mathcal{F}(\mathscr{D}, a, b)$.

Now let us suppose that for a certain node (a, b) with $\mathrm{desc}(a, b) = c$ the following conditions hold:

- $\mathrm{Hom}((E_{c,b}, V_{c,b}), (E_{a,c}, V_{a,c})) = 0$ on all the set $\hat{\mathcal{G}}(\mathscr{D}, a, c) \times \hat{\mathcal{G}}(\mathscr{D}, c, b)$;
- the dimension of $\mathrm{Ext}^1((E_{c,b}, V_{c,b}), (E_{a,c}, V_{a,c}))$ is constant on all $\hat{\mathcal{G}}(\mathscr{D}, a, c) \times \hat{\mathcal{G}}(\mathscr{D}, c, b)$;
- the set $L_{a,c}$ consists of a single index $l_{a,c}$ and $\hat{\mathcal{G}}(\mathscr{D}, a, c) = \hat{\mathcal{G}}(\mathscr{D}, l_{a,c})$ is integral;
- the same condition for the node $(c, b; -)$.

Then by lemma 4.6.1 we can choose the set $L_{a,b}$ so that it consists of a single index $l_{a,b}$ and we get a projective bundle $\hat{\mathcal{G}}(\mathscr{D}, a, b) = \hat{\mathcal{G}}(\mathscr{D}, l_{a,b})$ over

$$\hat{\mathcal{F}}(\mathscr{D}, a, b) = \hat{\mathcal{G}}(\mathscr{D}, a, c) \times \hat{\mathcal{G}}(\mathscr{D}, c, b).$$

Since the fiber is a projective space and the base is irreducible, we get that $\hat{\mathcal{G}}(\mathscr{D}, a, b)$ is again irreducible. Therefore, by induction we get the following result.

Corollary 5.0.8. *Let us suppose that the following 2 conditions hold:*

- *for all internal nodes (a, b) with $desc(a, b) = c$ and for all $(\alpha_{a,c}, \alpha_{c,b}) \in \mathcal{G}(\mathscr{D}, a, c) \times \mathcal{G}(\mathscr{D}, c, b)$ we have that*

  $$Hom((E_{c,b}, V_{c,b}), (E_{a,c}, V_{a,c})) = 0$$

  and $Ext^1((E_{c,b}, V_{c,b}), (E_{a,c}, V_{a,c}))$ has constant dimension e_c on $\mathcal{G}(\mathscr{D}, a, c) \times \mathcal{G}(\mathscr{D}, c, b)$ (this means that there exists only one "interesting" set of data $\mathscr{D}$);

- *for all leaves $(i-1,i)$ of the tree the schemes $G(\alpha_c; n_i, d_i, k_i)$ are irreducible.*

Then for all internal nodes (a,b) the set $\mathcal{G}(\mathscr{D}, a, b)$ has a natural scheme structure of projective bundle (with fibers isomorphic to $\mathbb{P}^{e_c-1}$) over $\mathcal{G}(\mathscr{D}, a, c) \times \mathcal{G}(\mathscr{D}, c, b)$.

Chapter 6

Objects with unique Jordan-Hölder filtration of length 3 and 4

In this chapter we describe how we can parametrize all the (E, V)'s that have a unique α_c-Jordan-Hölder filtration of length r equal to 3 or 4. We will not give a complete description of all possible cases, but we will focus only on those cases that will be needed in order to compute the Hodge-Deligne polynomials of $G(\alpha; n, d, k)$ for $n = 3, 4$, $k = 1$ and any d. We get complete results when $r = 3$; when $r = 4$ we get complete results only in the case when the second and third object of the graded are

6.1 Unique Jordan-Hölder filtration of length 3

Let us fix any triple (n, d, k), a critical value α_c for that triple and let (E, V) be any α_c-semistable coherent system of type (n, d, k). Let us suppose that at α_c the graded of a coherent system (E, V) is $\oplus_{i=1}^{3}(Q_i, W_i)$ and that (E, V) has a unique α_c-Jordan-Hölder filtration (therefore, that filtration coincides with the α_c-canonical filtration by lemma 2.1.2). We want to parametrize all the (E, V)'s of that type, having fixed the graded (and also its order, since the filtration is unique).

If the α_c-JHF is unique, then the only subobjects of (E, V) with α_c-slope equal to $\mu_{\alpha_c}(E, V)$ will be (Q_1, W_1) and an extension (E_2, V_2) of (Q_2, W_2) by (Q_1, W_1); the quotient $(E, V)/(E_2, V_2)$ will be isomorphic to (Q_3, W_3). Therefore, given any (E, V) with unique α_c-Jordan-Hölder filtration, we have that (E, V) belongs to $G^+(\alpha_c; n, d, k)$ if and only if the following numerical conditions are satisfied:

$$\frac{k_1}{n_1} < \frac{k}{n}, \quad \frac{k_1 + k_2}{n_1 + n_2} < \frac{k}{n}. \tag{6.1}$$

Analogously, given any (E, V) with unique α_c-Jordan-Hölder filtration, we have that (E, V) belongs to $G^-(\alpha_c; n, d, k)$ if and only if the following numerical conditions are satisfied:

$$\frac{k_1}{n_1} > \frac{k}{n}, \quad \frac{k_1 + k_2}{n_1 + n_2} > \frac{k}{n}. \tag{6.2}$$

In both cases, we need a way of parametrizing all the (E, V)'s with unique filtration, having fixed the graded. According to different relations between the various objects of the graded, we will need one of the 4 descriptions given below. Roughly speaking, the first 2 descriptions amount to considering (E, V) as obtained via a tree of type A, while the last 2 ones are obtained by considering a tree of type B as follows:

$$\begin{tikzcd}[column sep=small]
\boxed{A} & & (0,3) \arrow[dl, dash] \arrow[dr, dash] & \\
& (0,2) \arrow[dl, dash] \arrow[dr, dash] & & (2,3) \\
(0,1) & & (1,2) &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
\boxed{B} & (0,3) \arrow[dl, dash] \arrow[dr, dash] & & \\
(0,1) & & (1,3) \arrow[dl, dash] \arrow[dr, dash] & \\
& (1,2) & & (2,3).
\end{tikzcd}$$

Lemma 6.1.1. *Let us fix any triple $(Q_i, W_i)_{i=1,2,3} \in \prod_{i=1}^{3} G_i$ with numerical conditions (6.1), respectively (6.2), and let us suppose that $(Q_1, W_1) \not\simeq (Q_2, W_2) \not\simeq (Q_3, W_3)$. Then the (E, V)'s that belong to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, that have unique Jordan-Hölder filtration at α_c and graded $\oplus_{i=1}^{3}(Q_i, W_i)$ are parametrized by pairs $([\mu], [\nu])$ where:*

- $[\mu] \in \mathbb{P}(Ext^1((Q_2, W_2), (Q_1, W_1)))$ *and μ has a representative of the form*

$$0 \to (Q_1, W_1) \to (E_2, V_2) \to (Q_2, W_2) \to 0;$$

- $[\nu] \in \overline{M}([\mu]) := \mathbb{P}(Ext^1((Q_3, W_3), (E_2, V_2))) \smallsetminus \mathbb{P}(Ext^1((Q_3, W_3), (Q_1, W_1)))$.

Proof. Given the graded, a filtration of (E, V) is

$$0 \subset (E_1, V_1) := (Q_1, W_1) \subset (E_2, V_2) \subset (E, V)$$

where we have exact sequences:

$$0 \to (Q_1, W_1) \xrightarrow{\sigma} (E_2, V_2) \xrightarrow{\kappa} (Q_2, W_2) \to 0, \tag{6.3}$$

$$0 \to (E_2, V_2) \xrightarrow{\varepsilon} (E, V) \xrightarrow{\delta} (Q_3, W_3) \to 0. \tag{6.4}$$

Now by proposition 2.2.1, (E, V) has a unique α_c-Jordan-Hölder filtration if and only if all the sequences

$$0 \to (Q_k, W_k) \to (E_{k+1}, V_{k+1})/(E_{k-1}, V_{k-1}) \to (Q_{k+1}, W_{k+1}) \to 0$$

for $k = 1, 2$ are non-split. This amounts to imposing that both (6.3) and

$$0 \to (Q_2, W_2) \to (E, V)/(Q_1, W_1) \to (Q_3, W_3) \to 0 \tag{6.5}$$

are non-split. Since $(Q_1, W_1) \not\cong (Q_2, W_2)$ and since both objects are α_c-stable, then for all extensions (6.3) that are non-split, $\mathrm{Aut}(E_2, V_2) = \mathbb{C}^*$. Moreover, the objects of the form (E_2, V_2) in (6.4) are parametrized by $\mathbb{P}(\mathrm{Ext}^1((Q_2, W_2), (Q_1, W_1)))$.

If we apply the functor $\mathrm{Hom}((Q_3, W_3), -)$ to (6.3), we get a long exact sequence:

$$\begin{aligned} \cdots \to \mathrm{Hom}\left((Q_3, W_3), (Q_2, W_2)\right) \to \mathrm{Ext}^1\left((Q_3, W_3), (Q_1, W_1)\right) \xrightarrow{\overline{\sigma}} \\ \xrightarrow{\overline{\sigma}} \mathrm{Ext}^1\left((Q_3, W_3), (E_2, V_2)\right) \xrightarrow{\overline{\kappa}} \mathrm{Ext}^1\left((Q_3, W_3), (Q_2, W_2)\right) \to \cdots \end{aligned} \tag{6.6}$$

If we denote by ν the class of (6.4) and by ν' the class of (6.5), then we get that $\nu' = \overline{\kappa}(\nu)$. Then $\nu' \neq 0$ if and only if ν is not in the image of $\overline{\sigma}$. By hypothesis, we have that $\overline{\sigma}$ is injective, so it makes sense to consider

$$M([\mu]) := \mathrm{Ext}^1\left((Q_3, W_3), (E_2, V_2)\right) \smallsetminus \mathrm{Ext}^1\left((Q_3, W_3), (Q_1, W_1)\right).$$

Now given any sequence of the form (6.4), we have that $\mathrm{Aut}(E_2, V_2) = \mathrm{Aut}(Q_3, W_3) = \mathbb{C}^*$, so the (E, V)'s we are interested in are parametrized by equivalence classes of points in $M([\mu])$, modulo the action of $\mathbb{C}^*$. □

Proposition 6.1.2. *Let us fix any triple (n, d, k), a critical value α_c for it and any triple $(n_i, d_i, k_i)_{i=1,2,3}$ compatible with $(\alpha_c; n, d, k)$, i.e. such that*

$$\sum_{i=1}^{3} n_i = n,\ \sum_{i=1}^{3} d_i = d,\ \sum_{i=1}^{3} k_i = k,\ \mu_{\alpha_c}(n_i, d_i, k_i) = \mu_{\alpha_c}(n, d, k) \quad \forall\, i = 1, 2, 3. \tag{6.7}$$

Let us assume that conditions (6.1), respectively (6.2), are satisfied. Moreover, let us suppose that for every triple of points $(Q_i, W_i)_{i=1,2,3} \in \prod_{i=1}^{3} G_i$ we have:

$$Hom((Q_3, W_3), (Q_2, W_2)) = 0 = Hom((Q_2, W_2), (Q_1, W_1))$$

(in particular, this holds if $(n_1, k_1) \neq (n_2, k_2) \neq (n_3, k_3)$). Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in $G^-(\alpha_c; n, d, k)$, with unique *α_c-JHF and graded in $\prod_{i=1}^{3} G_i$. Then there exists a finite family $\{R_{a,b,c;i,j}\}$ of schemes for $(a, b, c) \in \mathbb{N}^2 \times \mathbb{N}_0$, $c \leq b$ and i, j varying in finite sets (for a, b, c fixed), together with injective morphisms to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Every $R_{a,b,c;i,j}$ comes with a sequence of 2 morphisms:*

$$\begin{aligned} \varphi_{a,b,c;i,j} : R_{a,b,c;i,j} &\longrightarrow U_{a,b,c;i,j} \subset R_{a;i} \times G_3, \\ \varphi_{a;i} : R_{a;i} &\longrightarrow U_{a;i} \subset G_1 \times G_2, \end{aligned}$$

where:

- *$\varphi_{a,b,c;i,j}$ has fibers isomorphic to $\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}$, and $\varphi_{a;i}$ has fibers isomorphic to $\mathbb{P}^{a-1}$;*

- $\{U_{a;i}\}_i$ *is a finite disjoint locally closed covering of*

$$U_a := \{((Q_1, W_1), (Q_2, W_2)) \in G_1 \times G_2 \ s.t. \ dim \ Ext^1((Q_2, W_2), (Q_1, W_1)) = a\};$$

  U_a *is a locally closed subscheme of* $G_1 \times G_2$ *and so are all the* $U_{a;i}$*'s.*

- $\{U_{a,b,c;i,j}\}_j$ *is a finite disjoint locally closed covering of*

$$U_{a,b,c;i} := \{((E_2, V_2), (Q_3, W_3)) \in R_{a;i} \times G_3 \ s.t.$$
$$dim \ Ext^1((Q_3, W_3), (E_2, V_2)) = b, \quad dim \ Ext^1((Q_3, W_3), \widetilde{\varphi}_{a;i}(E_2, V_2)) = c\}$$

  where $\widetilde{\varphi}_{a;i}$ *is the composition of* $\varphi_{a;i}$ *with the projection to* G_1*.* $U_{a,b,c;i}$ *is locally closed in* $R_{a;i} \times G_3$ *and so are all the* $U_{a,b,c;i,j}$*.*

If $(n_1, k_1) = (n_2, k_2)$*, then all the previous results still hold on* $(G_1 \times G_2 \smallsetminus \Delta_{12}) \times G_3$ *instead of* $G_1 \times G_2 \times G_3$*.*

Proof. If $(n_1, k_1) = (n_2, k_2)$, then the last condition of (6.7) implies that $d_1 = d_2$, therefore $G_1 = G_2$. For simplicity, we do all the proof of the proposition in the case when this does not happen. If $(n_1, k_1) = (n_2, k_2)$, then the proof is exactly the same by restricting to $G_1 \times G_2 \smallsetminus \Delta_{12}$ whenever it is necessary.

Let us fix any triple $(a, b, c) \in \mathbb{N}^2 \times \mathbb{N}_0$. Then proposition 5.0.5 for $r = 3$ applied to a tree of type A and to data $\mathscr{D} = \{a, b\}$ gives projective bundles $\{\hat{\varphi}_{a;i} : \hat{R}_{a;i} \to \hat{U}_{a;i}\}_i$ and $\{\hat{\varphi}_{a,b;i,j} : \hat{R}_{a,b;i,j} \to \hat{U}_{a,b;i,j}\}_{i,j}$, where

- $\{\hat{U}_{a;i}\}_{a;i}$ is a disjoint locally closed covering of $\hat{G}_1 \times \hat{G}_2$; we denote by $\hat{p}_{12}$ and $\hat{q}_{12}$ the projections from $\hat{U}_{a;i}$ to $\hat{G}_1$ and $\hat{G}_2$ respectively;
- $\{\hat{U}_{a,b;i,j}\}_{b;j}$ a disjoint locally closed covering of $\hat{R}_{a;i} \times \hat{G}_3$; we denote by $\hat{p}_{12,3}$ and $\hat{q}_{12,3}$ the projections from $\hat{U}_{a,b;i}$ to $\hat{R}_{a;i}$ and $\hat{G}_3$ respectively;
- the fibers of $\hat{\varphi}_{a;i}$ are isomorphic to $\mathbb{P}^{a-1}$;
- the fibers of $\hat{\varphi}_{a,b;i,j}$ are isomorphic to $\mathbb{P}^{b-1}$.

Moreover, we have universal families of extensions over $\hat{R}_{a;i}$ and $\hat{R}_{a,b;i,j}$ respectively:

$$0 \to (\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \otimes_{\hat{R}_{a;i}} \mathcal{O}_{a;i}(1) \to$$
$$\to (\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i}) \to (\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{q}'_{12}, \hat{q}_{12})^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2) \to 0, \tag{6.8}$$

$$0 \to (\hat{\varphi}'_{a,b;i,j}, \hat{\varphi}_{a,b;i,j})^*(\hat{p}'_{12,3}, \hat{p}_{12,3})^*(\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i}) \otimes_{\hat{R}_{a,b;i,j}} \mathcal{O}_{a,b;i,j}(1) \to$$

$$\to (\hat{\mathcal{E}}_{a,b;i,j},\hat{\mathcal{V}}_{a,b;i,j}) \to (\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{q}'_{12,3},\hat{q}_{12,3})^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3) \to 0. \tag{6.9}$$

Now for every $0 \leq c \leq b$ we define:

$$\hat{U}_{a,b,c;i,j} := \{t \in \hat{U}_{a,b;i,j} \text{ s.t. } \dim \operatorname{Ext}^1((\hat{q}'_{12,3},\hat{q}_{12,3})^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)_t,$$
$$(\hat{p}'_{12,3},\hat{p}_{12,3})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a,i})^*(\hat{p}'_{12},\hat{p}_{12})^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)_t) = c\}$$

By proposition 1.0.5 $\hat{U}_{a,b,c;i,j}$ is locally closed in $\hat{U}_{a,b;i,j}$ and therefore also in $\hat{R}_{a;i} \times \hat{G}_3$. Let us apply the functor

$$\mathcal{H}om_{\hat{\pi}_{a,b;i,j}}((\hat{q}'_{12,3},\hat{q}_{12,3})^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3),-)$$

to the pullback of (6.8) via $\hat{p}_{12,3}$. Then we get a morphism of the form

$$\mathcal{H}om_{\hat{\pi}_{a,b;i,j}}((\hat{q}'_{12,3},\hat{q}_{12,3})^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_3),(\hat{p}'_{12,3},\hat{p}_{12,3})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a,i})^*(\hat{p}'_1,\hat{p}_1)^*(\hat{\mathcal{Q}}_1\hat{\mathcal{W}}_1)) \to$$
$$\to \mathcal{H}om_{\hat{\pi}_{a,b;i,j}}((\hat{q}'_{12,3},\hat{q}_{12,3})^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_3),(\hat{p}'_{12,3},\hat{p}_{12,3})^*(\hat{\mathcal{E}}_{a;i},\hat{\mathcal{V}}_{a;i})).$$

By base change and the previous lemma, this morphism is injective, therefore we can rewrite it as

$$\hat{F}_{a,b,c;i,j} \hookrightarrow \hat{E}_{a,b,c;i,j}$$

Therefore, we can consider the subbundle

$$\hat{Q}_{a,b,c;i,j} := \mathbb{P}((\hat{F}_{a,b,c;i,j})^\vee) \subset \mathbb{P}((\hat{E}_{a,b,c;i,j})^\vee) = \hat{R}_{a,b;i,j}|_{\hat{U}_{a,b,c;i,j}}.$$

Then we define $\hat{R}_{a,b,c;i,j} := \hat{R}_{a,b;i,j}|_{\hat{U}_{a,b,c;i,j}} \smallsetminus \hat{Q}_{a,b,c;i,j}$. This comes with a morphism $\hat{\varphi}_{a,b,c;i,j}$ to $\hat{U}_{a,b,c;i,j}$ (given by the restriction of $\hat{\varphi}_{a,b;i,j}$) with fibers isomorphic to $\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}$.

By pullback of (6.8) from $\hat{R}_{a;i}$ and by restriction of (6.9) from $\hat{R}_{a,b;i,j}$ we get 2 families of extensions of coherent systems parametrized by $\hat{R}_{a,b,c;i,j}$. If we fix any point $r \in \hat{R}_{a,b,c;i,j}$ and we write $(E,V) := (\hat{\mathcal{E}}_{a,b;i,j},\hat{\mathcal{V}}_{a,b;i,j})_r$, then the 2 extensions over $\hat{R}_{a,b,c;i,j}$ give rise to 2 exact sequences of the form

$$0 \to (Q_1,W_1) \to (E_2,V_2) \to (Q_2,W_2) \to 0,$$
$$0 \to (E_2,V_2) \to (E,V) \to (Q_3,W_3) \to 0.$$

Now let us suppose that the numerical conditions (6.1) are satisfied; then the previous lemma proves that for every point $r \in \hat{R}_{a,b,c;i,j}$ the coherent system

$$(E,V) := (\hat{\mathcal{E}}_{a,b;i,j},\hat{\mathcal{V}}_{a,b;i,j})_r$$

has a unique filtration and it belongs to $G^+(\alpha_c;n,d,k)$. Therefore, by using the universal property of the moduli space $G(\alpha_c^+;n,d,k)$, we get for every $(a,b,c;i,j)$ an induced morphism

$$\hat{\eta}_{a,b,c;i,j} : \hat{R}_{a,b,c;i,j} \longrightarrow G(\alpha_c^+; n, d, k)$$

with values in $G^+(\alpha_c; n, d, k)$. Such a morphism is invariant under the free action of $G_{13} = PGL(N_1) \times PGL(N_2) \times PGL(N_3)$, so we get an induced morphism

$$\eta_{a,b,c;i,j} : R_{a,b,c;i,j} \longrightarrow G(\alpha_c^+; n, d, k).$$

The previous lemma proves that every such morphism is injective and that the images of all the $\eta_{a,b,c;i,j}$'s form a locally closed disjoint covering of $G' \subset G^+(\alpha_c; n, d, k)$. An analogous conclusion holds if we assume conditions (6.2). □

Lemma 6.1.3. *Let us fix any triple $(Q_i, W_i)_{i=1,2,3} \in \prod_{i=1}^3 G_i$ and let us suppose that $(Q_1, W_1) \simeq (Q_2, W_2) \not\simeq (Q_3, W_3)$ and that*

$$Ext^2\left((Q_3, W_3), (Q_1, W_1)\right) = 0$$

(with a little abuse of notation we will simply write (Q_1, W_1) also for (Q_2, W_2)). Let us denote by μ any class of a non-split extension of the form

$$0 \to (Q_1, W_1) \xrightarrow{\sigma} (E_2, V_2) \xrightarrow{\kappa} (Q_1, W_1) \to 0 \tag{6.10}$$

and by ν any class of a non-split extension of the form

$$0 \to (E_2, V_2) \xrightarrow{\varepsilon} (E, V) \xrightarrow{\delta} (Q_3, W_3) \to 0. \tag{6.11}$$

Having fixed $[\mu] \in \mathbb{P}(Ext^1((Q_1, W_1), (Q_1, W_1))$, let us consider the space $Ext^1((Q_3, W_3), (E_2, V_2))$ and let us consider the action of $\mathbb{C} \times \mathbb{C}^*$ on it given as follows. For every pair of scalars (ξ, τ) and for every class of extension ν with representative of the form (6.11), we set $(\xi, \tau) \cdot \nu := \nu'$, where ν' is represented by

$$0 \to (E_2, V_2) \xrightarrow{\varepsilon'} (E, V) \xrightarrow{\delta} (Q_3, W_3) \to 0,$$

where $\varepsilon' := \varepsilon \circ (\xi \cdot \sigma \circ \kappa + \tau \cdot id_{(E_2, V_2)})$. Let us write

$$M([\mu]) := Ext^1((Q_3, W_3), (E_2, V_2)) \smallsetminus Ext^1((Q_3, W_3), (Q_1, W_1));$$

then the previous action sends $M([\mu])$ to itself, so it makes sense to consider $\overline{M}([\mu]) := M([\mu])/(\mathbb{C} \times \mathbb{C}^*)$. Then the (E, V)'s with unique Jordan-Hölder filtration at α_c and graded $\oplus_{i=1}^3 (Q_i, W_i)$ are parametrized by pairs $([\mu], [\nu])$ where:

- *$[\mu] \in \mathbb{P}(Ext^1((Q_1, W_1), (Q_1, W_1)))$ and μ has a representative of the form (6.10);*
- *$[\nu]$ is any object of $\overline{M}([\mu])$.*

Moreover, $\overline{M}([\mu]) \simeq \mathbb{C}^{b-1} \times \mathbb{P}^{b-1}$, where $b = dim\ Ext^1((Q_3, W_3), (Q_1, W_1))$.

Proof. As in the proof of lemma 6.1.1, we can consider exact sequences (6.3), (6.4) and (6.5), with (Q_2, W_2) replaced by (Q_1, W_1). Then (E, V) has a unique Jordan-Hölder filtration at α_c if and only if both (6.3) and (6.5) are non-split. This implies that $[\mu]$ varies in $\mathbb{P}(\mathrm{Ext}^1((Q_1, W_1), (Q_1, W_1)))$. As in the already cited lemma, let us consider the exact sequence (6.6). Since $\mathrm{Hom}((Q_3, W_3), (Q_2, W_2)) = 0 = \mathrm{Ext}^2((Q_3, W_3), (Q_1, W_1))$, we get a short exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1\left((Q_3, W_3), (Q_1, W_1)\right) \xrightarrow{\overline{\sigma}} \mathrm{Ext}^1\left((Q_3, W_3), (E_2, V_2)\right) \xrightarrow{\overline{\kappa}} \\ \xrightarrow{\overline{\kappa}} \mathrm{Ext}^1\left((Q_3, W_3), (Q_1, W_1)\right) \to 0. \end{aligned} \tag{6.12}$$

We need to consider all the elements $\nu \in \mathrm{Ext}^1((Q_3, W_3), (E_2, V_2))$ such that $\overline{\kappa}(\nu) \neq 0$, i.e. such that they are not in the image of $\overline{\sigma}$. So we need to consider the space

$$M([\mu]) := \mathrm{Ext}^1\left((Q_3, W_3), (E_2, V_2)\right) \smallsetminus \mathrm{Ext}^1\left((Q_3, W_3), (Q_1, W_1)\right).$$

From the previous exact sequence, we get a (non-canonical) isomorphism

$$M([\mu]) \simeq \mathrm{Ext}^1\left((Q_3, W_3), (Q_1, W_1)\right) \times \left(\mathrm{Ext}^1\left((Q_3, W_3), (Q_1, W_1)\right) \smallsetminus \{0\}\right). \tag{6.13}$$

By (6.10) we have that $\mathrm{Aut}(E_2, V_2) = \mathbb{C} \times \mathbb{C}^*$; to be more precise, given any pair of scalars (ξ, τ) the corresponding automorphism of (E_2, V_2) is given by $\xi \cdot \sigma \circ \kappa + \tau \cdot \mathrm{id}_{(E_2, V_2)}$. Therefore, there is an induced action of $\mathbb{C} \times \mathbb{C}^*$ on the space $\mathrm{Ext}^1((Q_3, W_3), (E_2, V_2))$ given as follows: for every extension (6.11), the image of such an extension via a pair (ξ, τ) is given by

$$0 \to (E_2, V_2) \xrightarrow{\varepsilon'} (E, V) \xrightarrow{\delta} (Q_3, W_3) \to 0$$

where $\varepsilon' := \varepsilon \circ (\xi \cdot \sigma \circ \kappa + \tau \cdot \mathrm{id}_{(E_2, V_2)})$. Now let us suppose that an extension ν represented by (6.11) is in the image of $\overline{\sigma}$. This is equivalent to say that we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & (Q_1, W_1) \arrow[r, "\widetilde{\varepsilon}"] \arrow[d, "\sigma"] \arrow[dr, phantom, "\curvearrowright"] & (E', V') \arrow[r, "\widetilde{\delta}"] \arrow[d, "\widetilde{\sigma}"] \arrow[dr, phantom, "\curvearrowright"] & (Q_3, W_3) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & (E_2, V_2) \arrow[r, "\varepsilon"] & (E, V) \arrow[r, "\delta"] & (Q_3, W_3) \arrow[r] & 0. \end{tikzcd}$$

Then

$$\varepsilon' \circ \sigma = \varepsilon \circ (\xi \cdot \sigma \circ \kappa + \tau \cdot \mathrm{id}_{(E_2, V_2)}) \circ \sigma =$$

$$= \xi \cdot \varepsilon \circ \sigma \circ \kappa \circ \sigma + \tau \cdot \varepsilon \circ \sigma = \tau \cdot \varepsilon \circ \sigma.$$

Therefore, we can induce a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (Q_1, W_1) \arrow[r, "\tau\cdot\widetilde{\varepsilon}"] \arrow[d, "\sigma"] \arrow[dr, phantom, "\curvearrowright"] & (E', V') \arrow[r, "\widetilde{\delta}"] \arrow[d, "\widetilde{\sigma}"] \arrow[dr, phantom, "\curvearrowright"] & (Q_3, W_3) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (E_2, V_2) \arrow[r, "\varepsilon'"] & (E, V) \arrow[r, "\delta"] & (Q_3, W_3) \arrow[r] & 0
\end{tikzcd}$$

where we set $\widetilde{\varepsilon}' := \tau \cdot \widetilde{\varepsilon}$. Therefore, $\mathbb{C} \times \mathbb{C}^*$ acts on $\mathrm{Ext}^1((Q_3, W_3), (E_2, V_2))$ by fixing the image of $\overline{\sigma}$, so we can consider the restricted action of that group on the set $M([\mu]) = \mathrm{Ext}^1((Q_3, W_3), (E_2, V_2)) \smallsetminus \mathrm{Im}(\overline{\sigma})$.

Now $\mathrm{Aut}(Q_3, W_3) = \mathbb{C}^*$ since (Q_3, W_3) is α_c-stable; moreover (E_2, V_2) and (Q_3, W_3) are not isomorphic (the first one is strictly α_c-semistable and the second one is α_c-stable), so the action of $\mathrm{Aut}(E_2, V_2) \times \mathrm{Aut}(Q_3, W_3)$ on $M([\mu])$ is simply given by the previous action. Then the (E, V)'s we are interested in are parametrized by equivalence classes of objects of $\overline{M}([\mu]) = M([\mu])/(\mathbb{C} \times \mathbb{C}^*)$. Under the isomorphism (6.13), the previous action on $M([\mu])$ is given by

$$(\xi, \tau) \cdot (\nu_1, \nu_2) := (\tau \cdot \nu_1 + \xi \cdot \nu_2, \tau \cdot \nu_2).$$

So it is easy to see that the quotient $\overline{M}([\mu])$ is isomorphic to $\mathbb{C}^{b-1} \times \mathbb{P}^{b-1}$ and we conclude. □

Proposition 6.1.4. *Let us fix any triple (n, d, k), a critical value α_c for it and any triple $(n_i, d_i, k_i)_{i=1,2,3}$ compatible with $(\alpha_c; n, d, k)$. Let us assume that conditions (6.1), respectively (6.2), are satisfied. Moreover, let us suppose that $(n_1, k_1) = (n_2, k_2)$ (so automatically $(n_i, k_i) \neq (n_3, k_3)$ for $i = 1, 2$) and that for every pair of points $((Q_1, W_1), (Q_3, W_3)) \in G_1 \times G_3$ we have:*

$$Ext^2((Q_3, W_3), (Q_1, W_1)) = 0.$$

Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in $G^-(\alpha_c; n, d, k)$, that have unique Jordan-Hölder filtration at α_c and graded $(Q_1, W_1) \oplus (Q_1, W_1) \oplus (Q_3, W_3)$. Then there exists a finite family $\{R_{a,b;i,j}\}$ of schemes for $(a, b) \in \mathbb{N}^2$ and i, j varying in finite sets (for a, b fixed), together with injective morphisms to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Every $R_{a,b;i,j}$ comes with a sequence of 2 morphisms

$$\varphi_{a,b;i,j} : R_{a,b;i,j} \longrightarrow U_{a,b:i,j} \subset R_{a;i} \times G_3,$$
$$\varphi_{a;i} : R_{a;i} \longrightarrow U_{a;i} \subset G_1 = G_2,$$

where:

- $\varphi_{a,b;i,j}$ *has fibers isomorphic to* $\mathbb{C}^{b-1}\times\mathbb{P}^{b-1}$ *and* $\varphi_{a;i}$ *has fibers isomorphic to* $\mathbb{P}^{a-1}$;
- $\{U_{a;i}\}_i$ *is a finite disjoint locally closed covering of*

$$U_a := \{(Q_1,W_1)\in G_1 \text{ s.t. } \dim Ext^1((Q_1,W_1),(Q_1,W_1)) = a\};$$

every U_a *is a locally closed subscheme of* G_1 *and so are all the* $U_{a;i}$*'s;*

- $\{U_{a,b;i,j}\}_j$ *is a finite disjoint locally closed covering of*

$$U_{a,b;i} := \{((E_2,V_2),(Q_3,W_3))\in R_{a;i}\times G_3 \text{ s.t. } \dim Ext^1((Q_3,W_3),\varphi_{a;i}(E_2,V_2)) = b\}.$$

$U_{a,b;i}$ *is locally closed in* $R_{a;i}\times G_3$ *and so are the* $U_{a,b;i,j}$*'s.*

Proof. Let us fix any pair $(a,b)\in\mathbb{N}^2$ and let us denote by $(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)$ the universal family over $\hat{G}_1=\hat{G}_2$. Since the objects of this scheme are α_c-stable, we get that for each point $t\in\hat{G}_1$ we have

$$\dim \mathrm{Hom}((\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)_t,(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)_t) = 1.$$

Let us apply lemma 4.6.1 on $T=\hat{G}_1$ with the two families of coherent systems both coinciding with $(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)$. Then we get that the set

$$\hat{U}_a := \{t\in\hat{G}_1 \text{ s.t. } \dim \mathrm{Ext}^1((\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)_t,(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)_t) = a\}$$

is a locally closed subscheme of $\hat{G}_1$ and there exists a finite disjoint covering of $\hat{U}_a$ by locally closed subschemes $\hat{U}_{a;i}$. Moreover, by proposition 4.6.2 we get that on each $\hat{U}_{a;i}$ there is a locally free sheaf

$$\hat{\mathcal{H}}_{a;i} := \mathcal{E}xt^1_{\pi_{\hat{U}_{a;i}}}((\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)|_{\hat{U}_{a;i}},(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)|_{\hat{U}_{a;i}})^\vee$$

and a projective bundle $\hat{\varphi}_{a;i}:\hat{R}_{a;i}:=\mathbb{P}(\hat{\mathcal{H}}_{a;i})\to\hat{U}_{a;i}\subset\hat{G}_1$, together with a family of non-split extensions $\{e_r\}_{r\in\hat{R}_{a;i}}$ parametrized by $\hat{R}_{a;i}$ of $(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)$ by $(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)\otimes_{\hat{R}_{a;i}}\mathcal{O}_{\hat{R}_{a;i}}(1)$. Such a family of extensions is universal on the category of reduced $\hat{U}_{a;i}$-schemes. Now by definition of family of extensions, for each i there is an open covering $\{\hat{R}^k_{a;i}\}_{k\in K}$ of $\hat{R}_{a;i}$; for each k there is an extension

$$0\to((\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)\otimes_{\hat{R}_{a;i}}\mathcal{O}_{\hat{R}_{a;i}}(1))|_{\hat{R}^k_{a;i}}\xrightarrow{\sigma^k_{a;i}}(\hat{\mathcal{E}}^k_{a;i},\hat{\mathcal{V}}^k_{a;i})\xrightarrow{\kappa^k_{a;i}}(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)|_{\hat{R}^k_{a;i}}\to 0 \tag{6.14}$$

over $\hat{R}^k_{a;i}$ such that e_r is the restriction of (6.14) for each $r\in\hat{R}^k_{a;i}$. Since $\hat{R}_{a;i}$ is noetherian, then we can assume that K is finite and we denote its elements by $\{k_1<\cdots<k_r\}$. Then for each set $k_\bullet=\{k'_1<\cdots<k'_s\}\subset K$ we define the locally closed subscheme of $\hat{R}_{a;i}$:

$$\hat{R}^{k_\bullet}_{a;i} := (\hat{R}^{k'_1}_{a;i} \cap \cdots \cap \hat{R}^{k'_s}_{a;i}) \smallsetminus (\hat{R}^{k'_{s+1}}_{a;i} \cup \cdots \cup \hat{R}^{k'_r}_{a;i}),$$

where $\{k'_{s+1} < \cdots < k'_r\}$ is the complement of $k_\bullet$ in K. Since K is finite, we get that these schemes form a finite locally closed disjoint covering of $\hat{R}_{a;i}$. For each $k_\bullet$ we consider the embedding of $\hat{R}^{k_\bullet}_{a;i}$ in $\hat{PR}^{k'_1}_{a;i}$ and we pullback (6.14) using that morphism. So for each $k_\bullet$ we get an extension

$$0 \to ((\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \otimes_{\hat{R}_{a;i}} \mathcal{O}_{\hat{R}_{a;i}}(1))|_{\hat{R}^{k_\bullet}_{a;i}} \xrightarrow{\sigma^{k_\bullet}_{a;i}} (\hat{\mathcal{E}}^{k_\bullet}_{a;i}, \hat{\mathcal{V}}^{k_\bullet}_{a;i}) \xrightarrow{\kappa^{\bullet}_{a;i}} (\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)|_{\hat{R}^{k_\bullet}_{a;i}} \to 0 \tag{6.15}$$

over $\hat{R}^{k_\bullet}_{a;i}$. Now for each pair of indices $(i, k_\bullet)$ we consider the scheme $T' := \hat{R}^{k_\bullet}_{a;i} \times \hat{G}_3$, together with the projections $\hat{p}_1$ and $\hat{p}_2$ to the 2 factors. Then we define $\hat{U}^{k_\bullet}_{a,b;i}$ as the subscheme of T' defined by

$$\{t' \in T' \text{ s.t. } \dim \operatorname{Ext}^1((\hat{p}'_2, \hat{p}_2)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)_{t'}, (\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i}, \hat{\mathcal{V}}^{k_\bullet}_{a;i})_{t'} = b\}.$$

By proposition 1.0.5 we get that $\hat{U}^{k_\bullet}_{a,b;i}$ is locally closed in $\hat{R}^{k_\bullet}_{a;i} \times \hat{G}_3$. By construction, we have that each $\hat{R}^{k_\bullet}_{a;i}$ is locally closed in $\hat{R}_{a;i}$, so each $\hat{U}^{k_\bullet}_{a,b;i}$ is locally closed in $\hat{R}_{a;i} \times \hat{G}_3$. Then let us consider the scheme $\hat{U}_{a,b;i}$ defined as the subscheme of $\hat{R}_{a;i} \times \hat{G}_3$ covered by all the schemes of the form $\hat{U}^{k_\bullet}_{a,b;i}$. This is a disjoint covering of such a scheme by locally closed subschemes of $\hat{R}_{a;i} \times \hat{G}_3$. Therefore, this gives a disjoint locally closed covering of $\hat{U}_{a,b;i}$.

By construction, for each $k_\bullet$ and for each $t' \in \hat{U}^{k_\bullet}_{a,b;i}$, we have that both $(\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i}, \hat{\mathcal{V}}^{k_\bullet}_{a;i})_{t'}$ and $(\hat{p}'_2, \hat{p}_2)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)_{t'}$ are α_c-semistable. Moreover, since $(n_1, k_1) \neq (n_3, k_3)$ by hypothesis, then we have that:

$$\operatorname{Hom}((\hat{p}'_2, \hat{p}_2)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)_{t'}, (\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i}, \hat{\mathcal{V}}^{k_\bullet}_{a;i})_{t'}) = 0$$

for all $t' \in \hat{U}^{k_\bullet}_{a,b;i}$. Then we can apply proposition 4.6.3 for such a scheme and we get that there is a finite disjoint locally closed covering $\{\hat{U}^{k_0,k_\bullet}_{a,b;i}\}_{k_0}$ of $\hat{U}^{k_\bullet}_{a,b;i}$. For each k_0 there is a locally free sheaf on each $\hat{U}^{k_0,k_\bullet}_{a,b;i}$:

$$\hat{\mathcal{H}}^{k_0,k_\bullet}_{a,b;i} := \mathcal{E}xt^1_{\hat{\pi}^{k_0,k_\bullet}_{a,b;i}}((\hat{p}'_2, \hat{p}_2)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3), (\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i}, \hat{\mathcal{V}}^{k_\bullet}_{a;i}))^\vee.$$

We denote by

$$\hat{\theta}^{k_0,k_\bullet}_{a,b;i} : \hat{Q}^{k_0,k_\bullet}_{a,b;i} \longrightarrow \hat{U}^{k_0,k_\bullet}_{a,b;i} \subset \hat{R}_{a;i} \times \hat{G}_3$$

the vector bundle associated to that locally free sheaf. Using the same proposition, we have that there exists a universal extension parametrized by $\hat{Q}^{k_0,k_\bullet}_{a,b;i}$, as follows:

$$\begin{aligned} 0 \to (\hat{\theta}'^{k_0,k_\bullet}_{a,b;i}, \hat{\theta}^{k_0,k_\bullet}_{a,b;i})^*(\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i}, \hat{\mathcal{V}}^{k_\bullet}_{a;i}) \xrightarrow{\varepsilon^{k_0,k_\bullet}_{a,b;i}} (\hat{\mathcal{E}}^{k_0,k_\bullet}_{a,b;i}, \hat{\mathcal{V}}^{k_0,k_\bullet}_{a,b;i}) \xrightarrow{\delta^{k_0,k_\bullet}_{a,b;i}} \\ \xrightarrow{\delta^{k_0,k_\bullet}_{a,b;i}} (\hat{\theta}'^{k_0,k_\bullet}_{a,b;i}, \hat{\theta}^{k_0,k_\bullet}_{a,b;i})^*(\hat{p}'_2, \hat{p}_2)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3) \to 0 \end{aligned} \tag{6.16}$$

Now the set $\{\hat{U}^{k_0,k_\bullet}_{a,b;i}\}_{k_0,k_\bullet}$ is a finite disjoint locally closed covering of $\hat{U}_{a,b;i}$, so we rename it as $\{\hat{U}_{a,b;i,j}\}_j$. According to this notation, we denote by $\hat{Q}_{a,b;i,j}$, $(\hat{\mathcal{E}}_{a,b;i,j}, \hat{\mathcal{V}}_{a,b;i,j})$, etc. all the various objects we have defined so far.

Now let us fix any $j = (k_0, k_\bullet) = (k_0, k'_1, \cdots, k'_s)$ and let us consider the morphisms:

$$\hat{\gamma}_{a,b;i,j} : \hat{U}_{a,b;i,j} = \hat{U}^{k_0,k_\bullet}_{a,b;i} \hookrightarrow \hat{R}^{k_\bullet}_{a;i} \times \hat{G}_3 \xrightarrow{\hat{p}_1} \hat{R}^{k_\bullet}_{a;i} \hookrightarrow \hat{R}^{k'_1}_{a;i}$$

and

$$\hat{\chi}_{a,b;i,j} : \hat{Q}_{a,b;i,j} \xrightarrow{\hat{\theta}_{a,b;i,j}} \hat{U}_{a,b;i,j} \xrightarrow{\hat{\gamma}_{a,b;i,j}} \hat{R}^{k'_1}_{a;i}.$$

Let us consider the pullback of (6.15) (with k replaced by k'_1) via $\hat{\chi}_{a,b;i,j}$:

$$\begin{aligned} 0 \to (\hat{\chi}'_{a,b;i,j}, \hat{\chi}_{a,b;i,j})^*((\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \otimes_{\hat{R}_{a;i}} \mathcal{O}_{\hat{R}_{a;i}}(1)) \xrightarrow{\hat{\chi}^*_{a,b;i,j}\sigma^{k'_1}_{a;i}} \\ \xrightarrow{\chi^*_{a,b;i,j}\sigma^{k'_1}_{a;i}} \chi^*_{a,b;i,j}(\hat{\mathcal{E}}^{k'_1}_{a;i}, \hat{\mathcal{V}}^{k'_1}_{a;i}) \xrightarrow{\chi^*_{a,b;i,j}\kappa^{k'_1}_{a;i}} (\hat{\chi}'_{a,b;i,j}, \hat{\chi}_{a,b;i,j})^*(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \to 0. \end{aligned} \tag{6.17}$$

Eventually by restricting again the set $\hat{U}_{a,b;i,j}$, we can suppose that the pullback of the line bundle $\mathcal{O}_{\hat{R}_{a;i}}(1)$ is free on $\hat{U}_{a,b;i,j}$, so we can assume also that

$$(\hat{\chi}'_{a,b;i,j}, \hat{\chi}_{a,b;i,j})^*\mathcal{O}_{\hat{R}_{a;i}}(1)$$

is free on $\hat{Q}_{a,b;i,j}$. Therefore, we can identify the first an the last term of the previous exact sequence.

Then for every pair of scalars $(\xi, \tau) \in \mathbb{C} \times \mathbb{C}^*$, let us consider the new extension

$$\begin{aligned} 0 \to (\hat{\theta}'_{a,b;i,j}, \hat{\theta}_{a,b;i,j})^*(\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i}) \xrightarrow{\varepsilon_{a,b;i,j}(\xi,\tau)} (\hat{\mathcal{E}}_{a,b;i,j}, \hat{\mathcal{V}}_{a,b;i,j}) \xrightarrow{\delta_{a,b;i,j}} \\ \xrightarrow{\delta_{a,b;i,j}} (\hat{\theta}'_{a,b;i,j}, \hat{\theta}_{a,b;i,j})^*(\hat{p}'_2, \hat{p}_2)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3) \to 0 \end{aligned} \tag{6.18}$$

where we write:

$$\varepsilon_{a,b;i,j}(\xi, \tau) := \varepsilon \circ \left(\xi \cdot \chi^*_{a,b;i,j}(\sigma^{k'_1}_{a;i} \circ \kappa^{k'_1}_{a;i}) + \tau \cdot \mathrm{id}_{(\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i})} \right).$$

Now (6.18) is again an extension parametrized by $\hat{Q}_{a,b;i,j}$ of the same 2 objects of (6.16). Therefore, by the universal property of (6.16) (see corollary 4.3.3), we get that there is a unique morphism

$$\hat{\varrho}_{a,b;i,j}(\xi, \tau) = \hat{\varrho}(\xi, \tau) : \hat{Q}_{a,b;i,j} \longrightarrow \hat{Q}_{a,b;i,j}$$

over $\hat{U}_{a,b;i,j}$, such that (6.18) is the pullback of (6.16) via $\hat{\varrho}(\xi, \tau)$. In particular, we have a commutative diagram

$$\begin{tikzcd} \hat{Q}_{a,b;i,j} \arrow[rr, "\hat{\varrho}(\xi,\tau)"] \arrow[dr, "\hat{\theta}_{a,b;i,j}"'] & \curvearrowright & \hat{Q}_{a,b;i,j}. \arrow[dl, "\hat{\theta}_{a,b;i,j}"] \\ & \hat{U}_{a,b;i,j} & \end{tikzcd} \tag{6.19}$$

Now we want to prove that such a morphism is an automorphism of $\hat{Q}_{a,b;i,j}$, so we need to find an inverse for such a morphism; we claim that for every $(\xi,\tau)\in\mathbb{C}\times\mathbb{C}^*$ the inverse of $\hat{\varrho}(\xi,\tau)$ is given by $\hat{\varrho}(-\xi\tau^{-2},\tau^{-1})$. First of all, a direct check proves that if we apply the pair $(-\xi\tau^{-2},\tau^{-1})$ to the sequence (6.18), then we get the extension (6.16). Therefore, we get that the morphism

$$f := \hat{\varrho}(-\xi\tau^{-2},\tau^{-1})\circ\hat{\varrho}(\xi,\tau):\ \hat{Q}_{a,b;i,j}\longrightarrow\hat{Q}_{a,b;i,j}$$

is such that the pullback via f of (6.16) is again the same extension. Obviously, also the identity of $\hat{Q}_{a,b;i,j}$ has the same property; since (6.16) is universal, then we get that f coincides with the identity. The same argument proves also that

$$\hat{\varrho}(\xi,\tau)\circ\hat{\varrho}(-\xi\tau^{-2},\tau^{-1}):\ \hat{Q}_{a,b;i,j}\longrightarrow\hat{Q}_{a,b;i,j}$$

is the identity of $\hat{Q}_{a,b;i,j}$. Now let us consider the group action on $\mathbb{C}\times\mathbb{C}^*$ given by $(\xi',\tau')\cdot(\xi,\tau)=(\xi\tau'+\xi'\tau,\tau\tau')$. The previous observation then proves that we have a natural action of $\mathbb{C}\times\mathbb{C}^*$ on $\hat{Q}_{a,b;i,j}$ given by

$$\begin{aligned} \mathbb{C}\times\mathbb{C}^* &\to \mathrm{Aut}(\hat{Q}_{a,b;i,j}) \\ (\xi,\tau) &\mapsto \hat{\varrho}_{a,b;i,j}(\xi,\tau). \end{aligned}$$

Then we define a subvector bundle $\hat{Q}'_{a,b;i,j}$ of $\hat{Q}_{a,b;i,j}$ as follows. Let us consider the pullback of the sequence (6.14) via the morphism $\hat{\gamma}_{a,b;i,j}:\hat{U}_{a,b;i,j}\to\hat{R}^{k'_1}_{a;i}$ and let us apply to it the functor $\mathcal{H}om_{\hat{\pi}_{a,b;i,j}}((\hat{p}'_2,\hat{p}_2)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3),-)$. Then we get a long exact sequence:

$$\begin{aligned} \cdots\to\mathcal{H}om_{\hat{\pi}_{a,b;i,j}}((\hat{p}'_2,\hat{p}_2)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3),(\hat{p}'_1,\hat{p}_1)^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1))\to \\ \to\mathcal{E}xt^1_{\hat{\pi}_{a,b;i,j}}((\hat{p}'_2,\hat{p}_2)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3),(\hat{p}'_1,\hat{p}_1)^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1))\otimes_{\hat{U}_{a,b;i,j}}(\hat{p}'_1,\hat{p}_1)^*\mathcal{O}_{\hat{R}_{a;i}}(1))\xrightarrow{\zeta} \\ \xrightarrow{\zeta}\mathcal{E}xt^1_{\hat{\pi}_{a,b;i,j}}((\hat{p}'_2,\hat{p}_2)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3),(\hat{p}'_1,\hat{p}_1)^*(\hat{\mathcal{E}}^{k'_1}_{a;i},\hat{\mathcal{V}}^{k'_1}_{a;i}))\to\cdots \end{aligned} \tag{6.20}$$

Now the first sheaf of this sequence is actually zero using base change and the hypothesis that $(n_1,k_1)\neq(n_3,k_3)$. Then we get that ζ is injective. Moreover, for every point t of $\hat{U}_{a,b;i,j}$, we have:

$$\mathrm{Ext}^l_{\hat{\pi}_{a,b;i,j}}((p'_2,p_2)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)_t,(\hat{p}'_1,\hat{p}_1)^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1))\otimes_{\hat{U}_{a,b;i,j}}(\hat{p}'_1,\hat{p}_1)^*\mathcal{O}_{\hat{R}_{a;i}}(1)_t)=0$$

for $l = 0$ and $l = 2$; the case $l = 0$ is a consequence of the hypothesis that $(n_1, k_1) \neq (n_3, k_3)$, while the case $l = 2$ is a consequence of the hypothesis that

$$\mathrm{Ext}^1((Q_3, W_3), (Q_1, W_1)) = 0 \quad \forall\, ((Q_1, W_1), (Q_3, W_3)) \in G_1 \times G_3.$$

Then by base change we get that the second sheaf of (6.20) is locally free. We denote by $\hat{Q}'_{a,b;i,j}$ the vector bundle on $\hat{U}_{a,b;i,j}$ associated to its dual. Since ζ is injective, using the definition of $\hat{Q}_{a,b;i,j}$ we get that $\hat{Q}'_{a,b;i,j}$ is a subvector bundle of $\hat{Q}_{a,b;i,j}$.

Now on each fiber of the bundle $\hat{Q}_{a,b;i,j}$ over $\hat{U}_{a,b;i,j}$ we have the description given in lemma 6.1.3. In particular, every $\hat{\varrho}_{a,b;i,j}(\xi, \tau)$ acts on $\hat{Q}'_{a,b;i,j}$ by fixing it. Therefore, we get that the previous action of $\mathbb{C} \times \mathbb{C}^*$ induces an action of the same group on $\hat{Q}_{a,b;i,j} \smallsetminus \hat{Q}'_{a,b;i,j}$.

Now for every $(a, b; i, j)$ and (ξ, τ) as before we have a commutative diagram (6.19), so the action of $\mathbb{C} \times \mathbb{C}^*$ is compatible with the projection $\hat{\theta}_{a,b;i,j}$. So it makes sense to consider the quotient

$$\hat{R}_{a,b;i,j} := (\hat{Q}_{a,b;i,j} \smallsetminus \hat{Q}'_{a,b;i,j})/(\mathbb{C} \times \mathbb{C}^*)$$

and the induced fibration

$$\hat{\varphi}_{a,b;i,j} : \hat{R}_{a,b;i,j} \longrightarrow \hat{U}_{a,b;i,j}.$$

The fibers of such a morphism are described as in lemma 6.1.3, so each fiber is isomorphic to $\mathbb{C}^{b-1} \times \mathbb{P}^{b-1}$. Now we recall that on $\hat{Q}_{a,b;i,j}$ we have a family $(\hat{\mathcal{E}}_{a,b;i,j}, \hat{\mathcal{V}}_{a,b;i,j})$ such that if we denote by q any point of $\hat{Q}_{a,b;i,j}$ and by (E, V) the restriction of such a family to q, then we have a pair of exact sequences:

$$0 \to (Q_1, W_1) \to (E_2, V_2) \to (Q_1, W_1) \to 0,$$
$$0 \to (E_2, V_2) \to (E, V) \to (Q_3, W_3) \to 0.$$

The first one is always non-split by construction of $\hat{R}_{a;i}$; if q belongs to $\hat{Q}_{a,b;i,j} \smallsetminus \hat{Q}'_{a,b;i,j}$, then also the induced sequence

$$0 \to (Q_1, W_1) \to (E, V)/(Q_1, W_1) \to (Q_3, W_3) \to 0$$

is non-split. If we assume also conditions (6.1), then by lemma 6.1.3 we get that (E, V) belongs to $G' \subset G^+(\alpha_c; n, d, k) \subset G(\alpha_c^+; n, d, k)$. Then by the universal property of the scheme $G(\alpha_c^+; n, d, k)$ we get that the previous family induces a morphism

$$\hat{\omega}_{a,b;i,j} : \hat{Q}_{a,b;i,j} \smallsetminus \hat{Q}'_{a,b;i,j} \longrightarrow G(\alpha_c^+; n, d, k).$$

By the previous lemma, we have that $\hat{\omega}_{a,b;i,j}$ is invariant under the action of $\mathbb{C} \times \mathbb{C}^*$, so it induces a morphism

$$\overline{\omega}_{a,b;i,j} : \hat{R}_{a,b;i,j} \longrightarrow G(\alpha_c^+; n, d, k).$$

Finally, there are free actions of $PGL(N_1)$ on $\hat{G}_1$, $\hat{U}_{a;i}$ and $\hat{R}_{a;i}$. Moreover, there are free actions of $PGL(N_1) \times PGL(N_3)$ on $\hat{R}_{a;i} \times \hat{G}_3$, $\hat{U}_{a,b;i,j}$ and $\hat{R}_{a,b;i,j}$. Then there are induced fibrations

$$\varphi_{a;i} : R_{a;i} \longrightarrow U_{a;i},$$

$$\varphi_{a,b;i,j} : R_{a,b;i,j} \longrightarrow U_{a,b;i,j}.$$

with the properties stated in the claim of the lemma. The morphism $\overline{\omega}_{a,b;i,j}$ is invariant under the action on $\hat{R}_{a,b;i,j}$, so there is an induced morphism

$$\omega_{a,b;i,j} : R_{a,b;i,j} \longrightarrow G(\alpha_c^+; n, d, k).$$

Such a morphism is injective (and has values in G') because of the previous lemma, so we conclude. An analogous conclusion holds if we assume conditions (6.2) instead of (6.1). □

Lemma 6.1.5. *Let us fix any triple $(Q_i, W_i)_{i=1,2,3} \in \prod_{i=1}^3 G_i$ and let us suppose that $(Q_1, W_1) \not\simeq (Q_2, W_2) \not\simeq (Q_3, W_3)$. Then the (E, V)'s with unique Jordan-Hölder filtration at α_c and graded $\oplus_{i=1}^3 (Q_i, W_i)$ are parametrized by pairs $([\mu], [\nu])$ where:*

- *$[\mu] \in \mathbb{P}(Ext^1((Q_3, W_3), (Q_2, W_2)))$ and μ has a representative of the form*

$$0 \to (Q_2, W_2) \xrightarrow{\sigma} (E'', V'') \xrightarrow{\kappa} (Q_3, W_3) \to 0; \tag{6.21}$$

- *$[\nu] \in \overline{M}([\mu]) := \mathbb{P}(Ext^1((E'', V''), (Q_1, W_1))) \smallsetminus \mathbb{P}(Ext^1((Q_3, W_3), (Q_1, W_1)))$.*

The proof of this lemma is analogous to the proof of lemma 6.1.1, so we omit it. Using such a lemma we can prove the following proposition. The proof is analogous to that of proposition 6.1.2.

Proposition 6.1.6. *Let us fix any triple (n, d, k), a critical value α_c for it and a triple $(n_i, d_i, k_i)_{i=1,2,3}$ compatible with $(\alpha_c; n, d, k)$. Let us assume that conditions (6.1), respectively (6.2), are satisfied. Moreover, let us suppose that for every triple of points $(Q_i, W_i)_{i=1,2,3} \in \prod_{i=1}^3 G_i$ we have:*

$$Hom((Q_2, W_2), (Q_1, W_1)) = 0 = Hom((Q_3, W_3), (Q_2, W_2))$$

(in particular, this holds if $(n_1, k_1) \neq (n_2, k_2) \neq (n_3, k_3)$). Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in $G^-(\alpha_c; n, d, k)$, with unique *Jordan-Hölder filtration at α_c and graded in $\prod_{i=1}^3 G_i$. Then there exists a finite family $\{R_{a,b,c;i,j}\}$ of schemes for $(a, b, c) \in \mathbb{N}^2 \times \mathbb{N}_0$, $c \leq b$ and i, j varying in finite sets (for a, b, c fixed), together with injective morphisms to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Every $R_{a,b,c;i,j}$ comes with a sequence of 2 morphisms:*

$$\varphi_{a,b,c;i,j} : R_{a,b,c;i,j} \longrightarrow U_{a,b,c;i,j} \subset G_1 \times R_{a;i},$$
$$\varphi_{a;i} : R_{a;i} \longrightarrow U_{a;i} \subset G_2 \times G_3,$$

where:

- $\varphi_{a,b,c;i,j}$ *has fibers isomorphic to* $\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}$ *and* $\varphi_{a;i}$ *has fibers isomorphic to* $\mathbb{P}^{a-1}$*;*
- $\{U_{a;i}\}_i$ *is a locally closed covering of*

$$U_a := \{((Q_2,W_2),(Q_3,W_3)) \in G_2 \times G_3 \text{ s.t. } \dim Ext^1((Q_3,W_3),(Q_2,W_2)) = a\};$$

every U_a *is a locally closed subscheme of* $G_2 \times G_3$ *and so are all the* $U_{a;i}$*'s;*

- $\{U_{a,b,c;i,j}\}_j$ *is a locally closed covering of*

$$U_{a,b,c;i} := \{((Q_1,W_1),(E'',V'')) \in G_1 \times R_{a;i} \text{ s.t.}$$
$$\dim Ext^1((E'',V''),(Q_1,W_1)) = b, \quad \dim Ext^1(\widetilde{\varphi}_{a;i}(E'',V''),(Q_1,W_1)) = c\};$$

where $\widetilde{\varphi}_{a;i}$ *is the composition of* $\varphi_{a;i}$ *with the projection to* G_3*. Every* $U_{a,b,c;i}$ *is locally closed in* $G_1 \times R_{a;i}$ *and so are the* $U_{a,b,c;i,j}$*'s.*

If $(n_2,k_2) = (n_3,k_3)$*, then the previous results hold on* $G_1 \times (G_2 \times G_3 \smallsetminus \Delta_{23})$ *instead of* $G_1 \times G_2 \times G_3$*.*

Remark 6.1.1. If we assume that

$$(Q_1,W_1) \not\simeq (Q_2,W_2) \not\simeq (Q_3,W_3), \quad \mathrm{Ext}^1((Q_3,W_3),(Q_1,W_1))$$

then we can easily compare the descriptions of propositions 6.1.2 and 6.1.6 and we get that those descriptions give rise to the same Hodge-Deligne polynomials.

Lemma 6.1.7. *Let us fix any triple* $(Q_i,W_i)_{i=1,2,3} \in \prod_{i=1}^3 G_i$ *and let us suppose that* $(Q_1,W_1) \not\simeq (Q_2,W_2) \simeq (Q_3,W_3)$ *and that*

$$Ext^2\left((Q_2,W_2),(Q_1,W_1)\right) = 0$$

(with a little abuse of notation we will simply write (Q_2,W_2) *also for* (Q_3,W_3)*). Let us denote by* μ *any class of a non-split extension of the form*

$$0 \to (Q_2,W_2) \xrightarrow{\sigma} (E'',V'') \xrightarrow{\kappa} (Q_2,W_2) \to 0 \tag{6.22}$$

and by ν *any class of a non-split extension of the form*

$$0 \to (Q_1, W_1) \xrightarrow{\varepsilon} (E, V) \xrightarrow{\delta} (E'', V'') \to 0. \tag{6.23}$$

Having fixed $[\mu] \in \mathbb{P}(Ext^1((Q_2, W_2), (Q_2, W_2)))$, *let us consider the space* $Ext^1((E'', V''), (Q_1, W_1))$ *and let us consider the action of* $\mathbb{C} \times \mathbb{C}^*$ *on it given as follows. For every pair of scalars* (ξ, τ) *and for every class of extension* ν *with representative of the form* (6.23), *we set* $(\xi, \tau) \cdot \nu := \nu'$ *where* ν' *is represented by*

$$0 \to (Q_1, W_1) \xrightarrow{\varepsilon} (E, V) \xrightarrow{\delta'} (E'', V'') \to 0,$$

where $\delta' = (\xi \cdot \sigma \circ \kappa + \tau \cdot id_{(E'',V'')}) \circ \delta$. *Let us write*

$$M([\mu]) := Ext^1((E'', V''), (Q_1, W_1)) \smallsetminus Ext^1((Q_2, W_2), (Q_1, W_1));$$

then the previous action sends $M([\mu])$ *to itself, so it makes sense to consider* $\overline{M}([\mu]) := M([\mu])/(\mathbb{C} \times \mathbb{C}^*)$. *Then the* (E, V)*'s with unique Jordan-Hölder filtration at* α_c *and graded* $\oplus_{i=1}^3 (Q_i, W_i)$ *are parametrized by pairs* $([\mu], [\nu])$ *where:*

- $[\mu] \in \mathbb{P}(Ext^1((Q_2, W_2), (Q_2, W_2)))$ *and* μ *has a representative of the form (6.22);*
- $[\nu]$ *is any object of* $\overline{M}([\mu])$.

Moreover, $\overline{M}([\mu]) \simeq \mathbb{C}^{b-1} \times \mathbb{P}^{b-1}$ *where* $b = dim\ Ext^1((Q_2, W_2), (Q_1, W_1))$.

The proof of this lemma is analogous to the proof of lemma 6.1.3, so we omit it. Using such a lemma we can prove the following proposition. The proof is analogous to that of proposition 6.1.4.

Proposition 6.1.8. *Let us fix any triple* (n, d, k), *a critical value* α_c *for it and any triple* $(n_i, d_i, k_i)_{i=1,2,3}$ *compatible with* $(\alpha_c; n, d, k)$. *Let us assume that conditions (6.1), respectively (6.2) are satisfied. Moreover, let us suppose that* $(n_1, k_1) \neq (n_2, k_2) = (n_3, k_3)$ *and that for every pair of points* $((Q_1, W_1), (Q_2, W_2)) \in G_1 \times G_2$ *we have:*

$$Ext^2((Q_2, W_2), (Q_1, W_1)) = 0.$$

Let us denote by G' *the set of all the* (E, V)*'s in* $G^+(\alpha_c; n, d, k)$, *respectively in* $G^-(\alpha_c; n, d, k)$, *that have unique Jordan-Hölder filtration at* α_c *and graded* $(Q_1, W_1) \oplus (Q_2, W_2) \oplus (Q_2, W_2)$. *Then there exists a finite family* $\{R_{a,b;i,j}\}$ *of schemes for* $(a, b) \in \mathbb{N}^2$ *and* i, j *varying in finite sets (for* a, b *fixed), together with injective morphisms to* $G^+(\alpha_c; n, d, k)$, *respectively to* $G^-(\alpha_c; n, d, k)$, *such that the images form a disjoint covering of* G' *by locally closed subschemes. Every* $R_{a,b;i,j}$ *comes with a sequence of 2 morphisms*

$$\varphi_{a,b;i,j} : R_{a,b;i,j} \longrightarrow U_{a,b:i,j} \subset G_1 \times R_{a;i},$$
$$\varphi_{a;i} : R_{a;i} \longrightarrow U_{a;i} \subset G_2 = G_3,$$

where:

- $\varphi_{a,b;i,j}$ *has fibers isomorphic to* $\mathbb{C}^{b-1} \times \mathbb{P}^{b-1}$*;*
- $\varphi_{a;i}$ *has fibers isomorphic to* $\mathbb{P}^{a-1}$*;*
- $\{U_{a;i}\}_i$ *is a locally closed covering of*

$$U_a := \{(Q_2, W_2) \in G_2 \text{ s.t. } \dim Ext^1((Q_2, W_2), (Q_2, W_2)) = a\};$$

every U_a *is a locally closed subscheme of* G_2 *and so are all the* $U_{a;i}$*'s;*

- $\{U_{a,b;i,j}\}_j$ *is a locally closed covering of*

$$U_{a,b;i} := \{((Q_1, W_1), (E'', V'')) \in G_1 \times R_{a;i} \text{ s.t. } \dim Ext^1(\varphi_{a;i}(E'', V''), (Q_1, W_1)) = b\}.$$

$U_{a,b;i}$ *is locally closed in* $G_1 \times R_{a;i}$ *and so are the* $U_{a,b;i,j}$*'s.*

6.2 Jordan-Hölder filtration of length 4

Let us suppose that at α_c the graded of a coherent system (E, V) is $\oplus_{i=1}^4 (Q_i, W_i)$ and that (E, V) has a unique α_c-Jordan-Hölder filtration (therefore, that filtration coincides with its α_c-canonical filtration). We want to parametrize all the corresponding (E, V)'s, having fixed the graded (and also its order, since the filtration is unique).

If the α_c-JHF is unique, then the only proper α_c-semistable subobjects of (E, V) with its same α_c-slope are:

- (Q_1, W_1), that is the only α_c-stable one;
- an extension (E_2, V_2) of (Q_2, W_2) by (Q_1, W_1);
- an extension (E_3, V_3) of (Q_3, W_3) by (E_2, V_2);

the quotient $(E, V)/(E_3, V_3)$ will be isomorphic to (Q_4, W_4). Let us denote by (n_i, d_i, k_i) the type of each (Q_i, W_i). If (E, V) has unique α_c-Jordan-Hölder filtration, then it belongs to $G^+(\alpha_c; n, d, k)$ if and only if the following numerical conditions are satisfied:

$$\frac{k_1}{n_1} < \frac{k}{n}, \quad \frac{k_1 + k_2}{n_1 + n_2} < \frac{k}{n}, \quad \frac{k_1 + k_2 + k_3}{n_1 + n_2 + n_3} < \frac{k}{n}. \tag{6.24}$$

Using the fact that $\mu_{\alpha_c}(E, V) = \mu_{\alpha_c}(Q_i, W_i)$ for $i = 1, \cdots, 4$, the last inequality can be rewritten as $k_4/n_4 > k/n$.

Analogously, if we assume that (E, V) has a unique α_c-JHF, then it belongs to $G^-(\alpha_c; n, d, k)$ if and only if the following numerical conditions are verified:

$$\frac{k_1}{n_1} > \frac{k}{n}, \quad \frac{k_1 + k_2}{n_1 + n_2} > \frac{k}{n}, \quad \frac{k_1 + k_2 + k_3}{n_1 + n_2 + n_3} > \frac{k}{n}. \tag{6.25}$$

In both cases, we need a way of parametrizing all the (E, V)'s with unique filtration, having fixed the graded. According to different cases, we will need one of the 4 descriptions given below. A priori there are 5 possible trees associated to (E, V), but for our purposes (that is, the study of the case $n = 4, k = 1$) we will need only the following one.

$\boxed{C}$

$$\begin{tikzcd}[column sep=small]
 & & & (0,4) \arrow[dll, dash] \arrow[drr, dash] & & & \\
 & (0,2) \arrow[dl, dash] \arrow[dr, dash] & & & & (2,4) \arrow[dl, dash] \arrow[dr, dash] & \\
(0,1) & & (1,2) & & (2,3) & & (3,4).
\end{tikzcd}$$

We will basically divide our description in the following 4 cases:

(i) $(Q_1, W_1) \simeq (Q_2, W_2)$ and $(Q_4, W_4) \not\simeq (Q_i, W_i)$ for $i = 1, 2, 3$ (lemma 6.2.1 and proposition 6.2.2);

(ii) $(Q_1, W_1) \not\simeq (Q_2, W_2)$ and $(Q_4, W_4) \not\simeq (Q_i, W_i)$ for $i = 1, 2, 3$ (lemma 6.2.3 and proposition 6.2.4);

(iii) $(Q_3, W_3) \simeq (Q_4, W_4)$ and $(Q_1, W_1) \not\simeq (Q_i, W_i)$ for $i = 2, 3, 4$ (lemma 6.2.5 and proposition 6.2.6);

(iv) $(Q_3, W_3) \not\simeq (Q_4, W_4)$ and $(Q_1, W_1) \not\simeq (Q_i, W_i)$ for $i = 2, 3, 4$ (lemma 6.2.7 and proposition 6.2.8).

Remark 6.2.1. For each of these cases one should consider 2 different subcases according to the fact that (Q_2, W_2) and (Q_3, W_3) are isomorphic or not. In the second subcase we get complete results as stated below, while the first subcase is still an open problem.

Lemma 6.2.1. *Let us fix any quadruple $(Q_i, W_i)_{i=1,\cdots,4} \in \prod_{i=1}^4 G_i$ and let us assume that*

$$(Q_1, W_1) \simeq (Q_2, W_2) \not\simeq (Q_3, W_3), \quad (Q_4, W_4) \not\simeq (Q_i, W_i) \quad \forall\, i = 1, 2, 3,$$
$$Ext^2\left((Q_4, W_4), (Q_1, W_1)\right) = 0 = Ext^2\left((E'', V''), (Q_1, W_1)\right),$$

where (E'', V'') is any non-split extension of (Q_4, W_4) by (Q_3, W_3). Let us denote by μ any class of a non-split extension of the form

$$0 \to (Q_1, W_1) \xrightarrow{\sigma} (E_2, V_2) \xrightarrow{\kappa} (Q_2, W_2) \to 0 \tag{6.26}$$

and by ν any class of a non-split extension of the form

$$0 \to (Q_3, W_3) \xrightarrow{\varepsilon} (E'', V'') \xrightarrow{\delta} (Q_4, W_4) \to 0. \tag{6.27}$$

Having fixed $[\mu] \in \mathbb{P}(Ext^1((Q_1, W_1), (Q_1, W_1)))$ and $[\nu] \in \mathbb{P}(Ext^1((Q_4, W_4), (Q_3, W_3))$, let us consider the space $M_1([\mu], [\nu]) := Ext^1((E'', V''), (E_2, V_2))$, let us denote by η any object in that space and let us choose a representative of η as follows:

$$0 \to (E_2, V_2) \xrightarrow{\iota} (E, V) \xrightarrow{\lambda} (E'', V'') \to 0. \tag{6.28}$$

Then let us consider the action of $\mathbb{C} \times \mathbb{C}^*$ on $M_1([\mu], [\nu])$ given as follows. For every pair of scalars (ξ, τ) and for every class η of an extension with representative (6.28), we set $(\xi, \tau) \cdot \eta := \eta'$, where η' is represented by

$$0 \to (E_2, V_2) \xrightarrow{\iota(\xi,\tau)} (E, V) \xrightarrow{\lambda} (E'', V'') \to 0, \tag{6.29}$$

where $\iota(\xi, \tau) := \iota \circ (\xi \cdot \sigma \circ \kappa + \tau \cdot id_{(E_2,V_2)})$. Let us denote by $M_2([\mu], [\nu])$ the image of the linear map $A + F$ where A and F are the maps induced by (6.27) and (6.26) respectively, as follows:

$$\begin{aligned} A &: Ext^1((Q_4, W_4), (E_2, V_2)) \longrightarrow Ext^1((E'', V''), (E_2, V_2)) = M_1([\mu], [\nu]), \\ F &: Ext^1((E'', V''), (Q_1, W_1)) \longrightarrow Ext^1((E'', V''), (E_2, V_2)) = M_1([\mu], [\nu]). \end{aligned}$$

Let us write $M([\mu], [\nu]) := M_1([\mu], [\nu]) \setminus M_2([\mu], [\nu])$. The previous action sends $M_2([\mu], [\nu])$ to itself, so it makes sense to consider $\overline{M}([\mu], [\nu]) := M([\mu], [\nu])/(\mathbb{C} \times \mathbb{C}^*)$. Then the objects (E, V)'s with unique α_c-Jordan-Hölder filtration and graded $\oplus_{i=1}^4 (Q_i, W_i)$ are parametrized by triples $([\mu], [\nu], [\eta])$ where:

- *$[\mu] \in \mathbb{P}(Ext^1((Q_2, W_2), (Q_1, W_1)))$ and μ has a representative of the form (6.26);*
- *$[\nu] \in \mathbb{P}(Ext^1((Q_4, W_4), (Q_3, W_3)))$ and ν has a representative of the form (6.27);*
- *$[\eta] \in \overline{M}([\mu], [\nu])$.*

Moreover, if we write:

$$\begin{aligned} c &:= dim\ M_1([\mu], [\nu]), \quad d := dim\ Ext^1((Q_4, W_4), (E_2, V_2)), \\ e &:= dim\ Ext^1((E'', V''), (Q_1, W_1)), \quad f := dim\ Ext^1((Q_4, W_4), (Q_1, W_1)), \end{aligned}$$

then $dim\ M_2([\mu], [\nu]) = d + e - f$ and $\overline{M}([\mu], [\nu]) \simeq \mathbb{C}^{e-1} \times (\mathbb{P}^{c-e-1} \setminus \mathbb{P}^{d-f-1})$.

Proof. Having fixed $\oplus_{i=1}^4 (Q_i, W_i)$, let us denote by μ and ν the classes of any two extensions as follows:

$$0 \to (Q_1, W_1) \xrightarrow{\sigma} (E_2, V_2) \xrightarrow{\kappa} (Q_2, W_2) \to 0, \tag{6.30}$$

$$0 \to (Q_3, W_3) \xrightarrow{\varepsilon} (E'', V'') \xrightarrow{\delta} (Q_4, W_4) \to 0. \tag{6.31}$$

Having fixed $[\mu]$ and $[\nu]$ as in the claim of the lemma, we set $M_1([\mu],[\nu]) := \mathrm{Ext}^1((E'',V''),(E_2,V_2))$ and let η be any object of this space; let us consider a representative of it as follows:

$$0 \to (E_2,V_2) \xrightarrow{\iota} (E,V) \xrightarrow{\lambda} (E'',V'') \to 0. \tag{6.32}$$

Then let us consider the following 6 long exact sequences, induced by (6.30) and (6.31).

$$\begin{aligned}0 &\to \mathrm{Hom}((Q_4,W_4),(E_2,V_2)) \to \mathrm{Hom}((E'',V''),(E_2,V_2)) \to \mathrm{Hom}((Q_3,W_3),(E_2,V_2)) \to \\ &\to \mathrm{Ext}^1((Q_4,W_4),(E_2,V_2)) \xrightarrow{A} \mathrm{Ext}^1((E'',V''),(E_2,V_2)) \xrightarrow{B} \mathrm{Ext}^1((Q_3,W_3),(E_2,V_2)) \to \\ &\to \mathrm{Ext}^2((Q_4,W_4),(E_2,V_2));\end{aligned} \tag{6.33}$$

$$\cdots \to \mathrm{Ext}^1((Q_3,W_3),(Q_1,W_1)) \xrightarrow{C} \mathrm{Ext}^1((Q_3,W_3),(E_2,V_2)) \xrightarrow{D} \mathrm{Ext}^1((Q_3,W_3),(Q_2,W_2)); \tag{6.34}$$

$$\begin{aligned}0 &\to \mathrm{Hom}((E'',V''),(Q_1,W_1)) \to \mathrm{Hom}((E'',V''),(E_2,V_2)) \to \mathrm{Hom}((E'',V''),(Q_2,W_2)) \to \\ &\to \mathrm{Ext}^1((E'',V''),(Q_1,W_1)) \xrightarrow{F} \mathrm{Ext}^1((E'',V''),(E_2,V_2)) \to \mathrm{Ext}^1((E'',V''),(Q_2,W_2)) \to \\ &\to \mathrm{Ext}^2((E'',V''),(Q_1,W_1));\end{aligned} \tag{6.35}$$

$$\cdots \to \mathrm{Ext}^1((E'',V''),(Q_1,W_1)) \xrightarrow{I} \mathrm{Ext}^1((Q_3,W_3),(Q_1,W_1)) \to \mathrm{Ext}^2((Q_4,W_4),(Q_1,W_1)); \tag{6.36}$$

$$\begin{aligned}0 &\to \mathrm{Hom}((Q_4,W_4),(Q_1,W_1)) \to \mathrm{Hom}((Q_4,W_4),(E_2,V_2)) \to \mathrm{Hom}((Q_4,W_4),(Q_2,W_2)) \to \\ &\to \mathrm{Ext}^1((Q_4,W_4),(Q_1,W_1)) \xrightarrow{L} \mathrm{Ext}^1((Q_4,W_4),(E_2,V_2));\end{aligned} \tag{6.37}$$

$$\begin{aligned}0 &\to \mathrm{Hom}((Q_4,W_4),(Q_1,W_1)) \to \mathrm{Hom}((E'',V''),(Q_1,W_1)) \to \mathrm{Hom}((Q_3,W_3),(Q_1,W_1)) \to \\ &\to \mathrm{Ext}^1((Q_4,W_4),(Q_1,W_1)) \xrightarrow{M} \mathrm{Ext}^1((E'',V''),(Q_1,W_1)).\end{aligned} \tag{6.38}$$

Let $\overline{\eta} := B(\eta)$ and let

$$0 \to (E_2,V_2) \to (E_3,V_3) \to (Q_3,W_3) \to 0 \tag{6.39}$$

be a representative of it. In particular, we have a commutative diagram with exact lines:

$$\begin{tikzcd}
0 \arrow[r] & (E_2,V_2) \arrow[r, "\iota"] \arrow[dr, phantom, "\curvearrowright"] & (E,V) \arrow[r, "\lambda"] \arrow[dr, phantom, "\curvearrowright"] & (E'',V'') \arrow[r] & 0. \\
0 \arrow[r] & (E_2,V_2) \arrow[r, "\alpha"] \arrow[u, equal] & (E_3,V_3) \arrow[r, "\beta"] \arrow[u, "\varepsilon'"'] & (Q_3,W_3) \arrow[r] \arrow[u, hook, "\varepsilon"'] & 0
\end{tikzcd} \tag{6.40}$$

Since ε is injective, using the snake lemma we get that also ε' is injective, so we get a filtration:

$$0 \subset (Q_1, W_1) =: (E_1, V_1) \subset (E_2, V_2) \subset (E_3, V_3) \subset (E, V) \tag{6.41}$$

and it is easy to see that this is a Jordan-Hölder filtration of (E, V) at α_c with graded $\oplus_{i=1}^4 (Q_i, W_i)$. Let us define $\widetilde{\eta} := D(\overline{\eta}) = D \circ B(\eta)$ and let

$$0 \to (Q_2, W_2) \to (\overline{E}, \overline{V}) \to (Q_3, W_3) \to 0$$

be a representative of it. In particular, by definition of D, we have a commutative diagram with exact lines:

$$\begin{tikzcd}
0 \arrow[r] & (E_2, V_2) \arrow[r, "\alpha"] \arrow[d, "\kappa"] \arrow[dr, phantom, "\curvearrowright"] & (E_3, V_3) \arrow[r, "\beta"] \arrow[d, "\kappa'"] \arrow[dr, phantom, "\curvearrowright"] & (Q_3, W_3) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (Q_2, W_2) \arrow[r, "\theta"] & (\overline{E}, \overline{V}) \arrow[r, "\zeta"] & (Q_3, W_3) \arrow[r] & 0.
\end{tikzcd} \tag{6.42}$$

Using the snake lemma, we get that κ' is surjective and its kernel coincides with the kernel of κ, i.e. with $(Q_1, W_1) = (E_1, V_1)$. Then we get that $\widetilde{\eta}$ has a representative of the form:

$$0 \to (Q_2, W_2) \to (E_3, V_3)/(E_1, V_1) \to (Q_3, W_3) \to 0. \tag{6.43}$$

Now let us apply proposition 2.2.1. Having fixed the Jordan-Hölder filtration (6.41), this is the unique α_c-Jordan-Hölder filtration of (E, V) if and only if all the sequences

$$0 \to (Q_i, W_i) \to (E_{i+1}, V_{i+1})/(E_{i-1}, V_{i-1}) \to (Q_{i+1}, W_{i+1}) \to 0$$

are non-split for $i = 1, 2, 3$. This is equivalent to imposing that μ, ν and $\widetilde{\eta}$ are all non-zero. So having fixed μ and ν both non-zero, we have to remove from $M_1([\mu], [\nu]) = \mathrm{Ext}^1((E'', V''), (E_2, V_2))$ all those η's such that $\widetilde{\eta} = D \circ B(\eta)$ is zero. Using the exactness of (6.34), $D \circ B(\eta) = 0$ if and only if $B(\eta) \in \mathrm{Ker}\ D = \mathrm{Im}\ C$.

Now let us consider the following diagram.

$$\begin{tikzcd}
\mathrm{Ext}^1((E'', V''), (Q_1, W_1)) \arrow[r, "F"] \arrow[d, "I"] \arrow[dr, phantom, "\curvearrowright"] & \mathrm{Ext}^1((E'', V''), (E_2, V_2)) \arrow[d, "B"] \\
\mathrm{Ext}^1((Q_3, W_3), (Q_1, W_1)) \arrow[r, "C"] & \mathrm{Ext}^1((Q_3, W_3), (E_2, V_2)).
\end{tikzcd}$$

Such a diagram is commutative by naturality of the functors $\mathrm{Ext}^1(-,-)$'s. By the hypotheses of the lemma, we have that I is surjective. Therefore, $\mathrm{Im}(C) = \mathrm{Im}(C \circ I) = \mathrm{Im}(B \circ F)$. So $\widetilde{\eta} = 0$ if and only if $B(\eta) \in \mathrm{Im}(B \circ F)$, i.e. if and only if there exists

$\bar{e} \in \mathrm{Ext}^1((E'', V''), (Q_1, W_1))$ such that $B(\eta) = B \circ F(\bar{e})$, i.e. if and only if there exists such an $\bar{e}$ such that $\eta - F(\bar{e}) \in \mathrm{Ker}(B) = \mathrm{Im}(A)$ (using the exactness of (6.33)). In other terms, $\widetilde{\eta} = 0$ if and only if there is a pair of extensions $(\widetilde{e}, \bar{e})$

$$\widetilde{e} \in \mathrm{Ext}^1((Q_4, W_4), (E_2, V_2)),\ \bar{e} \in \mathrm{Ext}^1((E'', V''), (Q_1, W_1)),$$

such that $\eta = A(\widetilde{e}) + F(\bar{e})$. So the set of the η's that we have to remove from $M_1([\mu], [\nu])$ coincides with the image $M_2([\mu], [\nu])$ of the linear map

$$A + F : \mathrm{Ext}^1((Q_4, W_4), (E_2, V_2)) \oplus \mathrm{Ext}^1((E'', V''), (Q_1, W_1)) \to$$
$$\to \mathrm{Ext}^1((E'', V''), (E_2, V_2)) = M_1([\mu], [\nu]).$$

Now we need to give a better description of the linear space $M_2([\mu], [\nu])$; in particular, we need to compute its dimension:

$$\dim\, M_2([\mu], [\nu]) = \dim\, \mathrm{Im}(A) + \dim\, \mathrm{Im}(F) - \dim\, M_3([\mu], [\nu])$$

where we denote by $M_3([\mu], [\nu])$ the linear space $\mathrm{Im}(A) \cap \mathrm{Im}(F) \subset M_2([\mu], [\nu])$. We need to compute the 3 values in the right hand side of the previous identity.

The space Im (A). We claim that in (6.33) we have:

$$\mathrm{Hom}((E'', V''), (E_2, V_2)) = 0. \tag{6.44}$$

Indeed, let us fix any morphism γ in such a space. Let us consider $\gamma_2 := \kappa \circ \gamma : (E'', V'') \to (Q_2, W_2)$ and $\gamma_{32} := \gamma_2 \circ \varepsilon : (Q_3, W_3) \to (Q_2, W_2)$. If γ_{32} is non-zero, then we get that it is an isomorphism and so γ_2 gives a splitting of (6.31), but this is impossible because we assumed that such an extension is non-split. Therefore, $\gamma_{32} = 0$; by exactness of (6.31) we get an induced morphism $\gamma_{42} : (Q_4, W_4) \to (Q_2, W_2)$ such that $\gamma_2 = \gamma_{42} \circ \delta$. Since (Q_4, W_4) and (Q_2, W_2) are both α_c-stable of the same slope and not isomorphic, then $\gamma_{42} = 0$, so $\gamma_2 = 0$. Therefore, by exactness of (6.30) we get an induced morphism $\gamma_1 : (E'', V'') \to (Q_1, W_1)$ such that $\gamma = \sigma \circ \gamma_1$. Then let us consider $\gamma_{31} := \gamma_1 \circ \varepsilon : (Q_3, W_3) \to (Q_1, W_1)$. If it is non-zero, then γ_1 gives a splitting for (6.31), so we get a contradiction. So $\gamma_{31} = 0$. Therefore $\gamma \circ \varepsilon = \sigma \circ \gamma_{31} = 0$. By exactness of (6.31), we get that there is an induced morphism $\gamma' : (Q_4, W_4) \to (E_2, V_2)$ such that $\gamma = \gamma' \circ \delta$. Since the graded of (E_2, V_2) does not contain an object isomorphic to (Q_4, W_4), then γ' is necessarily zero. So $\gamma = 0$ and the claim is proved.

By hypothesis we have $(Q_1, W_1) \simeq (Q_2, W_2) \not\simeq (Q_3, W_3)$. Since (6.30) is non-split, then the only proper α_c-(semi)stable subsystem of (E_2, V_2) is (Q_1, W_1). Since $(Q_3, W_3) \not\simeq (Q_1, W_1)$, then

$$\mathrm{Hom}((Q_3, W_3), (E_2, V_2)) = 0, \tag{6.45}$$

so the map A is injective by (6.33).

The space Im(F). We claim that in (6.35) we have:

$$\mathrm{Hom}((E'', V''), (Q_2, W_2)) = 0. \tag{6.46}$$

By contradiction, let us suppose that it contains a non-zero morphism. Then we get an induced non-zero morphism from (Q_3, W_3) or from (Q_4, W_4) to (Q_2, W_2); since all the (Q_i, W_i)'s are α_c-stable, then we would get an isomorphism. But we cannot have an isomorphism from (Q_4, W_4) to (Q_2, W_2) because of the hypotheses of the lemma. Also, by hypothesis we cannot have an induced isomorphism from (Q_3, W_3) to (Q_2, W_2). So we get that the map F is injective.

The space $M_3([\mu], [\nu])$. Let us consider the following diagram, that is commutative by naturality of the functors $\mathrm{Ext}^1(-,-)$'s.

$$\begin{tikzcd}
\mathrm{Ext}^1((Q_4, W_4), (Q_1, W_1)) \arrow[r, "L"] \arrow[d, "M"'] \arrow[dr, phantom, "\curvearrowright"] & \mathrm{Ext}^1((Q_4, W_4), (E_2, V_2)) \arrow[d, "A"] \\
\mathrm{Ext}^1((E'', V''), (Q_1, W_1)) \arrow[r, "F"] & \mathrm{Ext}^1((E'', V''), (E_2, V_2)).
\end{tikzcd}$$

Since such a diagram commutes, we get that $\mathrm{Im}(A \circ L) = \mathrm{Im}(F \circ M)$. We denote such a vector space by $M_3'([\mu], [\nu]) \subset M_3([\mu], [\nu])$. We want to prove that actually $M_3([\mu], [\nu]) = M_3'([\mu], [\nu])$. In order to do that, let us fix any object η in $M_3([\mu], [\nu])$; by definition of this space there exists a pair of objects

$$\widetilde{e} \in \mathrm{Ext}^1((Q_4, W_4), (E_2, V_2)),\ \bar{e} \in \mathrm{Ext}^1((E'', V''), (Q_1, W_1))$$

such that $A(\widetilde{e}) = \eta = F(\bar{e})$. Let us fix representatives for $\eta, \widetilde{e}$ and $\bar{e}$ given respectively by

$$\begin{gathered}
0 \to (E_2, V_2) \xrightarrow{\iota} (E, V) \xrightarrow{\lambda} (E'', V'') \to 0, \\
0 \to (E_2, V_2) \xrightarrow{\widetilde{\iota}} (\widetilde{E}, \widetilde{V}) \xrightarrow{\widetilde{\lambda}} (Q_4, W_4) \to 0, \\
0 \to (Q_1, W_1) \xrightarrow{\bar{\iota}} (\overline{E}, \overline{V}) \xrightarrow{\bar{\lambda}} (E'', V'') \to 0.
\end{gathered}$$

Since $A(\widetilde{e}) = \eta = F(\bar{e})$, we have a commutative diagram with exact rows as follows.

$$\begin{tikzcd}
0 \arrow[r] & (Q_1,W_1) \arrow[r, "\bar{\iota}"] \arrow[d, hook, "\sigma"] \arrow[dr, phantom, "\curvearrowright"] & (\overline{E},\overline{V}) \arrow[r, "\overline{\lambda}"] \arrow[d, hook, "\sigma'"] \arrow[dr, phantom, "\curvearrowright"] & (E'',V'') \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (E_2,V_2) \arrow[r, "\iota"] \arrow[d, equal] \arrow[dr, phantom, "\curvearrowright"] & (E,V) \arrow[r, "\lambda"] \arrow[d, two heads, "\delta'"] \arrow[dr, phantom, "\curvearrowright"] & (E'',V'') \arrow[r] \arrow[d, two heads, "\delta"] & 0 \\
0 \arrow[r] & (E_2,V_2) \arrow[r, "\widetilde{\iota}"] & (\widetilde{E},\widetilde{V}) \arrow[r, "\widetilde{\lambda}"] & (Q_4,W_4) \arrow[r] & 0.
\end{tikzcd}$$

By the snake lemma applied to the first 2 lines, we get that σ' is injective with cokernel (Q_2, W_2); again by the snake lemma applied to the second and to the third line we get that $\mathrm{Ker}\ \delta' \simeq \mathrm{Ker}\ \delta = (Q_3, W_3)$. Let us set $\gamma := \delta' \circ \sigma' : (\overline{E}, \overline{V}) \to (\widetilde{E}, \widetilde{V})$.

We claim that γ is not injective. By contradiction, let us suppose that γ is injective. If we apply the snake lemma on the whole diagram, we get a long exact sequence:

$$0 = \mathrm{Ker}\ \sigma \to \mathrm{Ker}\ \gamma \to \mathrm{Ker}\ \delta = (Q_3, W_3) \xrightarrow{\Theta}$$
$$\xrightarrow{\Theta} \mathrm{Coker}\ \sigma = (Q_2, W_2) \to \mathrm{Coker}\ \gamma \to \mathrm{Coker}\ \delta = 0.$$

If we assume that $\mathrm{Ker}\ \gamma = 0$; then we have that Θ is injective, so it is an isomorphism by lemma 1.0.4. So $\mathrm{Coker}\ \gamma = 0$, so γ is an isomorphism. Then this implies that

$$(Q_1,W_1)\oplus(Q_3,W_3)\oplus(Q_4,W_4) = \mathrm{gr}_{\alpha_c}(\overline{E},\overline{V}) \simeq \mathrm{gr}_{\alpha_c}(\overline{E},\overline{V})) = (Q_1,W_1)\oplus(Q_2,W_2)\oplus(Q_4,W_4).$$

This implies that $(Q_2, W_2) \simeq (Q_3, W_3)$, but this contradicts the hypotheses. Therefore we conclude that γ is not injective.

Now σ' is injective and $\mathrm{Ker}\ \delta' = \mathrm{Ker}\ \delta = (Q_3, W_3)$, so γ is not the zero morphism. Both the source and the target of such a morphism are α_c-semistable with the same slope by construction. Therefore by proposition 1.0.1 there exists a coherent system $\mathrm{Im}(\gamma) =: (\hat{E}, \hat{V}) \subset (\widetilde{E}, \widetilde{V})$, that is again α_c-semistable with the same slope as the previous objects. Then we consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & (Q_1,W_1) \arrow[r, "\bar{\iota}"] \arrow[d, equal] \arrow[dr, phantom, "\curvearrowright"] & (\overline{E},\overline{V}) \arrow[r, "\overline{\lambda}"] \arrow[d, two heads, "\gamma|_{(\hat{E},\hat{V})}"] \arrow[dr, phantom, "\curvearrowright"] & (E'',V'') \arrow[r] \arrow[d, two heads, "\delta"] & 0 \\
0 \arrow[r] & (Q_1,W_1) \arrow[r, "\gamma|_{(\hat{E},\hat{V})}\circ\bar{\iota}"] & (\hat{E},\hat{V}) \arrow[r, "\widetilde{\lambda}|_{(\hat{E},\hat{V})}"] & (Q_4,W_4) \arrow[r] & 0.
\end{tikzcd} \tag{6.47}$$

Using the previous diagram, this diagram is obviously commutative. Moreover, the second line is exact. Indeed:

- $\gamma|_{(\hat{E},\hat{V})} \circ \bar{\imath} = \widetilde{\imath} \circ \sigma$, so it is injective since both $\widetilde{\imath}$ and σ are so;

- $\widetilde{\lambda}|_{(\hat{E},\hat{V})}$ is surjective because the square on the right is commutative and both δ and $\overline{\lambda}$ are surjective;

- $\widetilde{\lambda}|_{(\hat{E},\hat{V})} \circ \gamma|_{(\hat{E},\hat{V})} \circ \bar{\imath} = \delta \circ \overline{\lambda} \circ \bar{\imath} = 0$; so $\mathrm{Im}(\gamma|_{(\hat{E},\hat{V})} \circ \bar{\imath}) \subseteq \mathrm{Ker}(\widetilde{\lambda}|_{(\hat{E},\hat{V})}) \subsetneq (\hat{E},\hat{V})$. Since γ is not injective, then the length of any α_c-Jordan-Hölder filtration of $(\hat{E},\hat{V})$ is strictly less than the length of any α_c-Jordan-Hölder filtration of $(\overline{E},\overline{V})$, so it is less or equal than 2. So necessarily the first inclusion must be an identity.

If we denote by $\hat{e} \in \mathrm{Ext}^1((Q_4,W_4),(Q_1,W_1))$ the extension in the second line of (6.47), then such a diagram proves that $\bar{e} = M(\hat{e})$. So we have proved that for every $\eta \in M_3([\mu],[\nu])$ there exists $\hat{e} \in \mathrm{Ext}^1((Q_4,W_4),(Q_1,W_1))$ such that $\eta = F \circ M(\hat{e})$. So this proves that

$$M_3([\mu],[\nu]) = M_3'([\mu],[\nu]). \tag{6.48}$$

The space $M_2([\mu],[\nu])$. We need to compute dim $M_2([\mu],[\nu])$. In order to do that, we fix the following notation:

$$\dim \mathrm{Ext}^1((Q_4,W_4),(E_2,V_2)) := d, \quad \dim \mathrm{Ext}^1((E'',V''),(Q_1,W_1)) := e,$$
$$\dim \mathrm{Ext}^1((Q_4,W_4),(Q_1,W_1)) := f.$$

We have already proved that the maps A and F and M are injective, so dim $\mathrm{Im}(A) = d$, dim $\mathrm{Im}(F) = e$. Moreover, using the hypotheses together with (6.38) we get that also M is injective. Therefore, dim $M_3([\mu],[\nu]) = \dim \mathrm{Im}(F \circ M) = f$. Then we get:

$$\dim M_2([\mu],[\nu]) = d + e - f. \tag{6.49}$$

For the moment we have proved that for all triples (μ,ν,η) as before with $\mu,\nu \neq 0$ and $\eta \in M([\mu],[\nu]) = M_1([\mu],[\nu]) \smallsetminus M_2([\mu],[\nu])$, we have that the induced (E,V) has unique α_c-Jordan-Hölder filtration and graded $\oplus_{i=1}^4 (Q_i,W_i)$. The set parametrizing all the (E_2,V_2)'s is given by $\mathbb{P}(\mathrm{Ext}^1((Q_2,W_2),(Q_1,W_1)))$; analogously, the set parametrizing all the (E'',V'')'s is given by $\mathbb{P}(\mathrm{Ext}^1((Q_4,W_4),(Q_3,W_3)))$. Moreover, for every (E_2,V_2) and (E'',V'') in those 2 spaces, we have that $\mathrm{Aut}(E'',V'') = \mathbb{C}^*$ and $\mathrm{Aut}(E_2,V_2) = \mathbb{C} \times \mathbb{C}^*$. Moreover, we have proved in (6.44) that $\mathrm{Hom}((E'',V''),(E_2,V_2)) = 0$.

So we have an induced action of $\mathbb{C} \times \mathbb{C}^*$ on $M_1([\mu],[\nu])$ as follows

$$\begin{array}{rcl} \mathbb{C} \times \mathbb{C}^* \times M_1([\mu],[\nu]) & \to & M_1([\mu],[\nu]) \\ (\xi,\tau,\eta) & \mapsto & \eta(\xi,\tau), \end{array}$$

where $\eta(\xi,\tau)$ has a representative of the form

$$0 \to (E_2, V_2) \xrightarrow{\iota(\xi,\tau)} (E, V) \xrightarrow{\lambda} (E'', V'') \to 0,$$

where $\iota(\xi, \tau) := \iota \circ (\xi \cdot \sigma \circ \kappa + \tau \cdot \mathrm{id}_{(E_2,V_2)})$. Now we want to prove that $M_2([\mu], [\nu])$ is sent to itself by such an action. So let us suppose that $\eta \in M_2([\mu], [\nu])$ with representative (6.32); by definition of $M_2([\mu], [\nu])$ we have $D \circ B(\eta) = 0$, i.e. we have a commutative diagram with exact rows (obtained by (6.40), (6.42) and (6.43)) and such that the last line is split, as follows.

$$\begin{tikzcd}
0 \arrow[r] & (E_2, V_2) \arrow[r, "\iota"] & (E, V) \arrow[r, "\lambda"] & (E'', V'') \arrow[r] & 0 \\
0 \arrow[r] & (E_2, V_2) \arrow[r, "\alpha"] \arrow[u, equal] \arrow[d, two heads, "\kappa"] \arrow[ur, phantom, "\curvearrowright"] & (E_3, V_3) \arrow[r, "\beta"] \arrow[u, hook, "\varepsilon'"] \arrow[d, two heads, "\kappa'"] \arrow[ur, phantom, "\curvearrowright"] & (Q_3, W_3) \arrow[r] \arrow[u, hook, "\varepsilon"] \arrow[d, equal] & 0 \\
0 \arrow[r] & (Q_2, W_2) \arrow[r, "\theta"] \arrow[ur, phantom, "\curvearrowright"] & (E_3, V_3)/(Q_1, W_1) \arrow[r, "\zeta"] \arrow[ur, phantom, "\curvearrowright"] & (Q_3, W_3) \arrow[r] & 0.
\end{tikzcd} \tag{6.50}$$

Now let us fix any pair $(\xi, \tau) \in \mathbb{C} \times \mathbb{C}^*$ and let us set $\alpha(\xi, \tau) := \alpha \circ (\xi \cdot \sigma \circ \kappa + \tau \cdot \mathrm{id}_{(E_2,V_2)})$. Then we have:

$$\iota(\xi, \tau) = \varepsilon' \circ \alpha(\xi, \tau), \quad \kappa' \circ \alpha(\xi, \tau) = \tau \cdot \theta \circ \kappa.$$

Then we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & (E_2, V_2) \arrow[r, "\iota(\xi,\tau)"] & (E, V) \arrow[r, "\lambda"] & (E'', V'') \arrow[r] & 0 \\
0 \arrow[r] & (E_2, V_2) \arrow[r, "\alpha(\xi,\tau)"] \arrow[u, equal] \arrow[d, two heads, "\kappa"] \arrow[ur, phantom, "\curvearrowright"] & (E_3, V_3) \arrow[r, "\beta"] \arrow[u, hook, "\varepsilon'"] \arrow[d, two heads, "\kappa'"] \arrow[ur, phantom, "\curvearrowright"] & (Q_3, W_3) \arrow[r] \arrow[u, hook, "\varepsilon"] \arrow[d, equal] & 0 \\
0 \arrow[r] & (Q_2, W_2) \arrow[r, "\tau\cdot\theta"] \arrow[ur, phantom, "\curvearrowright"] & (E_3, V_3)/(Q_1, W_1) \arrow[r, "\zeta"] \arrow[ur, phantom, "\curvearrowright"] & (Q_3, W_3) \arrow[r] & 0.
\end{tikzcd} \tag{6.51}$$

The last line is a representative of $D \circ B(\eta(\xi, \tau))$. Since the last line of (6.50) is split, then we get that also the last line of (6.51) is split. So we get that $\eta(\xi, \tau) \in M_2([\mu], [\nu])$. So we have that $\mathbb{C} \times \mathbb{C}^*$ acts on $M([\mu], [\nu]) := M_1([\mu], [\nu]) \setminus M_2([\mu], [\nu])$. Moreover, we can also prove easily that $(\mathbb{C} \times \mathbb{C}^*)(\mathrm{Im}(F)) \subset \mathrm{Im}(F) \subset M_2([\mu], [\nu])$. Therefore, such a group acts also on the subsets

$$M' := M_1([\mu],[\nu]) \smallsetminus \mathrm{Im}(F), \quad M'' := M_2([\mu],[\nu]) \smallsetminus \mathrm{Im}(F).$$

By construction we have:

$$M([\mu],[\nu]) = M_1([\mu],[\nu]) \smallsetminus M_2([\mu],[\nu]) = M' \smallsetminus M''.$$

Now let us consider the exact sequence (6.35). We have already proved in (6.46) that $\mathrm{Hom}((E'',V''),(Q_2,W_2)) = 0$. Moreover, by hypothesis

$$\mathrm{Ext}^2((E'',V''),(Q_1,W_1)) = 0.$$

Since $(Q_1,W_1) \simeq (Q_2,W_2)$, we get a short exact sequence

$$0 \to \mathrm{Ext}^1((E'',V''),(Q_1,W_1)) \xrightarrow{F}$$
$$\xrightarrow{F} \mathrm{Ext}^1((E'',V''),(E_2,V_2)) \to \mathrm{Ext}^1((E'',V''),(Q_1,W_1)) \to 0.$$

So we have a (non-canonical) isomorphism

$$M' = M_1([\mu],[\nu]) \smallsetminus \mathrm{Im}(F) \simeq$$
$$\simeq \mathrm{Ext}^1((E'',V''),(Q_1,W_1)) \times (\mathrm{Ext}^1((E'',V''),(Q_1,W_1)) \smallsetminus \{0\}).$$

Under this isomorphism the action of $\mathbb{C} \times \mathbb{C}^*$ on M' is given for every pair of scalars (ξ,τ) and for every $(\eta_1,\eta_2) \in M'$ as: $(\xi,\tau)\cdot(\eta_1,\eta_2) = (\tau\cdot\eta_1 + \xi\cdot\eta_2, \tau\cdot\eta_2)$. Now if we write $c := \dim\, M_1([\mu],[\nu])$, we get:

$$M' \simeq \mathbb{C}^c \smallsetminus \mathbb{C}^e = \mathbb{C}^e \times (\mathbb{C}^{c-e} \smallsetminus \{0\}),$$
$$M'' \simeq \mathbb{C}^{d+e-f} \smallsetminus \mathbb{C}^e = \mathbb{C}^e \times (\mathbb{C}^{d-f} \smallsetminus \{0\}).$$

Then:

$$\overline{M}([\mu],[\nu]) := M([\mu],[\nu])/(\mathbb{C}\times\mathbb{C}^*) = (M' \smallsetminus M'')/(\mathbb{C}\times\mathbb{C}^*) =$$
$$= (\mathbb{C}^{e-1} \times \mathbb{P}^{c-e-1}) \smallsetminus (\mathbb{C}^{e-1} \times \mathbb{P}^{d-f-1}).$$

So we conclude. □

Proposition 6.2.2. *Let us fix any triple (n,d,k), a critical value α_c for it and any quadruple $(n_i,d_i,k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c;n,d,k)$. Let us assume that conditions (6.24), respectively (6.25), are satisfied. Moreover, let us suppose that $(n_1,k_1) = (n_2,k_2)$. Let us denote by G' the set of all the (E,V)'s in $G^+(\alpha_c;n,d,k)$, respectively in $G^-(\alpha_c;n,d,k)$, that have unique α_c-Jordan-Hölder filtration at α_c and graded given by $\oplus_{i=1}^4(Q_i,W_i)$, such that*

$$(Q_1,W_1) \simeq (Q_2,W_2) \not\simeq (Q_3,W_3), \quad (Q_4,W_4) \not\simeq (Q_i,W_i) \quad \forall\, i=1,3,$$
$$Ext^2\left((Q_4,W_4),(Q_1,W_1)\right) = 0 = Ext^2\left((E'',V''),(Q_1,W_1)\right),$$

where (E'', V'') is any non-split extension of (Q_4, W_4) by (Q_3, W_3). Then there exists a finite family $\{R_{a,b,c,d,e,f;i,j,l}\}$ of schemes for (a, b, c, d, e, f) in $\mathbb{N}^6$, and i, j, l varying in finite sets (for a, b, c, d, e, f fixed), together with injective morphisms to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Every scheme $R_{a,b,c,d,e,f;i,j,l}$ comes with a triple of morphisms

$$\begin{aligned}\varphi_{a,b,c,d,e,f;i,j,l} : R_{a,b,c,d,e,f;i,j,l} &\longrightarrow U_{a,b,c,d,e,f:i,j,l} \subset R_{a;i} \times R^{b;j},\\ \varphi_{a;i} : R_{a;i} &\longrightarrow U_{a;i} \subset G_1 = G_2,\\ \varphi^{b;j} : R^{b;j} &\longrightarrow U^{b;j} \subset G_3 \times G_4\end{aligned}$$

where:

- *$\varphi_{a,b,c,d,e,f;i,j,l}$ has fibers isomorphic to $\mathbb{C}^{e-1} \times (\mathbb{P}^{c-e-1} \smallsetminus \mathbb{P}^{d-f-1})$, $\varphi_{a;i}$ has fibers isomorphic to $\mathbb{P}^{a-1}$ and $\varphi^{b;j}$ has fibers isomorphic to $\mathbb{P}^{b-1}$;*
- *$\{U_{a;i}\}_i$ is a finite disjoint locally closed covering of*

$$U_a := \{(Q_1, W_1) \in G_1 \text{ s.t. } \dim \mathrm{Ext}^1((Q_1, W_1), (Q_1, W_1)) = a\};$$

  every U_a is a locally closed subscheme of G_1 and so are all the $U_{a;i}$'s;
- *$\{U^{b;j}\}_j$ is a finite disjoint locally closed covering of*

$$U^b := \{((Q_3, W_3), (Q_4, W_4)) \in G_3 \times G_4 \text{ s.t. } \dim \mathrm{Ext}^1((Q_4, W_4), (Q_3, W_3)) = b\};$$

  every U^b is a locally closed subscheme of $G_3 \times G_4$ and so are all the $U^{b;j}$'s;
- *$\{U_{a,b,c,d,e,f;i,j,l}\}_l$ is a finite disjoint locally closed covering of*

$$\begin{aligned}U_{a,b,c,d,e,f;i,j} := \{((E_2, V_2), (E'', V'')) \in R_{a;i} \times R^{b;j} \text{ s.t. } \dim \mathrm{Ext}^1((E'', V''), (E_2, V_2)) = c,\\ \dim \mathrm{Ext}^1(\widetilde{\varphi}^{b;j}(E'', V''), (E_2, V_2)) = d, \dim \mathrm{Ext}^1((E'', V''), \varphi_{a;i}(E_2, V_2)) = e,\\ \dim \mathrm{Ext}^1(\widetilde{\varphi}^{b;j}(E'', V''), \varphi_{a;i}(E_2, V_2)) = f, \quad \varphi_{a;i}(E_2, V_2) \not\simeq \overline{\varphi}^{b;j}(E'', V'')\}, \qquad (6.52)\end{aligned}$$

  where $\widetilde{\varphi}^{b;j}$ is the composition of $\varphi^{b;j}$ with the projection to G_4 and $\overline{\varphi}^{b;j}$ is the composition of $\varphi^{b;j}$ with the projection to G_3. Every $U_{a,b,c,d,e,f;i,j}$ is locally closed in $R_{a;i} \times R^{b;j}$ and so are all the $U_{a,b,c,d,e,f;i,j,l}$'s. The last condition of (6.52) can be omitted if $(n_1, k_1) = (n_2, k_2) \neq (n_3, k_3)$; it is necessary if $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$.

Proof. Let us fix any sequence $(a, b, c, d, e, f) \in \mathbb{N}^6$ and let us denote by $(\hat{\mathcal{Q}}_i, \hat{\mathcal{W}}_i)$ the local universal families over the various Quot schemes $\hat{G}_i$'s for $i = 1, \cdots, 4$. Since $(n_1, k_1) = (n_2, k_2)$, then $d_1 = d_2$ and $\hat{G}_1 = \hat{G}_2$, so in particular, $(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) = (\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2)$.

Let us consider the scheme $\hat{G}_3 \times \hat{G}_4$ and let us denote by $\hat{p}_3$ and $\hat{p}_4$ the projections from such a scheme to its factors. Having fixed b, let us consider the locally closed subscheme of $\hat{G}_3 \times \hat{G}_4$ defined as:

$$\hat{U}^b := \{t \in \hat{G}_3 \times \hat{G}_4 \text{ s.t. } \dim \operatorname{Ext}^1((\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)_t, ((\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)_t) = b\}.$$

By hypothesis for each $t \in \hat{G}_3 \times \hat{G}_4$ we have:

$$\operatorname{Hom}((\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)_t, (\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)_t) = 0.$$

So we can apply the usual results on families of classes of extensions and we get that there is a finite disjoint covering $\{\hat{U}^{b;j}\}_j$ of $\hat{U}^b$ by locally closed subschemes; for every j there is a locally free sheaf on $\hat{U}^{b;j}$

$$\hat{\mathcal{H}}^{b;j} := \mathcal{E}xt^1_{\pi_{\hat{U}^{b;j}}}\Big((\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4), (\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)\Big)^\vee$$

and a projective bundle

$$\hat{\varphi}^{b;j} : \hat{R}^{b;j} := \mathbb{P}(\hat{\mathcal{H}}^{b;j}) \longrightarrow \hat{U}^{b;j} \subset \hat{G}_3 \times \hat{G}_4$$

with fibers isomorphic to $\mathbb{P}^{b-1}$. By abuse of notation, we denote by $\hat{\varphi}^{b;j}$ also the composition $\hat{R}^{b;j} \to \hat{G}_3 \times \hat{G}_4$. Moreover, there exists a family $(\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j})$ parametrized by $\hat{R}^{b;j}$ and a family of classes of non-split extensions parametrized by $\hat{R}^{b;j}$:

$$\begin{aligned} 0 \to (\hat{\varphi}'^{b;j}, \hat{\varphi}^{b;j})^*(\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3) \otimes_{\hat{R}^{b;j}} \mathcal{O}_{\hat{R}^{b;j}}(1) \xrightarrow{\varepsilon^{b;j}} \\ \xrightarrow{\varepsilon^{b;j}} (\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j}) \xrightarrow{\delta^{b;j}} (\hat{\varphi}'^{b;j}, \hat{\varphi}^{b;j})^*(\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4) \to 0. \end{aligned} \tag{6.53}$$

Such an extension is universal in the sense of corollary 4.4.4. Now let us pass to the scheme $\hat{G}_1 = \hat{G}_2$. Since the objects of this scheme are α_c-stable, we get that for each point $t \in \hat{G}_1$ we have

$$\dim \operatorname{Hom}((\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t, (\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = 1.$$

Let us apply proposition 4.6.2 on $\hat{G}_1$ with the two families of coherent systems both coinciding with $(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)$. Then we get that the set

$$\hat{U}_a := \{t \in \hat{G}_1 \text{ s.t. } \dim \operatorname{Ext}^1((\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t, (\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = a\}$$

is a locally closed subscheme of $\hat{G}_1$ and there exists a finite disjoint covering of $\hat{U}_a$ by locally closed subschemes $\hat{U}_{a;i}$; on each $\hat{U}_{a;i}$ there is a locally free sheaf

$$\hat{\mathcal{H}}_{a;i} := \mathcal{E}xt^1_{\pi_{\hat{U}_{a;i}}}\Big((\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1), (\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)\Big)^\vee$$

and a projective bundle

$$\hat{\varphi}_{a;i} : \hat{R}_{a;i} := \mathbb{P}(\hat{\mathcal{H}}_{a;i}) \longrightarrow \hat{U}_{a;i} \subset \hat{G}_1$$

with fibers isomorphic to $\mathbb{P}^{a-1}$. By abuse of notation we denote by $\hat{\varphi}_{a;i}$ also the composition $\hat{R}_{a;i} \to \hat{G}_1$. Moreover, there is a family of classes of extensions $\{e_r\}_{r\in\hat{R}_{a;i}}$ over $\hat{R}_{a;i}$ of $(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)$ by $(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \otimes_{\hat{R}_{a;i}} \mathcal{O}_{\hat{R}_{a;i}}(1)$. Such a family of extensions is universal on the category of reduced $\hat{U}_{a;i}$-schemes. Now by definition of family of classes of extensions, for each i there is an open covering $\{\hat{R}^k_{a;i}\}_{k\in K}$ of $\hat{R}_{a;i}$ together with a family of classes of non-split extensions:

$$0 \to ((\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \otimes_{\hat{R}_{a;i}} \mathcal{O}_{\hat{R}_{a;i}}(1))|_{\hat{R}^k_{a;i}} \xrightarrow{\sigma^k_{a;i}} (\hat{\mathcal{E}}^k_{a;i}, \hat{\mathcal{V}}^k_{a;i}) \xrightarrow{\kappa^k_{a;i}} (\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)|_{\hat{R}^k_{a;i}} \to 0 \tag{6.54}$$

over $\hat{R}^k_{a;i}$ such that e_r is the restriction of (6.54) for each $r \in \hat{R}_{a;i}$. Since $\hat{R}^k_{a;i}$ is noetherian, then we can assume that K is a finite set and we denote its elements by $\{k_1 < \dots < k_r\}$. Then for each set $k_\bullet = \{k'_1 < \dots < k'_s\} \subset K$ we define the locally closed subscheme

$$\hat{R}^{k_\bullet}_{a;i} := (\hat{R}^{k'_1}_{a;i} \cap \dots \cap \hat{R}^{k'_s}_{a;i}) \smallsetminus (\hat{R}^{k'_{s+1}}_{a;i} \cup \dots \cup \hat{R}^{k'_r}_{a;i}) \subset \hat{R}_{a;i},$$

where $\{k'_{s+1} < \dots < k'_r\}$ is the complement of $k_\bullet$ in K. Since K is finite, we get that these schemes form a finite locally closed disjoint covering of $\hat{R}_{a;i}$. For each $k_\bullet$ we consider the embedding of $\hat{R}^{k_\bullet}_{a;i}$ in $\hat{R}^{k'_1}_{a;i}$ and we pullback (6.54) (for $k = k'_1$) via that morphism. So for each $k_\bullet$ we get a family of non-split extensions:

$$0 \to ((\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \otimes_{\hat{R}_{a;i}} \mathcal{O}_{\hat{R}_{a;i}}(1))|_{\hat{R}^{k_\bullet}_{a;i}} \xrightarrow{\sigma^{k_\bullet}_{a;i}} (\hat{\mathcal{E}}^{k_\bullet}_{a;i}, \hat{\mathcal{V}}^{k_\bullet}_{a;i}) \xrightarrow{\kappa^{k_\bullet}_{a;i}} (\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)|_{\hat{R}^{k_\bullet}_{a;i}} \to 0 \tag{6.55}$$

over $\hat{R}^{k_\bullet}_{a;i}$. Now for every choice of indices $(a, b; i, j, k_\bullet)$ we consider the scheme $T := \hat{R}^{k_\bullet}_{a;i} \times \hat{R}^{b;j}$ together with the projections $\hat{p}_{12}$ and $\hat{p}_{34}$ to the 2 factors. Then we define $\hat{U}^{k_\bullet}_{a,b,c,d,e,f;i;j}$ as the subscheme of T described by

$$\begin{gathered}
\{t \in T \text{ s.t. } \dim \operatorname{Ext}^1((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j})_t, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i}, \hat{\mathcal{V}}^{k_\bullet}_{a;i})_t) = c,\\
\dim \operatorname{Ext}^1((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\varphi}'^{b;j}, \hat{\varphi}^{b;j})^*(\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)_t, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i}, \hat{\mathcal{V}}^{k_\bullet}_{a;i})_t) = d,\\
\dim \operatorname{Ext}^1((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j})_t, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = e,\\
\dim \operatorname{Ext}^1((p'_{34}, p_{34})^*(\hat{\varphi}'^{b;j}, \hat{\varphi}^{b;j})^*(p'_4, p_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)_t, (p'_{12}, p_{12})^*(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = f,\\
\operatorname{Hom}((p'_{34}, p_{34})^*(\hat{\varphi}'^{b;j}, \hat{\varphi}^{b;j})^*(p'_3, p_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)_t, (p'_{12}, p_{12})^*(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = 0\}.
\end{gathered}$$

By proposition 1.0.5, this is a locally closed subscheme of $\hat{R}^{k_\bullet}_{a;i} \times \hat{R}^{b;j}$. By construction, we have that each $\hat{R}^{k_\bullet}_{a;i}$ is locally closed in $\hat{R}_{a;i}$, so each $\hat{U}^{k_\bullet}_{a,b,c,d,e,f;i,j}$ is locally closed in $\hat{R}_{a;i} \times \hat{R}^{b;j}$. Then let us consider the scheme $\hat{U}_{a,b,c,d,e,f;i,j}$ defined as the subscheme of $\hat{R}_{a;i} \times \hat{R}^{b;j}$ covered by all the schemes of the form $\hat{U}^{k_\bullet}_{a,b,c,d,e,f;i,j}$. This is a disjoint covering of such a scheme by locally closed subschemes of $\hat{R}_{a;i} \times \hat{R}^{b;j}$. Therefore, this gives a disjoint locally closed covering of $\hat{U}_{a,b,c,d,e,f;i,j}$.

By construction, for each $t \in \hat{U}^{k_\bullet}_{a,b,c,d,e,f;i,j}$, we have that the coherent systems $(\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i}, \hat{\mathcal{V}}^{k_\bullet}_{a;i})_t$ and $(\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}^{b,j}, \hat{\mathcal{V}}^{b;j})_t$ are both α_c-semistable. Moreover the numerical conditions we are considering and the condition that both (6.55) and (6.53) are non-split over each fiber prove that:

$$\mathrm{Hom}((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j})_t, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i}, \hat{\mathcal{V}}^{k_\bullet}_{a;i})_t) = 0$$

for all $t \in \hat{U}^{k_\bullet}_{a,b,c,d,e,f;i,j}$ (see (6.44)). Moreover, by construction

$$\dim \mathrm{Ext}^1((p'_{34}, p_{34})^*(\hat{\mathcal{E}}'^{b;j}, \hat{\mathcal{V}}'^{b;j})_t, (p'_{12}, p_{12})^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i}, \hat{\mathcal{V}}^{k_\bullet}_{a;i})_t)$$

is constant on $\hat{U}^{k_\bullet}_{a,b,c,d,e,f;i,j}$. Then we can apply proposition 4.6.3 for such a scheme and we get that there is a finite disjoint locally closed covering $\{\hat{U}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}\}_{l_1}$ of $\hat{U}^{k_\bullet}_{a,b,c,d,e,f;i,j}$. For each l_1 there is a locally free sheaf on the corresponding scheme:

$$\hat{\mathcal{H}}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j} := \mathcal{E}xt^1_{\pi_{\hat{U}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}}}\Big((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j}), (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i}, \hat{\mathcal{V}}^{k_\bullet}_{a;i})\Big)^\vee$$

and a vector bundle

$$\hat{\varphi}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j} : \hat{V}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j} := \mathbb{V}(\hat{\mathcal{H}}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}) \longrightarrow \hat{U}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}$$

together with a family $(\hat{\mathcal{E}}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}, \hat{\mathcal{V}}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j})$ parametrized by $\hat{V}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}$. Moreover, there is a universal extension over that scheme given by:

$$\begin{aligned} 0 \to (\hat{\varphi}'^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}, \hat{\varphi}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j})^*(\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i}, \hat{\mathcal{V}}^{k_\bullet}_{a;i}) &\xrightarrow{\iota^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}} \\ \xrightarrow{\iota^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}} (\hat{\mathcal{E}}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}, \hat{\mathcal{V}}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}) &\xrightarrow{\lambda^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}} \\ \xrightarrow{\lambda^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}} (\hat{\varphi}'^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}, \hat{\varphi}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j})^*(\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j}) &\to 0. \end{aligned} \tag{6.56}$$

Now let us consider the pullback of the sequence (6.53) via the morphism $\hat{U}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j} \hookrightarrow \hat{R}_{a,i} \times \hat{R}^{b;j} \stackrel{\hat{p}_{34}}{\to} \hat{R}^{b;j}$ (we denote again by $\hat{p}_{34}$ this morphism) and let us apply to the new exact sequence the functor

$$\mathcal{H}om_{\pi_{\hat{U}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}}}\Big(-, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i}, \hat{\mathcal{V}}^{k_\bullet}_{a;i})\Big).$$

Then we get a long exact sequence:

$$\mathcal{H}om_{\pi_{\hat{U}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}}}\Big((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\varphi}'^{b;j}, \hat{\varphi}^{b;j})^*(\hat{p}'_3, \hat{p}_3)^*((\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3) \otimes_{\hat{R}^{b;j}} \mathcal{O}_{\hat{R}^{b;j}}(1)), (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i}, \hat{\mathcal{V}}^{k_\bullet}_{a;i})\Big) \to$$

$$\to \mathcal{E}xt^1_{\pi_{\hat{U}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'^{b;j},\hat{\varphi}^{b;j})^*(\hat{p}'_4,\hat{p}_4)^*(\hat{\mathcal{Q}}_4,\hat{\mathcal{W}}_4),(\hat{p}'_{12},\hat{p}_{12})^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i},\hat{\mathcal{V}}^{k_\bullet}_{a;i})\Big)\xrightarrow{A}$$

$$\xrightarrow{A}\mathcal{E}xt^1_{\pi_{\hat{U}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}}}\Big((p'_{34},p_{34})^*(\hat{\mathcal{E}}^{b;j},\hat{\mathcal{V}}^{b;j}),(\hat{p}'_{12},\hat{p}_{12})^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i},\hat{\mathcal{V}}^{k_\bullet}_{a;i})\Big)\to\cdots$$

Let us consider the first sheaf in that exact sequence. As we said in the previous lemma (identity (6.45)), for each pair $((Q_3,W_3),(E_2,V_2))$ we get that $\mathrm{Hom}((Q_3,W_3),(E_2,V_2))=0$. Then by base change and if necessary by restricting to a subcovering of $\hat{U}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}$ the first sheaf is zero. Moreover, the second sheaf of the previous sequence is locally free if restricted to any such subscheme (this is because of the definition of $\hat{U}^{k_\bullet}_{a,b,c,d,e,f;i,j}$).

Analogously, let us consider the following long exact sequence obtained by applying the functor

$$\mathcal{H}om_{\pi_{\hat{U}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j},\hat{\mathcal{V}}^{b;j}),-\Big)$$

to the pullback of (6.55) from $\hat{R}^{k_\bullet}_{a;i}$ to $\hat{U}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}$:

$$\cdots\to\mathcal{H}om_{\pi_{\hat{U}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j},\hat{\mathcal{V}}^{b;j}),(\hat{p}'_{12},\hat{p}_{12})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)|_{\hat{R}^{k_\bullet}_{a;i}}\Big)\to$$

$$\to\mathcal{E}xt^1_{\pi_{\hat{U}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j},\hat{\mathcal{V}}^{b;j}),$$

$$(\hat{p}'_{12},\hat{p}_{12})^*((\hat{\varphi}_{a;i},\hat{\varphi}_{a;i})^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)\otimes_{\hat{R}_{a;i}}\mathcal{O}_{\hat{R}_{a;i}}(1))|_{\hat{R}^{k_\bullet}_{a;i}}\Big)\xrightarrow{F}$$

$$\xrightarrow{F}\mathcal{E}xt^1_{\pi_{\hat{U}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j},\hat{\mathcal{V}}^{b;j}),(\hat{p}'_{12},\hat{p}_{12})^*(\hat{\mathcal{E}}^{k_\bullet}_{a;i},\hat{\mathcal{V}}^{k_\bullet}_{a;i})\Big).$$

Using base change together with (6.46) and the fact that $(Q_2,W_2)\simeq(Q_1,W_1)$, we get that the first sheaf is zero. Moreover, having fixed the invariant e, the second sheaf of this exact sequence is locally free.

Let us denote by l any collection of indices $(k_\bullet,l_1)$. By construction l varies over a finite set, so we get a finite disjoint locally closed covering $\{\hat{U}_{a,b,c,d,e,f;i,j,l}\}_l$ of $\hat{U}_{a,b,c,d,e,f;i,j}$. According to this notation, we denote by $\hat{V}_{a,b,c,d,e,f;i,j,l}$, $(\hat{\mathcal{E}}_{a,b,c,d,e,f;i,j,l},\hat{\mathcal{V}}_{a,b,c,d,e,f;i,j,l})$, etc. all the various objects we have defined so far.

Moreover, by restricting to any subscheme $\hat{U}_{a,b,c,d,e,f;i,j,l}$ we can rewrite the previous 2 long exact sequences as injective morphisms of vector bundles as follows:

$$0\to\hat{V}^1_{a,b,c,d,e,f;i,j,l}\xrightarrow{A}\hat{V}_{a,b,c,d,e,f;i,j,l}$$

and

$$0\to\hat{V}^2_{a,b,c,d,e,f;i,j,l}\xrightarrow{F}\hat{V}_{a,b,c,d,e,f;i,j,l}.$$

According to the computations of the previous lemma, the induced morphism of vector bundles

$$A+F:\hat{V}^1_{a,b,c,d,e,f;i,j,l}\oplus\hat{V}^2_{a,b,c,d,e,f;i,j,l}\to\hat{V}_{a,b,c,d,e,f;i,j,l}$$

has constant rank (equal to $d+e-f$), so its image is a subvector bundle of $\hat{V}_{a,b,c,d,e,f;i,j,l}$ (see for example, [LP2, proposition 1.7.2]). We denote such a vector bundle by $\hat{V}'_{a,b,c,d,e,f;i,j,l}$.

Now we perform the same construction we did in the proof of proposition 6.1.4 in order to get an action of $\mathbb{C}\times\mathbb{C}^*$ on $\hat{V}_{a,b,c,d,e,f;i,j,l}$; using lemma 6.2.1, such an action restricts to an action of the same group on the scheme

$$\hat{Q}_{a,b,c,d,e,f;i,j,l}:=\hat{V}_{a,b,c,d,e,f;i,j,l}\smallsetminus\hat{V}'_{a,b,c,d,e,f;i,j,l}.$$

As in the already cited proposition we get that such an action is compatible with the fibration to $\hat{U}_{a,b,c,d,e,f;i,j,l}$. So it makes sense to consider the quotient

$$\hat{R}_{a,b,c,d,e,f;i,j,l}:=\hat{Q}_{a,b,c,d,e,f;i,j,l}/(\mathbb{C}\times\mathbb{C}^*)$$

and the induced fibration

$$\hat{\varphi}_{a,b,c,d,e,f;i,j,l}:\hat{R}_{a,b,c,d,e,f;i,j,l}\longrightarrow\hat{U}_{a,b,c,d,e,f;i,j,l}.$$

The fibers of such a fibration are described as in lemma 6.2.1, so each fiber is isomorphic to $\mathbb{C}^{e-1}\times(\mathbb{P}^{c-e-1}\smallsetminus\mathbb{P}^{d-f-1})$. Now we recall that on $\hat{Q}_{a,b,c,d,e,f;i,j,l}$ we have a family $(\hat{\mathcal{E}}_{a,b,c,d,e,f;i,j,l},\hat{\mathcal{V}}_{a,b,c,d,e,f;i,j,l})$ (given by restriction of the central term of (6.56)), such that if we denote by q any point of $\hat{Q}_{a,b,c,d,e,f;i,j,l}$ and by (E,V) the restriction of such a family to q, then by lemma 6.2.1 we have a triple of exact sequences:

$$\begin{gathered}0\to(Q_1,W_1)\to(E_2,V_2)\to(Q_1,W_1)\to 0,\\ 0\to(Q_3,W_3)\to(E'',V'')\to(Q_4,W_4)\to 0,\\ 0\to(E_2,V_2)\to(E,V)\to(E'',V'')\to 0,\end{gathered}$$

such that both the first 2 sequences and the induced sequence

$$0\to(Q_2,W_2)\to(E_3,V_3)/(Q_1,W_1)\to(Q_3,W_3)\to 0$$

are non-split. Then by lemma 6.2.1 (E,V) has a unique α_c-JHF. Now let us assume conditions (6.24) (an analogous proof holds for conditions (6.25)). Then (E,V) belongs to

$$G'\subset G^+(\alpha_c;n,d,k)\subset G(\alpha_c^+;n,d,k).$$

Then by the universal property of the scheme $G(\alpha_c^+;n,d,k)$ we get that the previous family induces a morphism

$$\hat{\omega}_{a,b,c,d,e,f;i,j,l}:\hat{Q}_{a,b,c,d,e,f;i,j,l}\longrightarrow G(\alpha_c^+;n,d,k).$$

By lemma 6.2.1 we have that $\hat{\omega}_{a,b,c,d,e,f;i,j,l}$ is invariant under the action of $\mathbb{C} \times \mathbb{C}^*$, so it induces a morphism

$$\overline{\omega}_{a,b,c,d,e,f;i,j,l} : \hat{R}_{a,b,c,d,e,f;i,j,l} \longrightarrow G(\alpha_c^+; n, d, k).$$

Finally, there are free actions as follows.

- $PGL(N_1)$ acts freely on $\hat{G}_1$, $\hat{U}_{a;i}$ and $\hat{R}_{a;i}$; this induces a projective fibration

$$\varphi_{a;i} : R_{a,i} \longrightarrow U_{a,i}$$

  with fibers isomorphic to $\mathbb{P}^{a-1}$. The family $\{U_{a;i}\}_i$ is a finite disjoint locally closed covering of the subscheme $U_a \subset G_1$ described in the claim of the proposition.

- $PGL(N_3) \times PGL(N_4)$ acts freely on $\hat{G}_3 \times \hat{G}_4$, $\hat{U}^{b;j}$ and $\hat{R}^{b;j}$; this induces a projective fibration

$$\varphi^{b;j} : R^{b,j} \longrightarrow U^{b;j}$$

  with fibers isomorphic to $\mathbb{P}^{b-1}$. The family $\{U^{b;j}\}_j$ is a finite disjoint locally closed covering of the subscheme $U^b \subset G_3 \times G_4$ described in the claim of the proposition.

- $PGL(N_1) \times PGL(N_3) \times PGL(N_4)$ acts freely on $\hat{R}_{a;i} \times \hat{R}^{b;j}$, $\hat{U}_{a,b,c,d,e,f;i,j,l}$ and $\hat{R}_{a,b,c,d,e,f;i,j,l}$; this induces a fibration

$$\varphi_{a,b,c,d,e,f;i,j,l} : R_{a,b,c,d,e,f;i,j,l} \longrightarrow U_{a,b,c,d,e,f;i,j,l}$$

  with fibers isomorphic to $\mathbb{C}^{e-1} \times (\mathbb{P}^{c-e-1} \smallsetminus \mathbb{P}^{d-f-1})$. The family $\{U_{a,b,c,d,e,f;i,j,l}\}_l$ is a finite disjoint locally closed covering of the subscheme $U_{a,b,c,d,e,f;i,j} \subset R_{a;i} \times R^{b;j}$ described in the claim of the proposition.

The morphism $\overline{\omega}_{a,b,c,d,e,f;i,j,l}$ is invariant under the action on $\hat{R}_{a,b,c,d,e,f;i,j,l}$, so there is an induced morphism

$$\omega_{a,b,c,d,e,f;i,j,l} : R_{a,b,c,d,e,f;i,j,l} \longrightarrow G(\alpha_c^+; n, d, k).$$

Such a morphism is injective (and has values in G') because of the previous lemma, so we conclude. $\square$

Lemma 6.2.3. *Let us fix any quadruple $(Q_i, W_i)_{i=1,\cdots,4} \in \prod_{i=1}^4 G_i$ and let us assume that*

$$(Q_1, W_1) \not\simeq (Q_2, W_2) \not\simeq (Q_3, W_3), \quad (Q_4, W_4) \not\simeq (Q_i, W_i) \quad \forall\, i = 1, 2, 3,$$
$$Ext^2\left((Q_4, W_4), (Q_1, W_1)\right) = 0 = Ext^2\left((E'', V''), (Q_1, W_1)\right),$$

where (E'', V'') is any non-split extension of (Q_4, W_4) by (Q_3, W_3). Let us denote by μ any class of a non-split extension of the form

$$0 \to (Q_1, W_1) \xrightarrow{\sigma} (E_2, V_2) \xrightarrow{\kappa} (Q_2, W_2) \to 0 \tag{6.57}$$

and by ν any class of a non-split extension of the form

$$0 \to (Q_3, W_3) \xrightarrow{\varepsilon} (E'', V'') \xrightarrow{\delta} (Q_4, W_4) \to 0. \tag{6.58}$$

Having fixed $[\mu] \in \mathbb{P}(Ext^1((Q_2, W_2), (Q_1, W_1)))$ and $[\nu] \in \mathbb{P}(Ext^1((Q_4, W_4), (Q_3, W_3)))$, let us consider the space $M_1([\mu], [\nu]) := Ext^1((E'', V''), (E_2, V_2))$ and let us denote by $M_2([\mu], [\nu])$ the image of the linear map $A + F$ where A and F are the maps induced by (6.58) and (6.57) respectively, as follows:

$$\begin{aligned} A &: Ext^1((Q_4, W_4), (E_2, V_2)) \longrightarrow Ext^1((E'', V''), (E_2, V_2)) = M_1([\mu], [\nu]), \\ F &: Ext^1((E'', V''), (Q_1, W_1)) \longrightarrow Ext^1((E'', V''), (E_2, V_2)) = M_1([\mu], [\nu]). \end{aligned}$$

Then the objects (E, V)'s with unique α_c-Jordan-Hölder filtration and graded $\oplus_{i=1}^4 (Q_i, W_i)$ are parametrized by triples $([\mu], [\nu], [\eta])$, where:

- $[\mu] \in \mathbb{P}(Ext^1((Q_2, W_2), (Q_1, W_1)))$ *and μ has a representative of the form (6.57);*
- $[\nu] \in \mathbb{P}(Ext^1((Q_4, W_4), (Q_3, W_3)))$ *and ν has a representative of the form (6.58);*
- $[\eta] \in \overline{M}([\mu], [\nu]) := \mathbb{P}(M_1([\mu], [\nu])) \smallsetminus \mathbb{P}(M_2([\mu], [\nu]))$.

Moreover, if we write:

$$\begin{aligned} c := dim\ M_1([\mu], [\nu]), \quad d := dim\ Ext^1((Q_4, W_4), (E_2, V_2)), \\ e := dim\ Ext^1((E'', V''), (Q_1, W_1)), \quad f := dim\ Ext^1((Q_4, W_4), (Q_1, W_1)), \end{aligned}$$

then $dim\ M_2([\mu], [\nu]) = d + e - f$ and $\overline{M}([\mu], [\nu])) \simeq \mathbb{P}^{c-1} \smallsetminus \mathbb{P}^{d+e-f-1}$.

Proof. The proof is analogous to the proof of lemma 6.2.1. In this proof we need to consider 2 subcases as follows:

(a) $\mathrm{Hom}\,((Q_3, W_3), (Q_1, W_1)) = 0$;

(b) $\mathrm{Hom}\,((Q_3, W_3), (Q_1, W_1)) = \mathbb{C}$.

Since (Q_1, W_1) and (Q_3, W_3) are both α_c-stable of the same slope, these are the only 2 possibilities. Using the hypotheses of this lemma, these conditions can be restated as:

(a) $(Q_1, W_1) \not\simeq (Q_2, W_2) \not\simeq (Q_3, W_3)$ and $(Q_1, W_1) \not\simeq (Q_3, W_3)$;

(b) $(Q_1, W_1) \simeq (Q_3, W_3) \not\simeq (Q_2, W_2)$.

Then the proof is on the same line of the proof of lemma 6.2.1, so we just describe briefly the relevant changes. We consider the same 6 long exact sequences (6.33)-(6.38) of that lemma. Exactly as in that lemma we get that the (E, V)'s we are interested in are those induced by extensions η that belong to $\mathrm{Ext}^1((E'', V''), (E_2, V_2))$ and not to the subspace

$$M_2([\mu], [\nu]) := \mathrm{Im}(A + F)$$

and we get that

$$\dim\, M_2([\mu], [\nu]) = \dim\, \mathrm{Im}(A) + \dim\, \mathrm{Im}(F) - \dim\, M_3([\mu], [\nu])$$

where we denote by $M_3([\mu], [\nu])$ the linear space $\mathrm{Im}(A) \cap \mathrm{Im}(F) \subset M_2([\mu], [\nu])$. As in (6.44) we get

$$\mathrm{Hom}((E'', V''), (E_2, V_2)) = 0. \tag{6.59}$$

If we assume condition (a), then we have that $(Q_3, W_3) \not\simeq (Q_i, W_i)$ for $i = 1, 2$, therefore

$$\mathrm{Hom}((Q_3, W_3), (E_2, V_2)) = 0, \tag{6.60}$$

so the map A is injective by (6.33). If we assume (b), then

$$\mathrm{Hom}((Q_3, W_3), (E_2, V_2)) = \mathbb{C}. \tag{6.61}$$

So in case (b) if we use (6.33), (6.59) and (6.61) we get that $\mathrm{Ker}(A) = \mathbb{C}$. As in lemma 6.2.1 we have that F is injective and that the space $M_3([\mu], [\nu])$ coincides with the image of the linear map $A + F$.

Now let us consider the map M: in case (a) using (6.38) we get that such a morphism is injective. In case (b) we get that $\mathrm{Hom}((Q_3, W_3), (Q_1, W_1)) = \mathbb{C}$. Moreover,

$$\mathrm{Hom}((E'', V''), (Q_1, W_1)) = 0. \tag{6.62}$$

Indeed, if we have a non-zero morphism in that set, by exactness of (6.58) we get a non-zero morphism from (Q_3, W_3) to (Q_1, W_1) or from (Q_4, W_4) to (Q_1, W_1). In the first case we will get a splitting of (6.58), while the second case will give an isomorphism from (Q_4, W_4) to (Q_1, W_1). Both cases are impossible by hypothesis and construction, so the claim is proved. Using (6.38) and (6.62) we get that in case (b) $\mathrm{Ker}(M) = \mathbb{C}$.

Then we need to compute $\dim\, M_2([\mu], [\nu])$. In order to do that, we fix the following notation:

$$\dim\, \mathrm{Ext}^1((Q_4, W_4), (E_2, V_2)) := d, \quad \dim\, \mathrm{Ext}^1((E'', V''), (Q_1, W_1)) := e,$$
$$\dim\, \mathrm{Ext}^1((Q_4, W_4), (Q_1, W_1)) := f.$$

Both in case (a) and (b) the map F is injective, so $\dim\, \mathrm{Im}(F) = e$. Moreover,

- in case (a) the map A is injective, so dim $\mathrm{Im}(A) = d$. Moreover, also the map M is injective and so is $F \circ M$. Since $M_3([\mu],[\nu])$ coincides with the image of $F \circ M$, then dim $M_3([\mu],[\nu]) = f$;

- in case (b) the kernel of A has dimension 1, so dim $\mathrm{Im}(A) = d-1$. Moreover, also the kernel of M has dimension 1. Since F is injective, then also the kernel of $F \circ M$ has dimension 1. Then dim $M_3([\mu],[\nu]) = f-1$.

Then both in case (a) and in case (b) we get

$$\dim\, M_2([\mu],[\nu]) = d+e-f = (d-1)+e-(f-1) \tag{6.63}$$

The set parametrizing all the (E_2,V_2)'s is given by $\mathbb{P}((Q_2,W_2),(Q_1,W_1))$ and analogously, the set parametrizing all the (E'',V'')'s is given by $\mathbb{P}((Q_4,W_4),(Q_3,W_3))$. Moreover, for every (E_2,V_2) and (E'',V'') we have $\mathrm{Aut}(E_2,V_2) = \mathrm{Aut}(E'',V'') = \mathbb{C}^*$. Moreover, we have proved in (6.44) that

$$\mathrm{Hom}((E'',V''),(E_2,V_2)) = 0.$$

So the induced action on $M_1([\mu],[\nu])$ and on $M_1([\mu],[\nu]) \smallsetminus M_2([\mu],[\nu])$ is given by multiplication of scalars in $\mathbb{C}^*$. So having fixed any pair of points

$$([\mu],[\nu]) \in \mathbb{P}(\mathrm{Ext}^1((Q_2,W_2),(Q_1,W_1)) \times \mathbb{P}(\mathrm{Ext}^1((Q_4,W_4),(Q_3,W_3)),$$

we get that the (E,V)'s with unique α_c-Jordan-Hölder filtration are parametrized by

$$\mathbb{P}(M_1([\mu],[\nu])) \smallsetminus \mathbb{P}(M_2([\mu],[\nu])).$$

□

Proposition 6.2.4. *Let us fix any triple (n,d,k), a critical value α_c for it and a quadruple $(n_i,d_i,k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c;n,d,k)$. Let us assume that conditions (6.24), respectively (6.25), are satisfied and that for every quadruple $(Q_i,W_i)_{i=1,\cdots,4} \in \prod_{i=1}^4 G_i$ we have $(Q_i,W_i) \not\simeq (Q_4,W_4)$ for $i=1,2,3$. Let us denote by G' the set of all the (E,V)'s in $G^+(\alpha_c;n,d,k)$, respectively in $G^-(\alpha_c;n,d,k)$, that have unique α_c-Jordan-Hölder filtration and graded at α_c in $\prod_{i=1}^4 G_i$ and such that*

$$(Q_1,W_1) \not\simeq (Q_2,W_2) \not\simeq (Q_3,W_3), \quad Ext^2\left((Q_4,W_4),(Q_1,W_1)\right) = 0 = Ext^2\left((E'',V''),(Q_1,W_1)\right),$$

where (E'',V'') is any non-split extension of (Q_4,W_4) by (Q_3,W_3). Then there exists a finite family $\{R_{a,b,c,d,e,f;i,j,l}\}$ of schemes for $(a,b,c,d,e,f) \in \mathbb{N}^6$, and i,j,l varying in finite sets (for a,b,c,d,e,f fixed), together with injective morphisms to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Every scheme $R_{a,b,c,d,e,f;i,j,l}$ comes with a triple of morphisms

$$\varphi_{a,b,c,d,e,f;i,j,l} : R_{a,b,c,d,e,f;i,j,l} \longrightarrow U_{a,b,c,d,e,f:i,j,l} \subset R_{a;i} \times R^{b;j},$$
$$\varphi_{a;i} : R_{a;i} \longrightarrow U_{a;i} \subset G_1 \times G_2,$$
$$\varphi^{b;j} : R^{b;j} \longrightarrow U^{b;j} \subset G_3 \times G_4,$$

where:

- $\varphi_{a,b,c,d,e,f;i,j,l}$ *has fibers isomorphic to* $\mathbb{P}^{c-1} \smallsetminus \mathbb{P}^{d+e-f-1}$, $\varphi_{a;i}$ *has fibers isomorphic to* $\mathbb{P}^{a-1}$ *and* $\varphi^{b;j}$ *has fibers isomorphic to* $\mathbb{P}^{b-1}$*;*

- $\{U_{a;i}\}_i$ *is a finite disjoint locally closed covering of*

$$U_a := \{((Q_1, W_1), (Q_2, W_2)) \in G_1 \times G_2 \text{ s.t.}$$
$$\dim Ext^1((Q_2, W_2), (Q_1, W_1)) = a, \quad Hom((Q_2, W_2), (Q_1, W_1)) = 0\};$$

  every U_a *is a locally closed subscheme of* $G_1 \times G_2$ *and so are all the* $U_{a;i}$*'s. The last condition on* U_a *can be dropped if* $(n_1, k_1) \neq (n_2, k_2)$*, otherwise it is necessary;*

- $\{U^{b;j}\}_j$ *is a finite disjoint locally closed covering of*

$$U^b := \{((Q_3, W_3), (Q_4, W_4)) \in G_3 \times G_4 \text{ s.t. } \dim Ext^1((Q_4, W_4), (Q_3, W_3)) = b\};$$

  every U^b *is a locally closed subscheme of* $G_3 \times G_4$ *and so are all the* $U^{b;j}$*'s;*

- $\{U_{a,b,c,d,e,f;i,j,l}\}_l$ *is a finite disjoint locally closed covering of*

$$U_{a,b,c,d,e,f;i,j} := \{((E_2, V_2), (E'', V'')) \in R_{a;i} \times R^{b;j} \text{ s.t. } \dim Ext^1((E'', V''), (E_2, V_2)) = c,$$
$$\dim Ext^1(\widetilde{\varphi}^{b;j}(E'', V''), (E_2, V_2)) = d, \quad \dim Ext^1((E'', V''), \widetilde{\varphi}_{a;i}(E_2, V_2)) = e,$$
$$\dim Ext^1(\widetilde{\varphi}^{b;j}(E'', V''), \widetilde{\varphi}_{a;i}(E_2, V_2)) = f, \quad Hom(\overline{\varphi}^{b;j}(E'', V''), \overline{\varphi}_{a;i}(E_2, V_2)) = 0\},$$

  where:

  - $\widetilde{\varphi}_{a;i}$ *is the composition of* $\varphi_{a;i}$ *with the projection to* G_1*;*
  - $\overline{\varphi}_{a;i}$ *is the composition of* $\varphi_{a;i}$ *with the projection to* G_2*;*
  - $\overline{\varphi}^{b;j}$ *is the composition of* $\varphi^{b;j}$ *with the projection to* G_3*;*
  - $\widetilde{\varphi}^{b;j}$ *is the composition of* $\varphi^{b;j}$ *with the projection to* G_4*.*

  Every $U_{a,b,c,d,e,f;i,j}$ *is locally closed in* $R_{a;i} \times R^{b;j}$ *and so are all the* $U_{a,b,c,d,e,f;i,j,l}$*'s. The last condition on* $U_{a,b,c,d,e,f;i,j}$ *can be dropped if* $(n_1, k_1) \neq (n_2, k_2)$*, otherwise it is necessary.*

Proof. Let us fix any sequence $(a, b, c, d, e, f) \in \mathbb{N}^6$ and let us denote by $(\hat{\mathcal{Q}}_i, \hat{\mathcal{W}}_i)$ the universal families over the various Quot schemes $\hat{G}_i$ for $i = 1, \cdots, 4$. Let us consider the scheme $\hat{G}_3 \times \hat{G}_4$ and let us denote by $\hat{p}_3$ and $\hat{p}_4$ the projections from such a scheme to its factors. Having fixed b, let us consider the locally closed subscheme of $\hat{G}_3 \times \hat{G}_4$ defined as:

$$\hat{U}^b := \{t \in \hat{G}_3 \times \hat{G}_4 \text{ s.t. } \dim \mathrm{Ext}^1((\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)_t, ((\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)_t) = b\}.$$

By hypothesis for each $t \in \hat{G}_3 \times \hat{G}_4$ we have:

$$\mathrm{Hom}((\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)_t, (\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)_t) = 0.$$

So we can apply corollary 4.3.3 and we get that there is a finite disjoint covering $\{\hat{U}^{b;j}\}_j$ of $\hat{U}^b$ by locally closed subschemes; for every j there is a locally free sheaf on $\hat{U}^{b;j}$

$$\hat{\mathcal{H}}^{b;j} := \mathcal{E}xt^1_{\pi_{\hat{U}^{b;j}}}((\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)|_{\hat{U}^{b;j}}, (p'_3, p_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)|_{\hat{U}^{b;j}})^\vee$$

and a projective bundle

$$\hat{\varphi}^{b;j} : \hat{R}^{b;j} := \mathbb{P}(\hat{\mathcal{H}}^{b;j}) \longrightarrow \hat{U}^{b;j} \subset \hat{G}_3 \times \hat{G}_4$$

with fibers isomorphic to $\mathbb{P}^{b-1}$. By abuse of notation, we denote by $\hat{\varphi}^{b;j}$ also the composition $\hat{R}^{b;j} \to \hat{G}_3 \times \hat{G}_4$. Moreover, there exists a family $(\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j})$ parametrized by $\hat{R}^{b;j}$ and a universal family of classes of non-split extensions over that scheme:

$$\begin{aligned} 0 \to (\hat{\varphi}'^{b;j}, \hat{\varphi}^{b;j})^*(\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3) \otimes_{\hat{R}^{b;j}} \mathcal{O}_{\hat{R}^{b;j}}(1) \xrightarrow{\varepsilon^{b;j}} \\ \xrightarrow{\varepsilon^{b;j}} (\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j}) \xrightarrow{\delta^{b;j}} (\hat{\varphi}'^{b;j}, \hat{\varphi}^{b;j})^*(\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4) \to 0. \end{aligned} \tag{6.64}$$

Now let us pass to the scheme $\hat{G}_1 \times \hat{G}_2$ and let us denote by $\hat{p}_1, \hat{p}_2$ the projections to the 2 factors. The set

$$\hat{U}_a := \{t \in \hat{G}_1 \times \hat{G}_2 \text{ s.t. } \dim \mathrm{Ext}^1((\hat{p}'_2, \hat{p}_2)^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2)_t, (\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = a,$$

$$\mathrm{Hom}((\hat{p}'_2, \hat{p}_2)^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2)_t, (\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = 0\}$$

is a locally closed subscheme of $\hat{G}_1 \times \hat{G}_2$. By corollary 4.3.3 there exists a finite disjoint covering of $\hat{U}_a$ by locally closed subschemes $\hat{U}_{a;i}$; on each $\hat{U}_{a;i}$ there is a locally free sheaf

$$\hat{\mathcal{H}}_{a;i} := \mathcal{E}xt^1_{\pi_{\hat{U}_{a;i}}}((\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2), (\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1))^\vee$$

and a projective bundle

$$\hat{\varphi}_{a;i} : \hat{R}_{a;i} := \mathbb{P}(\hat{\mathcal{H}}_{a;i}) \longrightarrow \hat{U}_{a;i} \subset \hat{G}_1 \times \hat{G}_2$$

with fibers isomorphic to $\mathbb{P}^{a-1}$. By abuse of notation we denote by $\hat{\varphi}_{a;i}$ also the composition $\hat{R}_{a;i} \to \hat{G}_1 \times \hat{G}_2$. Moreover, there exists a family $(\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i})$ parametrized by $\hat{R}_{a;i}$ and a universal family of non-split extensions over that scheme:

$$\begin{aligned} 0 \to (\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \otimes_{\hat{R}_{a;i}} \mathcal{O}_{\hat{R}_{a;i}}(1) \xrightarrow{\sigma_{a;i}} \\ \xrightarrow{\sigma_{a;i}} (\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i}) \xrightarrow{\kappa_{a;i}} (\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_2, \hat{p}_2)^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2) \to 0. \end{aligned} \tag{6.65}$$

Now for every choice of indices (i, j) we consider the scheme $T := \hat{R}_{a;i} \times \hat{R}^{b;j}$ together with the projections $\hat{p}_{12}$ and $\hat{p}_{34}$ to the 2 factors. Then we define $\hat{U}_{a,b,c,d,e,f;i;j}$ as the subscheme of T:

$$\begin{gathered} \{t \in T \text{ s.t. } \dim \operatorname{Ext}^1((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j})_t, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i})_t) = c, \\ \dim \operatorname{Ext}^1((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\varphi}'^{b;j}, \hat{\varphi}^{b;j})^*(\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)_t, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i})_t) = d, \\ \dim \operatorname{Ext}^1((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j})_t, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = e, \\ \dim \operatorname{Ext}^1((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\varphi}'^{b;j}, \hat{\varphi}^{b;j})^*(\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)_t, \\ (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(p'_1, p_1)^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = f \\ \operatorname{Hom}((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\varphi}'^{b;j}, \hat{\varphi}^{b;j})^*(\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)_t, \\ (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(p'_2, p_2)^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2)_t) = 0\}. \end{gathered}$$

By proposition 1.0.5, this is a locally closed subscheme of $\hat{R}_{a;i} \times \hat{R}^{b;j}$. By construction, for each $t \in \hat{U}_{a,b,c,d,e,f;i,j}$, we have that both $(\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i})_t$ and $(\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}^{b,j}, \hat{\mathcal{V}}^{b;j})_t$ are α_c-semistable. As in (6.44), we have:

$$\operatorname{Hom}((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j})_t, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i})_t) = 0$$

for all $t \in \hat{U}_{a,b,c,d,e,f;i,j}$. Moreover, by construction

$$\dim \operatorname{Ext}^1((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j})_t, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i})_t) = c$$

is constant on $\hat{U}_{a,b,c,d,e,f;i,j}$. Then we can apply proposition 4.6.3 for such a scheme and we get that there is a finite disjoint locally closed covering $\{\hat{U}^{l_1}_{a,b,c,d,e,f;i,j}\}_{l_1}$ of $\hat{U}_{a,b,c,d,e,f;i,j}$. For each l_1 there is a locally free sheaf on the corresponding scheme:

$$\hat{\mathcal{H}}^{l_1}_{a,b,c,d,e,f;i,j} := \mathcal{E}xt^1_{\pi_{\hat{U}^{l_1}_{a,b,c,d,e,f;i,j}}} \Big((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j}), (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i}) \Big)^\vee$$

and a vector bundle of rank c:

$$\hat{\varphi}^{l_1}_{a,b,c,d,e,f;i,j} : \hat{V}^{l_1}_{a,b,c,d,e,f;i,j} := \mathbb{V}(\hat{\mathcal{H}}^{l_1}_{a,b,c,d,e,f;i,j}) \longrightarrow \hat{U}^{l_1}_{a,b,c,d,e,f;i,j}$$

together with a family $(\hat{\mathcal{E}}^{l_1}_{a,b,c,d,e,f;i,j}, \hat{\mathcal{V}}^{l_1}_{a,b,c,d,e,f;i,j})$ parametrized by $\hat{V}^{l_1}_{a,b,c,d,e,f;i,j}$. Moreover, there is a universal family of extensions over that scheme given by:

$$
\begin{aligned}
0 \to (\hat{\varphi}'^{l_1}_{a,b,c,d,e,f;i,j}, \hat{\varphi}^{l_1}_{a,b,c,d,e,f;i,j})^*(\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i}) \xrightarrow{\iota^{l_1}_{a,b,c,d,e,f;i,j}} \\
\xrightarrow{\iota^{l_1}_{a,b,c,d,e,f;i,j}} (\hat{\mathcal{E}}^{l_1}_{a,b,c,d,e,f;i,j}, \hat{\mathcal{V}}^{l_1}_{a,b,c,d,e,f;i,j}) \xrightarrow{\lambda^{l_1}_{a,b,c,d,e,f;i,j}} \\
\xrightarrow{\lambda^{l_1}_{a,b,c,d,e,f;i,j}} (\hat{\varphi}'^{l_1}_{a,b,c,d,e,f;i,j}, \hat{\varphi}^{l_1}_{a,b,c,d,e,f;i,j})^*(\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j}) \to 0.
\end{aligned}
\tag{6.66}
$$

Now let us consider the pullback of the sequence (6.64) via the morphism $\hat{U}^{l_1}_{a,b,c,d,e,f;i,j} \hookrightarrow \hat{R}_{a;i} \times \hat{R}^{b;j} \xrightarrow{\hat{p}_{34}} \hat{R}^{b;j}$ (we denote again by $\hat{p}_{34}$ this morphism) and let us apply to the new exact sequence the functor

$$
\mathcal{H}om_{\pi_{\hat{U}^{l_1}_{a,b,c,d,e,f;i,j}}}\Big(-, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i})\Big).
$$

Then we get a long exact sequence:

$$
\begin{aligned}
\mathcal{H}om_{\pi_{\hat{U}^{l_1}_{a,b,c,d,e,f;i,j}}}\Big((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\varphi}'^{b;j}, \hat{\varphi}^{b;j})^*(\hat{p}'_3, \hat{p}_3)^*((\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)\otimes_{\hat{R}^{b;j}}\mathcal{O}_{\hat{R}^{b;j}}(1)), (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i})\Big) \to \\
\to \mathcal{E}xt^1_{\pi_{\hat{U}^{l_1}_{a,b,c,d,e,f;i,j}}}\Big((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\varphi}'^{b;j}, \hat{\varphi}^{b;j})^*(\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4), (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i})\Big) \xrightarrow{A} \\
\xrightarrow{A} \mathcal{E}xt^1_{\pi_{\hat{U}^{l_1}_{a,b,c,d,e,f;i,j}}}\Big((\hat{p}'_{34}, \hat{p}_{34})^*((\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j}), (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}_{a;i}, \hat{\mathcal{V}}_{a;i})\Big) \to \cdots
\end{aligned}
$$

Let us consider the first sheaf in that exact sequence. If we use (6.60) and (6.61)), for each pair $((Q_3, W_3), (E_2, V_2))$ we get that either $\mathrm{Hom}((Q_3, W_3), (E_2, V_2)) = 0$ or $\dim \mathrm{Hom}((Q_3, W_3), (E_2, V_2)) = 1$, according to the relation between (Q_1, W_1) and (Q_3, W_3). Then there exists a disjoint covering of $\hat{U}^{l_1}_{a,b,c,d,e,f;i,j}$ by locally closed subschemes indexed by $l_2 \in \{0, 1\}$, so that the sheaf we are considering is locally free if restricted to any such subscheme. In particular, it is the zero sheaf for $l_2 = 0$ and it is a line bundle if $l_2 = 1$. Moreover, by definition of $\hat{U}_{a,b,c,d,e,f;i,j}$ also the second sheaf of the previous sequence is locally free if restricted to any such subscheme.

Analogously, let us consider the following long exact sequence obtained by applying the functor

$$
\mathcal{H}om_{\pi_{\hat{U}^{l_1}_{a,b,c,d,e,f;i,j}}}\Big((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j}), -\Big)
$$

to the pullback of (6.65) via the morphism $\hat{U}^{l_1}_{a,b,c,d,e,f;i,j} \hookrightarrow \hat{R}_{a;i} \times \hat{R}^{b;j} \xrightarrow{\hat{p}_{12}} \hat{R}_{a;i}$.

$$
\begin{aligned}
\cdots \to \mathcal{H}om_{\pi_{\hat{U}^{l_1}_{a,b,c,d,e,f;i,j}}}\Big((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j}), (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_2, \hat{p}_2)^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2)\Big) \to \\
\to \mathcal{E}xt^1_{\pi_{\hat{U}^{l_1}_{a,b,c,d,e,f;i,j}}}\Big((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}^{b;j}, \hat{\mathcal{V}}^{b;j}),
\end{aligned}
$$

$$(\hat{p}'_{12},\hat{p}_{12})^*(\hat{\varphi}_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_1,\hat{p}_1)^*\Big((\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)\otimes_{\hat{R}_{a;i}}\mathcal{O}_{\hat{R}_{a;i}}(1))\Big)\xrightarrow{F}$$

$$\xrightarrow{F}\mathcal{E}xt^1_{\pi_{\hat{U}^{k_\bullet,l_1}_{a,b,c,d,e,f;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\mathcal{E}}^{b,j},\hat{\mathcal{V}}^{b;j}),(\hat{p}'_{12},\hat{p}_{12})^*(\hat{\mathcal{E}}_{a;i},\hat{\mathcal{V}}_{a;i})\Big)\to\cdots$$

Using base change together with (6.46), we get that the first sheaf is zero. Moreover, having fixed the invariant e, the second sheaf of this exact sequence is locally free of rank e.

Let us denote by l any collection of indices (l_1,l_2). By construction l varies over a finite set, so we get a finite disjoint locally closed covering $\{\hat{U}_{a,b,c,d,e,f;i,j,l}\}_l$ of $\hat{U}_{a,b,c,d,e,f;i,j}$. According to this notation, we denote by $\hat{V}_{a,b,c,d,e,f;i,j,l}$, $(\hat{\mathcal{E}}_{a,b,c,d,e,f;i,j,l},\hat{\mathcal{V}}_{a,b,c,d,e,f;i,j,l})$, etc. all the various objects we have defined so far.

Moreover, by restricting to any subscheme $\hat{U}_{a,b,c,d,e,f;i,j,l}$, we rewrite the previous 2 exact sequences as sequences of morphisms of vector bundles as follows:

$$0\to\hat{V}^3_{a,b,c,d,e,f;i,j,l}\longrightarrow\hat{V}^1_{a,b,c,d,e,f;i,j,l}\xrightarrow{A}\hat{V}_{a,b,c,d,e,f;i,j,l}$$

and

$$0\to\hat{V}^2_{a,b,c,d,e,f;i,j,l}\xrightarrow{F}\hat{V}_{a,b,c,d,e,f;i,j,l}.$$

If $l=(l_1,l_2,l_3,l_4,l_5)$ is such that $l_2=0$, then $V^3_{a,b,c,d,e,f;i,j,l}=0$, so we get that A is injective; if $l_2=1$, then $\mathrm{Ker}(A)=V^3_{a,b,c,d,e,f;i,j,l}$ is a line bundle. In both cases, the computations of lemma 6.2.3 prove that the induced morphism of vector bundles

$$A+F:\hat{V}^1_{a,b,c,d,e,f;i,j,l}\oplus\hat{V}^2_{a,b,c,d,e,f;i,j,l}\longrightarrow\hat{V}_{a,b,c,d,e,f;i,j,l}$$

has constant rank (equal to $d+e-f$), so its image is a subvector bundle of $\hat{V}_{a,b,c,d,e,f;i,j,l}$ (see for example, [LP2, proposition 1.7.2]). We denote such a vector bundle by $\hat{V}'_{a,b,c,d,e,f;i,j,l}$ and we write

$$\hat{Q}_{a,b,c,d,e,f;i,j,l}:=\hat{V}_{a,b,c,d,e,f;i,j,l}\smallsetminus\hat{V}'_{a,b,c,d,e,f;i,j,l}$$

Then we get an obvious action of $\mathbb{C}^*$ on both $\hat{V}_{a,b,c,d,e,f;i,j,l}$ and $\hat{V}'_{a,b,c,d,e,f;i,j,l}$. Such an action is compatible with the fibration to $\hat{U}_{a,b,c,d,e,f;i,j,l}$. So it makes sense to consider the quotient

$$\hat{R}_{a,b,c,d,e,f;i,j,l}:=\hat{Q}_{a,b,c,d,e,f;i,j,l}/\mathbb{C}^*$$

and the induced fibration

$$\hat{\varphi}_{a,b,c,d,e,f;i,j,l}:\hat{R}_{a,b,c,d,e,f;i,j,l}\longrightarrow\hat{U}_{a,b,c,d,e,f;i,j,l}.$$

The fibers of such a fibration are described in lemma 6.2.3, so each fiber is isomorphic to $\mathbb{P}^{c-1}\smallsetminus\mathbb{P}^{d+e-f-1}$. Now we recall that on $\hat{Q}_{a,b,c,d,e,f;i,j}$ we have a family $(\hat{\mathcal{E}}_{a,b,c,d,e,f;i,j,l},\hat{\mathcal{V}}_{a,b,c,d,e,f;i,j,l})$ (given by restriction of the central term of (6.66)) such that if we denote by q any point of

$\hat{Q}_{a,b,c,d,e,f;i,j,l}$ and by (E,V) the restriction of such a family to q, then we have a triple of exact sequences:

$$0 \to (Q_1, W_1) \to (E_2, V_2) \to (Q_1, W_1) \to 0,$$
$$0 \to (Q_3, W_3) \to (E'', V'') \to (Q_4, W_4) \to 0,$$
$$0 \to (E_2, V_2) \to (E, V) \to (E'', V'') \to 0$$

such that both the first 2 sequences and the induced sequence

$$0 \to (Q_2, W_2) \to (E_3, V_3)/(Q_1, W_1) \to (Q_3, W_3) \to 0$$

are non-split. Then by lemma 6.2.3 we have that (E,V) has a unique α_c-JHF. Now let us assume conditions (6.24) (an analogous proof holds for conditions (6.25)). Then (E,V) belongs to

$$G' \subset G^+(\alpha_c; n, d, k) \subset G(\alpha_c^+; n, d, k).$$

Then by the universal property of the scheme $G(\alpha_c^+; n, d, k)$ we get that the previous family induces a morphism

$$\hat{\omega}_{a,b,c,d,e,f;i,j,l} : \hat{Q}_{a,b,c,d,e,f;i,j,l} \to G(\alpha_c^+; n, d, k).$$

By construction, we have that $\hat{\omega}_{a,b,c,d,e,f;i,j,l}$ is invariant under the action of $\mathbb{C}^*$, so it induces a morphism

$$\overline{\omega}_{a,b,c,d,e,f;i,j,l} : \hat{R}_{a,b,c,d,e,f;i,j,l} \to G(\alpha_c^+; n, d, k).$$

Finally, there are free actions as follows.

- $PGL(N_1) \times PGL(N_2)$ acts on $\hat{G}_1 \times \hat{G}_2$, $\hat{U}_{a;i}$ and $\hat{R}_{a;i}$; this induces a projective fibration

$$\varphi_{a;i} : R_{a;i} \longrightarrow U_{a;i}$$

  with fibers isomorphic to $\mathbb{P}^{a-1}$. The family $\{U_{a;i}\}_i$ is a finite disjoint locally closed covering of the subscheme $U_a \subset G_1 \times G_2$ described in the claim of the proposition.

- $PGL(N_3) \times PGL(N_4)$ acts on $\hat{G}_3 \times \hat{G}_4$, $\hat{U}^{b;j}$ and $\hat{R}^{b;j}$; this induces a projective fibration

$$\varphi^{b;j} : R^{b;j} \longrightarrow U^{b;j}$$

  with fibers isomorphic to $\mathbb{P}^{b-1}$. The family $\{U^{b;j}\}_j$ is a finite disjoint locally closed covering of the subscheme $U_b \subset G_3 \times G_4$ described in the claim of the proposition.

- $PGL(N_1) \times PGL(N_2) \times PGL(N_3) \times PGL(N_4)$ acts on $\hat{R}_{a;i} \times \hat{R}^{b;j}$, $\hat{U}_{a,b,c,d,e,f;i,j,l}$ and $\hat{R}_{a,b,c,d,e,f;i,j,l}$; this induces a fibration

$$\varphi_{a,b,c,d,e,f;i,j,l} : R_{a,b,c,d,e,f;i,j,l} \longrightarrow U_{a,b,c,d,e,f;i,j,l}$$

  with fibers isomorphic to $\mathbb{P}^{c-1} \smallsetminus \mathbb{P}^{d+e-f-1}$. The family $\{U_{a,b,c,d,e,f;i,j,l}\}_l$ is a finite disjoint locally closed covering of the subscheme $U_{a,b,c,d,e,f;i,j} \subset R_{a;i} \times R^{b;j}$ described in the claim of the proposition.

The morphism $\overline{\omega}_{a,b,c,d,e,f;i,j,l}$ is invariant under the action on $\hat{R}_{a,b,c,d,e,f;i,j,l}$, so there is an induced morphism

$$\omega_{a,b,c,d,e,f;i,j,l} : R_{a,b,c,d,e,f;i,j,l} \longrightarrow G(\alpha_c^+; n, d, k).$$

Such a morphism is injective (and has values in G') because of the lemma 6.2.3, so we conclude.

□

Lemma 6.2.5. *Let us fix any quadruple $(Q_i, W_i)_{i=1,\cdots,4} \in \prod_{i=1}^4 G_i$ and let us assume that*

$$(Q_2, W_2) \not\simeq (Q_3, W_3) \simeq (Q_4, W_4), \quad (Q_1, W_1) \not\simeq (Q_i, W_i) \quad \forall\, i = 2, 3, 4,$$
$$Ext^2\left((Q_4, W_4), (Q_1, W_1)\right) = 0 = Ext^2\left((Q_4, W_4), (E_2, V_2)\right),$$

where (E_2, V_2) is any non-split extension of (Q_2, W_2) by (Q_1, W_1). Let us denote by μ any class of a non-split extension of the form

$$0 \to (Q_1, W_1) \xrightarrow{\varepsilon} (E_2, V_2) \xrightarrow{\delta} (Q_2, W_2) \to 0 \tag{6.67}$$

and by ν any class of a non-split extension of the form

$$0 \to (Q_3, W_3) \xrightarrow{\sigma} (E'', V'') \xrightarrow{\kappa} (Q_4, W_4) \to 0. \tag{6.68}$$

Having fixed $[\mu] \in \mathbb{P}(Ext^1((Q_2, W_2), (Q_1, W_1)))$ and $[\nu] \in \mathbb{P}(Ext^1((Q_3, W_3), (Q_4, W_4)))$, let us consider the space $M_1([\mu], [\nu]) := Ext^1((E'', V''), (E_2, V_2))$, let us denote by η any object in that space and let us choose a representative of η as follows:

$$0 \to (E_2, V_2) \xrightarrow{\iota} (E, V) \xrightarrow{\lambda} (E'', V'') \to 0. \tag{6.69}$$

Then let us consider the action of $\mathbb{C} \times \mathbb{C}^*$ on $M_1([\mu], [\nu])$ given as follows. For every pair of scalars (ξ, τ) and for every class η of an extension (6.69) we set $(\xi, \tau) \cdot \eta := \eta'$ where η' is represented by

$$0 \to (E_2, V_2) \xrightarrow{\iota} (E, V) \xrightarrow{\lambda(\xi,\tau)} (E'', V'') \to 0, \tag{6.70}$$

where $\lambda(\xi, \tau) := (\xi \cdot \sigma \circ \kappa + \tau \cdot id_{(E'',V'')}) \circ \lambda$. Let us denote by $M_2([\mu], [\nu])$ the image of the linear map $A + F$ where A and F are the maps induced by (6.68) and (6.67) respectively, as follows:

$$A : Ext^1((Q_4, W_4), (E_2, V_2)) \longrightarrow Ext^1((E'', V''), (E_2, V_2)) = M_1([\mu], [\nu]),$$
$$F : Ext^1((E'', V''), (Q_1, W_1)) \longrightarrow Ext^1((E'', V''), (E_2, V_2)) = M_1([\mu], [\nu]).$$

Let us write $M([\mu], [\nu]) := M_1([\mu], [\nu]) \smallsetminus M_2([\mu], [\nu])$. The previous action sends $M_2([\mu], [\nu])$ to itself, so it makes sense to consider $\overline{M}([\mu], [\nu]) := M([\mu], [\nu])/(\mathbb{C} \times \mathbb{C}^*)$. Then the objects (E, V)'s with unique α_c-Jordan-Hölder filtration and graded $\oplus_{i=1}^4 (Q_i, W_i)$ are parametrized by triples $([\mu], [\nu], [\eta])$, where:

- *$[\mu] \in \mathbb{P}(Ext^1((Q_2, W_2), (Q_1, W_1)))$ and μ has a representative of the form (6.67);*
- *$[\nu] \in \mathbb{P}(Ext^1((Q_4, W_4), (Q_3, W_3)))$ and ν has a representative of the form (6.68);*
- *$[\eta] \in \overline{M}([\mu], [\nu])$.*

Moreover, if we write:

$$c := dim\ M_1([\mu], [\nu]), \quad d := dim\ Ext^1((Q_4, W_4), (E_2, V_2)),$$
$$e := dim\ Ext^1((E'', V''), (Q_1, W_1)), \quad f := dim\ Ext^1((Q_4, W_4), (Q_1, W_1)),$$

then $dim\ M_2([\mu], [\nu]) = d + e - f$ and $\overline{M}([\mu], [\nu]) \simeq \mathbb{C}^{d-1} \times (\mathbb{P}^{c-d-1} \smallsetminus \mathbb{P}^{e-f-1})$.

Proof. The proof is on the same line of the proof of lemma 6.2.1 and we refer to that proof for most of the time. In particular, we still have that

$$\mathrm{Hom}((E'', V''), (E_2, V_2)) = 0 = \mathrm{Hom}((Q_4, W_4), (E_2, V_2)). \tag{6.71}$$

Also in this case we get that the morphisms A, F, M and L are all injective. Moreover, $M_3([\mu], [\nu]) = \mathrm{Im}(A \circ L) = \mathrm{Im}(F \circ M)$. Then we have that

$$\dim\ M_2([\mu], [\nu]) = d + e - f. \tag{6.72}$$

As in lemma 6.2.1 for all triples (μ, ν, η) as before with $\mu, \nu \neq 0$ and $\eta \in M([\mu], [\nu]) = M_1([\mu], [\nu]) \smallsetminus M_2([\mu], [\nu])$, we have that the induced (E, V) has unique Jordan-Hölder filtration at α_c and graded $\oplus_{i=1}^4 (Q_i, W_i)$. The set parametrizing all the (E_2, V_2)'s is given by $\mathbb{P}(\mathrm{Ext}^1((Q_2, W_2), (Q_1, W_1)))$ and analogously, the set parametrizing all the (E'', V'')'s is in bijection with $\mathbb{P}(\mathrm{Ext}^1((Q_4, W_4), (Q_3, W_3)))$. Moreover, for every (E_2, V_2) and (E'', V'') in those 2 spaces, we have that $\mathrm{Aut}(E_2, V_2) = \mathbb{C}^*$ and $\mathrm{Aut}(E'', V'') = \mathbb{C} \times \mathbb{C}^*$.

So we have an induced action of $\mathbb{C} \times \mathbb{C}^*$ on $M_1([\mu], [\nu])$ as follows

$$\begin{array}{rcl} \mathbb{C} \times \mathbb{C}^* \times M_1([\mu], [\nu]) & \to & M_1([\mu], [\nu]) \\ (\xi, \tau, \eta) & \mapsto & \eta(\xi, \tau), \end{array}$$

where $\eta(\xi,\tau)$ has a representative of the form

$$0 \to (E_2, V_2) \xrightarrow{\iota} (E, V) \xrightarrow{\lambda(\xi,\tau)} (E'', V'') \to 0$$

where $\lambda(\xi,\tau) := (\xi \cdot \sigma \circ \kappa + \tau \cdot \mathrm{id}_{(E'',V'')}) \circ \lambda$. Now we have that $M_2([\mu],[\nu])$ is sent to itself by such an action. The method for proving this is analogous to the one used in the same point in lemma 6.2.1, so we omit the details. So we have that $\mathbb{C}\times\mathbb{C}^*$ acts on $M([\mu],[\nu]) = M_1([\mu],[\nu]) \smallsetminus M_2([\mu],[\nu])$. Moreover, we can also prove easily that $(\mathbb{C}\times\mathbb{C}^*)(\mathrm{Im}(A)) \subset \mathrm{Im}(A) \subset M_2([\mu],[\nu])$. Therefore, such a group acts also on the subsets

$$M' := M_1([\mu],[\nu]) \smallsetminus \mathrm{Im}(A), \quad M'' := M_2([\mu],[\nu]) \smallsetminus \mathrm{Im}(A).$$

By construction we have:

$$M([\mu],[\nu]) = M_1([\mu],[\nu]) \smallsetminus M_2([\mu],[\nu]) = M' \smallsetminus M''.$$

Now let us consider the exact sequence (6.33). We have already proved in (6.76) that $\mathrm{Hom}((Q_3, W_3), (E_2, V_2)) = 0$. Moreover, by hypothesis

$$\mathrm{Ext}^2((Q_4, W_4), (E_2, V_2)) = 0.$$

Since $(Q_3, W_3) \simeq (Q_4, W_4)$, then (6.33) gives a short exact sequence as follows.

$$0 \to \mathrm{Ext}^1((Q_3, W_3), (E_2, V_2)) \xrightarrow{A} \mathrm{Ext}^1((E'', V''), (E_2, V_2)) \to \mathrm{Ext}^1((Q_3, W_3), (E_2, V_2)) \to 0.$$

So we have a (non-canonical) isomorphism

$$M' = M_1([\mu],[\nu]) \smallsetminus \mathrm{Im}(A) \simeq$$

$$\simeq \mathrm{Ext}^1((Q_3, W_3), (E_2, V_2)) \times (\mathrm{Ext}^1((Q_3, W_3), (E_2, V_2)) \smallsetminus \{0\}).$$

Under this isomorphism the action of $\mathbb{C}\times\mathbb{C}^*$ on M' is given for every pair of scalars (ξ,τ) and for every $(\eta_1,\eta_2) \in M'$ as $(\xi,\tau)\cdot(\eta_1,\eta_2) = (\tau\cdot\eta_1, \tau\cdot\eta_2 + \xi\cdot\eta_1)$. Now:

$$M' \simeq \mathbb{C}^c \smallsetminus \mathbb{C}^d = \mathbb{C}^d \times (\mathbb{C}^{c-d} \smallsetminus \{0\}),$$

$$M'' \simeq \mathbb{C}^{d+e-f} \smallsetminus \mathbb{C}^d = \mathbb{C}^d \times (\mathbb{C}^{e-f} \smallsetminus \{0\}).$$

Then:

$$\overline{M}([\mu],[\nu]) := M([\mu],[\nu])/(\mathbb{C}\times\mathbb{C}^*) = (H' \smallsetminus H'')/(\mathbb{C}\times\mathbb{C}^*) =$$

$$= (\mathbb{C}^{d-1} \times \mathbb{P}^{c-d-1}) \smallsetminus (\mathbb{C}^{d-1} \times \mathbb{P}^{e-f-1}).$$

So we conclude. □

Proposition 6.2.6. *Let us fix any triple (n, d, k), a critical value α_c for it and any quadruple $(n_i, d_i, k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c; n, d, k)$. Let us assume that conditions (6.24), respectively (6.25), are satisfied. Moreover, let us suppose that $(n_3, k_3) = (n_4, k_4)$. Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in $G^-(\alpha_c; n, d, k)$, that have unique α_c-Jordan-Hölder filtration at α_c and graded given by $\oplus_{i=1}^4 (Q_i, W_i)$ such that*

$$(Q_2, W_2) \not\simeq (Q_3, W_3) \simeq (Q_4, W_4), \quad (Q_1, W_1) \not\simeq (Q_i, W_i) \quad \forall\, i = 2, 3,$$
$$Ext^2\left((Q_4, W_4), (Q_1, W_1)\right) = 0 = Ext^2\left((Q_4, W_4), (E_2, V_2)\right),$$

where (E_2, V_2) is any non-split extension of (Q_2, W_2) by (Q_1, W_1).

$$(Q_1, W_1) \oplus (Q_2, W_2) \oplus (Q_3, W_3) \oplus (Q_3, W_3).$$

Then there exists a finite family $\{R_{a,b,c,d,e,f;i,j,l}\}$ of schemes for (a, b, c, d, e, f) in $\mathbb{N}^6$, and i, j, l varying in finite sets (for a, b, c, d, e, f fixed), together with injective morphisms to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Every scheme $R_{a,b,c,d,e,f;i,j,l}$ comes with a triple of morphisms

$$\varphi_{a,b,c,d,e,f;i,j,l} : R_{a,b,c,d,e,f;i,j,l} \longrightarrow U_{a,b,c,d,e,f:i,j,l} \subset R_{a;i} \times R^{b;j},$$
$$\varphi_{a;i} : R_{a;i} \longrightarrow U_{a;i} \subset G_1 \times G_2,$$
$$\varphi^{b;j} : R^{b;j} \longrightarrow U^{b;j} \subset G_3 = G_4$$

where:

- *$\varphi_{a,b,c,d,e,f;i,j,l}$ has fibers isomorphic to $\mathbb{C}^{d-1} \times (\mathbb{P}^{c-d-1} \smallsetminus \mathbb{P}^{e-f-1})$, $\varphi_{a;i}$ has fibers isomorphic to $\mathbb{P}^{a-1}$ and $\varphi^{b;j}$ has fibers isomorphic to $\mathbb{P}^{b-1}$;*
- *$\{U_{a;i}\}_i$ is a finite disjoint locally closed covering of*

$$U_a := \{((Q_1, W_1), (Q_2, W_2)) \in G_1 \times G_2 \ s.t. \ dim \ Ext^1((Q_2, W_2), (Q_1, W_1)) = a\};$$

  every U_a is a locally closed subscheme of $G_1 \times G_2$ and so are all the $U_{a;i}$'s;
- *$\{U^{b;j}\}_j$ is a finite disjoint locally closed covering of*

$$U^b := \{(Q_3, W_3) \in G_3 \ s.t. \ dim \ Ext^1((Q_3, W_3), (Q_3, W_3)) = b\};$$

  every U^b is a locally closed subscheme of G_3 and so are all the $U^{b;j}$'s;
- *$\{U_{a,b,c,d,e,f;i,j,l}\}_l$ is a finite disjoint locally closed covering of*

$$\begin{aligned} U_{a,b,c,d,e,f;i,j} := \{&((E_2, V_2), (E'', V'')) \in R_{a;i} \times R^{b;j} \ s.t. \ dim \ Ext^1((E'', V''), (E_2, V_2)) = c, \\ &dim \ Ext^1(\varphi^{b;j}(E'', V''), (E_2, V_2)) = d, \quad dim \ Ext^1((E'', V''), \widetilde{\varphi}_{a;i}(E_2, V_2)) = e, \\ &dim \ Ext^1(\varphi^{b;j}(E'', V''), \widetilde{\varphi}_{a;i}(E_2, V_2)) = f, \quad (\varphi^{b;j}(E'', V'') \not\simeq \overline{\varphi}_{a;i}(E_2, V_2))\}, \end{aligned} \tag{6.73}$$

where $\widetilde{\varphi}_{a;i}$ is the composition of $\varphi_{a;i}$ with the projection to G_1 and $\overline{\varphi}_{a;i}$ is the composition of $\varphi_{a;i}$ with the projection to G_2. Every $U_{a,b,c,d,e,f;i,j}$ is locally closed in $R_{a;i} \times R^{b;j}$ and so are all the $U_{a,b,c,d,e,f;i,j,l}$'s. The last condition of (6.73) can be omitted if $(n_2, k_2) \neq (n_3, k_3) = (n_4, k_4)$; it is necessary if $(n_2, k_2) = (n_3, k_3) = (n_4, k_4)$.

Proof. The proof is completely analogous to the one of proposition 6.2.2 using the results of lemma 6.2.5 instead of lemma 6.2.1. ☐

Lemma 6.2.7. *Let us fix any quadruple $(Q_i, W_i)_{i=1,\cdots,4} \in \prod_{i=1}^4 G_i$ and let us assume that*

$$(Q_2, W_2) \not\simeq (Q_3, W_3) \not\simeq (Q_4, W_4), \quad (Q_1, W_1) \not\simeq (Q_i, W_i) \quad \forall\, i = 2, 3, 4,$$
$$Ext^2\left((Q_4, W_4), (Q_1, W_1)\right) = 0 = Ext^2\left((Q_4, W_4), (E_2, V_2)\right),$$

where (E_2, V_2) is any non-split extension of (Q_2, W_2) by (Q_1, W_1). Let us denote by μ any class of a non-split extension of the form

$$0 \to (Q_1, W_1) \xrightarrow{\varepsilon} (E_2, V_2) \xrightarrow{\delta} (Q_2, W_2) \to 0 \tag{6.74}$$

and by ν any class of a non-split extension of the form

$$0 \to (Q_3, W_3) \xrightarrow{\sigma} (E'', V'') \xrightarrow{\kappa} (Q_4, W_4) \to 0. \tag{6.75}$$

Then let us consider the space $M_1([\mu], [\nu]) := Ext^1((E'', V''), (E_2, V_2))$ and let us denote by $M_2([\mu], [\nu])$ the image of the linear map $A + F$ where A and F are the maps induced by (6.75) and (6.74) respectively, as follows:

$$A : Ext^1((Q_4, W_4), (E_2, V_2)) \longrightarrow Ext^1((E'', V''), (E_2, V_2)) = M_1([\mu], [\nu]),$$
$$F : Ext^1((E'', V''), (Q_1, W_1)) \longrightarrow Ext^1((E'', V''), (E_2, V_2)) = M_1([\mu], [\nu]).$$

Then the objects (E, V)'s with unique α_c-Jordan-Hölder filtration and graded $\oplus_{i=1}^4 (Q_i, W_i)$ are parametrized by triples $([\mu], [\nu], [\eta])$, where:

- *$[\mu] \in \mathbb{P}(Ext^1((Q_2, W_2), (Q_1, W_1)))$ and μ has a representative of the form (6.74);*
- *$[\nu] \in \mathbb{P}(Ext^1((Q_4, W_4), (Q_3, W_3)))$ and ν has a representative of the form (6.75);*
- *$[\eta] \in \overline{M}([\mu], [\nu]) \in \mathbb{P}(M_1([\mu], [\eta])) \smallsetminus \mathbb{P}(M_2([\mu], [\nu]))$.*

Moreover, if we write:

$$c := dim\ M_1([\mu], [\nu]), \quad d := dim\ Ext^1((Q_4, W_4), (E_2, V_2)),$$
$$e := dim\ Ext^1((E'', V''), (Q_1, W_1)), \quad f := dim\ Ext^1((Q_4, W_4), (Q_1, W_1)),$$

then $dim\ M_2([\mu], [\nu]) = d + e - f$ and $\mathbb{P}(M_1([\mu], [\nu])) \smallsetminus \mathbb{P}(M_2([\mu], [\nu])) \simeq \mathbb{P}^{c-1} \smallsetminus \mathbb{P}^{d+e-f-1}$.

Proof. The proof is analogous to the proof of lemma 6.2.3. In this lemma we need to consider 2 subcases as follows:

(c) $\mathrm{Hom}\left((Q_4, W_4), (Q_2, W_2)\right) = 0$;

(d) $\mathrm{Hom}\left((Q_4, W_4), (Q_2, W_2)\right) = \mathbb{C}$.

Since (Q_2, W_2) and (Q_4, W_4) are both α_c-stable of the same slope, these are the only 2 possibilities. Using the hypotheses of this lemma, these conditions can be restated as:

(c) $(Q_2, W_2) \not\simeq (Q_3, W_3) \not\simeq (Q_4, W_4)$ and $(Q_2, W_2) \not\simeq (Q_4, W_4)$;

(d) $(Q_2, W_2) \not\simeq (Q_3, W_3) \not\simeq (Q_4, W_4)$ and $(Q_2, W_2) \simeq (Q_4, W_4)$.

Then in this case the roles of A and F are reversed with respect to the proof of lemma 6.2.3. To be more precise, with the same ideas of that lemma one can prove the following facts.

- In (6.33) we have that:

$$\mathrm{Hom}((Q_3, W_3), (E_2, V_2)) = 0. \tag{6.76}$$

  Therefore A is always injective, so dim $\mathrm{Im}(A) = d$.

- F is injective in case (c) and has a 1-dimensional kernel in case (d), so dim $\mathrm{Im}(F)$ is equal to e or $e - 1$ according to the 2 cases;

- $M_3([\mu], [\nu])$ again coincides with $\mathrm{Im}(A \circ L) = \mathrm{Im}(F \circ M)$. In this case using the sequence (6.37) we get that L is injective in case (c) and has a 1-dimensional kernel in case (d). Since A is injective, then dim $M_3([\mu], [\nu])$ = dim $\mathrm{Im}(A \circ L)$ is equal to f or to $f - 1$ according to the 2 cases.

Then both in case (c) and in case (d) we get that

$$\dim M_2([\mu], [\nu]) = d + e - f = d + (e - 1) - (f - 1). \tag{6.77}$$

Then the proof is on the same line of the proof of lemma 6.2.3, so we omit it. □

Proposition 6.2.8. *Let us fix any triple (n, d, k), a critical value α_c for it and a quadruple $(n_i, d_i, k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c; n, d, k)$. Let us assume that conditions (6.24), respectively (6.25), are satisfied and that for every quadruple $(Q_i, W_i)_{i=1,\cdots,4} \in \prod_{i=1}^{4} G_i$ we have $(Q_1, W_1) \not\simeq (Q_i, W_i)$ for $i = 2, 3, 4$. Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in $G^-(\alpha_c; n, d, k)$, that have unique α_c-Jordan-Hölder filtration at α_c and graded in $\prod_{i=1}^{4} G_i$, such that:*

$$(Q_2, W_2) \not\simeq (Q_3, W_3) \not\simeq (Q_4, W_4), \quad \mathit{Ext}^2\left((Q_4, W_4), (Q_1, W_1)\right) = 0 = \mathit{Ext}^2\left((Q_4, W_4), (E_2, V_2)\right),$$

where (E_2, V_2) is any non-split extension of (Q_2, W_2) by (Q_1, W_1). Then there exists a finite family $\{R_{a,b,c,d,e,f;i,j,l}\}$ of schemes for $(a,b,c,d,e,f) \in \mathbb{N}^6$, and i,j,l varying in finite sets (for a,b,c,d,e,f fixed), together with injective morphisms to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Every scheme $R_{a,b,c,d,e,f;i,j,l}$ comes with a triple of morphisms

$$\begin{aligned} \varphi_{a,b,c,d,e,f;i,j,l} : R_{a,b,c,d,e,f;i,j,l} &\longrightarrow U_{a,b,c,d,e,f:i,j,l} \subset R_{a;i} \times R^{b;j}, \\ \varphi_{a;i} : R_{a;i} &\longrightarrow U_{a;i} \subset G_1 \times G_2, \\ \varphi^{b;j} : R^{b;j} &\longrightarrow U^{b;j} \subset G_3 \times G_4 \end{aligned}$$

where:

- *$\varphi_{a,b,c,d,e,f;i,j,l}$ has fibers isomorphic to $\mathbb{P}^{c-1} \setminus \mathbb{P}^{d+e-f-1}$, $\varphi_{a;i}$ has fibers isomorphic to $\mathbb{P}^{a-1}$ and $\varphi^{b;j}$ has fibers isomorphic to $\mathbb{P}^{b-1}$;*
- *$\{U_{a;i}\}_i$ is a finite disjoint locally closed covering of*

$$U_a := \{((Q_1, W_1), (Q_2, W_2)) \in G_1 \times G_2 \ s.t. \ dim \ Ext^1((Q_2, W_2), (Q_1, W_1)) = a\};$$

  every U_a is a locally closed subscheme of $G_1 \times G_2$ and so are all the $U_{a;i}$'s.
- *$\{U^{b;j}\}_j$ is a finite disjoint locally closed covering of*

$$\begin{gathered} U^b := \{((Q_3, W_3), (Q_4, W_4)) \in G_3 \times G_4 \ s.t. \\ dim \ Ext^1((Q_4, W_4), (Q_3, W_3)) = b, \quad (Q_3, W_3) \not\simeq (Q_4, W_4)\}; \end{gathered}$$

  every U^b is a locally closed subscheme of $G_3 \times G_4$ and so are all the $U^{b;j}$'s. The last condition on U^b can be dropped if $(n_3, k_3) \neq (n_4, k_4)$, otherwise it is necessary;
- *$\{U_{a,b,c,d,e,f;i,j,l}\}_l$ is a finite disjoint locally closed covering of*

$$\begin{aligned} U_{a,b,c,d,e,f;i,j} := \{((E_2, V_2), (E'', V'')) \in R_{a;i} \times R^{b;j} \ s.t. \ dim \ Ext^1((E'', V''), (E_2, V_2)) = c, \\ dim \ Ext^1(\widetilde{\varphi}^{b;j}(E'', V''), (E_2, V_2)) = d, \quad dim \ Ext^1((E'', V''), \widetilde{\varphi}_{a;i}(E_2, V_2)) = e, \\ dim \ Ext^1(\widetilde{\varphi}^{b;j}(E'', V''), \widetilde{\varphi}_{a;i}(E_2, V_2)) = f, \quad \overline{\varphi}^{b;j}(E'', V'') \not\simeq \overline{\varphi}_{a;i}(E_2, V_2)\}, \end{aligned}$$

  where

  - *$\widetilde{\varphi}_{a;i}$ is the composition of $\varphi_{a;i}$ with the projection to G_1;*
  - *$\overline{\varphi}_{a;i}$ is the composition of $\varphi_{a;i}$ with the projection to G_2;*
  - *$\overline{\varphi}^{b;j}$ is the composition of $\varphi^{b;j}$ with the projection to G_3;*

– *$\widetilde{\varphi}^{b;j}$ is the composition of $\varphi^{b;j}$ with the projection to G_4.*

Every $U_{a,b,c,d,e,f;i,j}$ is locally closed in $R_{a;i} \times R^{b;j}$ and so are all the $U_{a,b,c,d,e,f;i,j,l}$'s. The last condition on $U_{a,b,c,d,e,f;i,j}$ can be dropped if $(n_2, k_2) \neq (n_3, k_3)$, otherwise it is necessary.

Proof. The proof is analogous to the proof of proposition 6.2.4. We simply use lemma 6.2.5 instead of lemma 6.2.3. □

Remark 6.2.2. Let us suppose that all the following conditions are satisfied:

- $(Q_1, W_1) \not\simeq (Q_i, W_i)$ for all $i = 2, 3, 4$;
- $(Q_4, W_4) \not\simeq (Q_i, W_i)$ for all $i = 2, 3$;
- $\mathrm{Ext}^2((Q_4, W_4), (Q_1, W_1)) = 0$;
- $\mathrm{Ext}^2((Q_4, W_4), (E_2, V_2)) = 0$ for all non-split extensions (E_2, V_2) of (Q_2, W_2) by (Q_1, W_1);
- $\mathrm{Ext}^2((E'', V''), (Q_1, W_1)) = 0$ for all non-split extensions (E'', V'') of (Q_4, W_4) by (Q_3, W_3).

Then we can apply both proposition 6.2.4 and proposition 6.2.8, and the 2 descriptions give rise to the same Hodge-Deligne polynomials, as they should do.

Remark 6.2.3. When the JHF is unique and has length equal to 4 there are still 4 subcases that we are not able to describe completely. These are in bijection with the subcases described in lemmas 6.2.1, 6.2.3, 6.2.5 and 6.2.7, with the only significant difference that in these subcases $(Q_2, W_2) \simeq (Q_3, W_3)$ instead of $(Q_2, W_2) \not\simeq (Q_3, W_3)$. To be more precise, the 4 subcases we still have to describe completely are as follows:

- $(Q_1, W_1) \simeq (Q_2, W_2) \simeq (Q_3, W_3), \quad (Q_4, W_4) \not\simeq (Q_i, W_i) \quad \forall\, i = 1, 2, 3,$
- $(Q_1, W_1) \not\simeq (Q_2, W_2) \simeq (Q_3, W_3), \quad (Q_4, W_4) \not\simeq (Q_i, W_i) \quad \forall\, i = 1, 2, 3,$
- $(Q_2, W_2) \simeq (Q_3, W_3) \simeq (Q_4, W_4), \quad (Q_1, W_1) \not\simeq (Q_i, W_i) \quad \forall\, i = 2, 3, 4,$
- $(Q_2, W_2) \simeq (Q_3, W_3) \not\simeq (Q_4, W_4), \quad (Q_1, W_1) \not\simeq (Q_i, W_i) \quad \forall\, i = 2, 3, 4.$

The point where the previous computations fail is where we need to prove that the morphisms of the form $\gamma = \delta' \circ \sigma'$ are not injective. Actually, in any of the previous cases such morphisms can actually be isomorphisms, so our construction does not work. In particular, in any of the previous 4 cases a direct computation shows that this implies that the space $M_3([\mu], [\nu])$ contains strictly the space $M_3'([\mu], [\nu])$, therefore it is not currently possible to compute the dimensions of $M_3([\mu], [\nu])$ and of $M_2([\mu], [\nu])$.

Chapter 7

Parametrization of coherent systems with non-unique Jordan-Hölder filtration

In this chapter we summarize the parametrizations for those coherent systems (E, V) that belong to $G^+(\alpha_c; n, d, k)$ or to $G^-(\alpha_c; n, d, k)$ and that have non-unique α_c-Jordan-Hölder filtration of length 3 or 4. For the proof of each result, see part II of this work.

7.1 Canonical filtration of type (1,2)

Proposition 7.1.1. *Let us fix any triple (n, d, k), a critical value α_c for it and a triple $(n_i, d_i, k_i)_{i=1,2,3}$ compatible with $(\alpha_c; n, d, k)$. Let us assume that $(n_2, k_2) \neq (n_3, k_3)$ and that*

$$\frac{k_i}{n_i} > \frac{k}{n} \quad \forall\, i = 2, 3, \tag{7.1}$$

respectively that

$$\frac{k_i}{n_i} < \frac{k}{n} \quad \forall\, i = 2, 3, \tag{7.2}$$

Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in $G^-(\alpha_c; n, d, k)$, such that

(i) they have graded at α_c given by $\oplus_{i=1}^3 (Q_i, W_i)$ of type $(n_i, d_i, k_i)_{i=1,2,3}$;

(ii) their α_c-canonical filtration is of type $(1, 2)$.

Then there is a finite family $\{R_{a,b;i,j}\}$ of schemes for $(a, b) \in \mathbb{N}^2$ and i, j varying in finite sets (for a, b fixed), together with injective morphisms to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c)$, such that the images form a disjoint covering of G' by locally closed subschemes. Every scheme $R_{a,b;i,j}$ comes with a sequence of 2 projective fibrations:

$$R_{a,b;i,j} \xrightarrow{\phi_1} A_{a,b;i,j} \xrightarrow{\phi_2} U_{a,b;i,j}$$

such that:

- ϕ_1 *has fibers isomorphic to* $\mathbb{P}^{b-1}$*, while* ϕ_2 *has fibers isomorphic to* $\mathbb{P}^{a-1}$*;*
- *every* $U_{a,b;i,j}$ *is the fiber product of* $U^2_{a;i}$ *and* $U^3_{b;j}$ *over* G_1*, where* $\{U^2_{a;i}\}_i$ *is a locally closed disjoint covering of*

$$U^2_a := \{((Q_1,W_1),(Q_2,W_2)) \in G_1 \times G_2 \ s.t. \ dim\ Ext^1((Q_2,W_2),(Q_1,W_1)) = a\}$$

and analogously for $U^3_{b;j} \subset G_1 \times G_3$*. In particular,* $\{U_{a;b;i,j}\}_{a,b;i,j}$ *is a disjoint covering of* $G_1 \times G_2 \times G_3$ *by locally closed subschemes.*

Proposition 7.1.2. *Let us fix any triple* (n,d,k)*, a critical value* α_c *for it and a triple* $(n_i,d_i,k_i)_{i=1,2,3}$ *compatible with* $(\alpha_c;n,d,k)$*. Let us assume that conditions (7.1), respectively (7.2), are satisfied. Moreover, let us suppose that* $(n_2,k_2)=(n_3,k_3)$*. Let us denote by* G' *the set of all the* (E,V)*'s in* $G^+(\alpha_c;n,d,k)$*, respectively in* $G^-(\alpha_c;n,d,k)$ *such that*

(i) they have graded at α_c *given by* $\oplus_{i=1}^3(Q_i,W_i)$ *of type* $(n_i,d_i,k_i)_{i=1,2,3}$*;*

(ii) $(Q_2,W_2)\not\simeq(Q_3,W_3)$*;*

(iii) their α_c*-canonical filtration is of type* $(1,2)$*.*

Then there is a finite family of schemes as follows:

(a) $R^1 = R_{a,b;i,j}$ *for every* $a<b\in\mathbb{N}_0$ *and* $(i,j)\in L^2_a\times L^2_b$*;*

(b) $R^2 = R_{a,a;i,j}|_{G_1\times(G_2\times G_3\smallsetminus\Delta_{23})\smallsetminus U_{a,a;i,j}\cap U_{a,a;j,i}}$ *for every* $a\in\mathbb{N}_0$ *and* $i<j\in L^2_a$*;*

(c) $R^3 = (R_{a,a;i,j}|_{G_1\times(G_2\times G_3\smallsetminus\Delta_{23})\cap U_{a,a;i,j}\cap U_{a,a;j,i}})/\mathbb{Z}_2$ *for every* $a\in\mathbb{N}_0$ *and* $i<j\in L^2_a$*;*

(d) $R^4 = (R_{a,a;i,i}|_{G_1\times(G_2\times G_3\smallsetminus\Delta_{23})})/\mathbb{Z}_2$ *for every* $a\in\mathbb{N}_0$ *and* $i\in L^2_a$*;*

where all the schemes of the form $R_{a,b;i,j}$ *are obtained exactly as in proposition 7.1.1, together with the same pairs of projective fibrations to the corresponding base* $U_{a,b;i,j}$*. Each scheme of type (a)-(d) comes with an injective morphism to* $G^+(\alpha_c;n,d,k)$*, respectively to* $G^-(\alpha_c;n,d,k)$*, such that the images form a disjoint locally closed covering of* G'*.*

The last 2 types of schemes come together with actions of $\mathbb{Z}_2$ *on the base space and on the fibers (compatible with the projective fibrations) as follows*

- $(Q_i,W_i)_{i=1,2,3}\mapsto(Q_i,W_i)_{i=1,3,2}$ *for every point of* $U_{a,a;i,j}$*;*

- $(\mu_2, \mu_3) \mapsto (\mu_3, \mu_2)$ *for every point* (μ_2, μ_3) *in the fiber over a triple* $(Q_i, W_i)_{i=1,2,3} \in U_{a,a;i,j}$.

Moreover, for every scheme R *of type (c) and (d) there exists a finite disjoint covering of its base space in* $G_1 \times G_2 \times G_3$ *by locally closed subschemes* T_l *that are invariant under the action of* $\mathbb{Z}_2$ *on* $G_1 \times G_2 \times G_3$*; in addition, there exist trivializations of the fibrations from* R *to* $U_{a,a;i,j}$

$$R|_{T_l} \xrightarrow{\sim} T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$$

that are compatible with the natural action of $\mathbb{Z}_2$ *on* $T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$.

Proposition 7.1.3. *Let us fix any triple* (n, d, k)*, a critical value* α_c *for it and a triple* $(n_i, d_i, k_i)_{i=1,2,3}$ *compatible with* $(\alpha_c; n, d, k)$*. Let us assume that conditions (7.1), respectively (7.2), are satisfied. Moreover, let us assume that* $(n_2, k_2) = (n_3, k_3)$*. Let us denote by* G' *the set of the* (E, V)*'s in* $G^+(\alpha_c; n, d, k)$*, respectively in* $G^-(\alpha_c; n, d, k)$*, such that:*

(i) they have graded at α_c *given by* $\oplus_{i=1}^3 (Q_i, W_i)$ *of type* $(n_i, d_i, k_i)_{i=1,2,3}$*;*

(ii) $(Q_2, W_2) \simeq (Q_3, W_3)$*;*

(iii) their α_c*-canonical filtration is of type* $(1, 2)$.

Then there is a finite family $\{R_{2;a;i} = Grass(2, R_{a;i})\}$ *for* $a \in \mathbb{N}$ *and* i *varying in a finite set (for a fixed), together with injective morphisms to* $G^+(\alpha_c; n, d, k)$*, respectively to* $G^-(\alpha_c; n, d, k)$*, such that the images form a disjoint locally closed covering of* G'*. Here each* $R_{a;i}$ *is a vector bundle over* $U_{a,i}$ *with fibers isomorphic to* $\mathbb{C}^a$ *and* $\{U_{a;i}\}_i$ *is a locally closed disjoint covering of*

$$U_a := \{((Q_1, W_1), (Q_2, W_2)) \in G_1 \times G_2 \text{ s.t } \dim \mathit{Ext}^1((Q_2, W_2), (Q_1, W_1)) = a\}.$$

7.2 Canonical filtration of type (2,1)

Proposition 7.2.1. *Let us fix any triple* (n, d, k) *a critical value* α_c *for it and a triple* $(n_i, d_i, k_i)_{i=1,2,3}$ *compatible with* $(\alpha_c; n, d, k)$*. Let us assume that*

$$\frac{k_i}{n_i} < \frac{k}{n} \quad \forall\, i = 1, 2. \tag{7.3}$$

respectively that

$$\frac{k_i}{n_i} > \frac{k}{n} \quad \forall\, i = 1, 2. \tag{7.4}$$

Moreover, let us suppose that $(n_1, k_1) \neq (n_2, k_2)$*. Then let us denote by* G' *the set of all the* (E, V)*'s in* $G^+(\alpha_c; n, d, k)$*, respectively in* $G^-(\alpha_c; n, d, k)$*, such that* *(i) they have graded at* α_c *given by* $\oplus_{i=1}^3(Q_i, W_i)$ *of type* $(n_i, d_i, k_i)_{i=1,2,3}$*;*

(ii) their α_c*-canonical filtration is of type* $(2, 1)$*.*

Then there is a finite family of schemes $\{R_{a,b;i,j}\}$ *for* $(a, b) \in \mathbb{N}^2$ *and* i, j *varying in finite sets (for* a, b *fixed), together with injective morphisms to* $G^+(\alpha_c; n, d, k)$*, respectively to* $G^-(\alpha_c; n, d, k)$*, such that the images form a disjoint locally closed covering of* G'*. Every* $R_{a,b;i,j}$ *comes with a sequence of 2 projective filtrations*

$$R_{a,b;i,j} \xrightarrow{\phi_1} A_{a,b;i,j} \xrightarrow{\phi_2} U_{a,b;i,j},$$

such that:

- ϕ_1 *has fibers isomorphic to* $\mathbb{P}^{b-1}$*, while* ϕ_2 *has fibers isomorphic to* $\mathbb{P}^{a-1}$*;*
- *every* $U_{a,b;i,j}$ *is the fiber product of* $U^1_{a;i}$ *and* $U^2_{b;j}$ *over* G_3*, where* $\{U^1_{a;i}\}_i$ *is a disjoint locally closed covering of*

$$U^1_a := \{((Q_1, W_1), (Q_3, W_3)) \in G_1 \times G_3 \text{ s.t. } \textit{dim } \textit{Ext}^1((Q_3, W_3), (Q_1, W_1)) = a\}$$

and analogously for $U^2_{b;j} \subset G_2 \times G_3$*.*

Proposition 7.2.2. *Let us fix any triple* (n, d, k)*, a critical value* α_c *for it and a triple* $(n_i, d_i, k_i)_{i=1,2,3}$ *compatible with* $(\alpha_c; n, d, k)$*. Let us suppose that conditions (7.3), respectively (7.4) are satisfied. Moreover, let us assume that* $(n_1, k_1) = (n_2, k_2)$*. Let us denote by* G' *the set of all the* (E, V)*'s in* $G^+(\alpha_c; n, d, k)$*, respectively in* $G^-(\alpha_c; n, d, k)$*, such that*

(i) they have graded at α_c *given by* $\oplus_{i=1}^3(Q_i, W_i)$ *of type* $(n_i, d_i, k_i)_{i=1,2,3}$*;*

(ii) $(Q_1, W_1) \not\simeq (Q_2, W_2)$*,*

(iii) their α_c*-canonical filtration is of type* $(2, 1)$*.*

Then there is a finite family of schemes as follows:

(a) $R^1 = R_{a,b;i,j}$ *for every* $a < b \in \mathbb{N}_0$ *and* $(i, j) \in L^1_a \times L^2_b$*;*

(b) $R^2 = R_{a,a;i,j}|_{(G_1\times G_2\smallsetminus\Delta_{12})\times G_3\smallsetminus U_{a,a;i,j}\cap U_{a,a;j,i}}$ *for every* $a \in \mathbb{N}_0$ *and* $i < j \in L^1_a$*;*

(c) $R^3 = (R_{a,a;i,j}|_{(G_1\times G_2\smallsetminus\Delta_{12})\times G_3\cap U_{a,a;i,j}\cap U_{a,a;j,i}})/\mathbb{Z}_2$ *for every* $a \in \mathbb{N}_0$ *and* $i < j \in L^1_a$*;*

(d) $R^4 = (R_{a,a;i,i}|_{(G_1\times G_2\smallsetminus\Delta_{12})\times G_3})/\mathbb{Z}_2$ *for every* $a \in \mathbb{N}_0$ *and* $i \in L^1_a$*;*

where all the schemes of the form $R_{a,b;i,j}$ *are obtained exactly as in proposition 7.2.1, together with the same pairs of projective fibrations to the corresponding base* $U_{a,b;i,j}$*. Each scheme of type (a)-(d) comes with an injective morphism to* $G^+(\alpha_c; n, d, k)$*, respectively to* $G^-(\alpha_c; n, d, k)$*, such that the images form a disjoint locally closed covering of* G'*.*

The last 2 types of schemes come together with actions of $\mathbb{Z}_2$ *on the base space and on the fibers (compatible with the projective fibrations) as follows*

- $(Q_i, W_i)_{i=1,2,3} \mapsto (Q_i, W_i)_{i=2,1,3}$ *for every point of* $U_{a,a;i,j}$*;*
- $(\mu_1, \mu_2) \mapsto (\mu_2, \mu_1)$ *for every point* (μ_1, μ_2) *in the fiber over a triple* $(Q_i, W_i)_{i=1,2,3} \in U_{a,a;i,j}$.

Moreover, for every scheme R *of type (c) and (d) there exists a finite disjoint covering of its base space in* $G_1 \times G_2 \times G_3$ *by locally closed subschemes* T_l *that are invariant under the action of* $\mathbb{Z}_2$ *on* $G_1 \times G_2 \times G_3$*; in addition, there exist trivializations of the fibrations from* R *to* $U_{a,a;i,j}$*:*

$$R|_{T_l} \xrightarrow{\sim} T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$$

that are compatible with the natural action of $\mathbb{Z}_2$ *on* $T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$.

Proposition 7.2.3. *Let us fix any triple* (n, d, k)*, a critical value* α_c *for it and a triple* $(n_i, d_i, k_i)_{i=1,2,3}$ *compatible with* $(\alpha_c; n, d, k)$*. Let us suppose that conditions (7.3), respectively (7.4), are satisfied. Moreover, let us assume that* $(n_1, k_1) = (n_2, k_2)$*. Let us denote by* G' *the set of all the* (E, V)*'s in* $G^+(\alpha_c; n, d, k)$*, respectively in* $G^-(\alpha_c; n, d, k)$*, such that*

(i) they have graded at α_c *given by* $\oplus_{i=1}^3 (Q_i, W_i)$ *of type* $(n_i, d_i, k_i)_{i=1,2,3}$*;*

(ii) $(Q_1, W_1) \simeq (Q_2, W_2)$*;*

(iii) their α_c*-canonical filtration is of type* $(2, 1)$.

Then there is a finite family $\{R_{2;a;i} = \mathit{Grass}(2, R_{a;i})\}$ *for* $a \in \mathbb{N}$ *and* i *varying in a finite set (for* a *fixed), together with injective morphisms to* $G^+(\alpha_c; n, d, k)$*, respectively to* $G^-(\alpha_c; n, d, k)$*, such that the images form a disjoint locally closed covering of* G'*. Here each* $R_{a;i}$ *is a vector bundle over* $U_{a;i}$ *with fibers isomorphic to* $\mathbb{C}^a$ *and* $\{U_{a;i}\}_i$ *is a locally closed disjoint covering of*

$$U_a := \{((Q_1, W_1), (Q_3, W_3)) \in G_1 \times G_3 \ \mathit{s.t}\ \mathit{dim}\ \mathit{Ext}^1((Q_3, W_3), (Q_1, W_1)) = a\}.$$

7.3 Canonical filtration of type (1,3)

Proposition 7.3.1. *Let us fix any triple* (n, d, k)*, a critical value* α_c *for it and a quadruple* $(n_i, d_i, k_i)_{i=1,\cdots,4}$ *compatible with* $(\alpha_c; n, d, k)$*. Let us assume that* $(n_i, k_i) \neq (n_j, k_j)$ *for* $i \neq j \in \{2, 3, 4\}$ *and that*

$$\frac{k_l}{n_l} > \frac{k}{n} \quad \forall\, l \in \{2, 3, 4\}, \tag{7.5}$$

respectively that

$$\frac{k_l}{n_l} < \frac{k}{n} \quad \forall\, l \in \{2, 3, 4\}. \tag{7.6}$$

Let us denote by G' *the set of all the* (E, V)*'s in* $G^+(\alpha_c; n, d, k)$*, respectively in* $G^-(\alpha_c; n, d, k)$*, such that* *(i) they have graded at α_c given by $\oplus_{i=1}^4(Q_i, W_i)$ of type $(n_i, d_i, k_i)_{i=1,\cdots,4}$;*

(ii) their α_c-canonical filtration is of type $(1, 3)$.

Then there is a finite family $\{R_{a,b,c;i,j,k}\}$ of schemes for $(a, b, c) \in \mathbb{N}^3$ and i, j, k varying in finite sets L_a^2, L_b^3, L_c^4 (for a, b, c fixed), together with injective morphisms to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Every scheme $R_{a,b,c;i,j,k}$ comes with a sequence of 3 morphisms:

$$R_{a,b,c;i,j} \xrightarrow{\phi_1} C_{a,b,c;i,j,k} \xrightarrow{\phi_2} B_{a,b,c;i,j,k} \xrightarrow{\phi_3} U_{a,b,c;i,j,k}$$

such that:

- *ϕ_1 has fibers isomorphic to $\mathbb{P}^{c-1}$, ϕ_2 has fibers isomorphic to $\mathbb{P}^{b-1}$ and ϕ_3 has fibers isomorphic to $\mathbb{P}^{a-1}$;*
- *every $U_{a,b,c;i,j,k}$ is the fiber product*

$$U_{a,b,c;i,j,k} = U_{a;i}^2 \times_{G_1} U_{b;j}^3 \times_{G_1} U_{c;k}^4,$$

where the set $\{U_{a;i}^2\}_i$ is a locally closed disjoint covering of

$$U_a^2 := \{((Q_1, W_1), (Q_2, W_2)) \in G_1 \times G_2 \ s.t. \ dim\ Ext^1((Q_2, W_2), (Q_1, W_1)) = a\}$$

and analogously for $U_{b;j}^3 \subset G_1 \times G_3$ and $U_{c;k}^4 \subset G_1 \times G_4$. In particular, $\{U_{a;b,c;i,j,k}\}_{a,b,c;i,j,k}$ is a disjoint covering of $G_1 \times G_2 \times G_3 \times G_4$ by locally closed subschemes.

Proposition 7.3.2. *Let us fix any triple (n, d, k), a critical value α_c for it and a quadruple $(n_i, d_i, k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c; n, d, k)$. Let us assume that conditions (7.5), respectively (7.6), are satisfied. Moreover, let us suppose that $(n_2, k_2) = (n_3, k_3) \neq (n_4, k_4)$. Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in $G^-(\alpha_c; n, d, k)$, such that*

(i) they have graded at α_c given by $\oplus_{i=1}^4(Q_i, W_i)$ of type $(n_i, d_i, k_i)_{i=1,\cdots,4}$;

(ii) $(Q_2, W_2) \not\simeq (Q_3, W_3)$;

(iii) their α_c-canonical filtration is of type $(1, 3)$.

Then there is a finite family of schemes as follows:

(a) $R^1 = R_{a,b,c;i,j,k}$ for every $a, b, c \in \mathbb{N}_0, a < b$ and $(i, j, k) \in L_a^2 \times L_b^3 \times L_c^4$;

(b) $R^2 = R_{a,a,c;i,j,k}|_{G_1 \times (G_2 \times G_3 \smallsetminus \Delta_{23}) \times G_4 \smallsetminus U_{a,a,c;i,j,k} \cap U_{a,a,c;j,i,k}}$ for every $a, c \in \mathbb{N}_0, i < j \in L_a^2$ and $k \in L_c^4$;

(c) $R^3 = (R_{a,a,c;i,j,k}|_{G_1 \times (G_2 \times G_3 \smallsetminus \Delta_{23}) \times G_4 \cap U_{a,a,c;i,j,k} \cap U_{a,a,c;j,i,k}})/\mathbb{Z}_2$ for every $a, c \in \mathbb{N}_0$ and $i < j \in L_a^2, k \in L_c^4$;

(d) $R^4 = (R_{a,a,c;i,i,k}|_{G_1 \times (G_2 \times G_3 \smallsetminus \Delta_{23}) \times G_4})/\mathbb{Z}_2$ *for every* $a, c \in \mathbb{N}_0$ *and* $i \in L^2_a, k \in L^4_c$*;*

where all the schemes of the form $R_{a,b,c;i,j,k}$ *are obtained exactly as in proposition 7.3.1, together with the same triples of projective fibrations to the corresponding base* $U_{a,b,c;i,j,k}$*. Each scheme of type (a)-(d) comes with an injective morphism to* $G^+(\alpha_c; n, d, k)$*, respectively to* $G^-(\alpha_c; n, d, k)$*, such that the images form a disjoint locally closed covering of* G'*.*

The last 2 types of schemes come together with actions of $\mathbb{Z}_2$ *on the base space and on the fibers (compatible with the projective fibrations) as follows:*

- $(Q_i, W_i)_{i=1,2,3,4} \mapsto (Q_i, W_i)_{i=1,3,2,4}$ *for every point of* $U_{a,a,c;i,j,k}$*,*
- $(\mu_2, \mu_3, \mu_4) \mapsto (\mu_3, \mu_2, \mu_4)$ *for every point* (μ_2, μ_3, μ_4) *in the fiber over a quadruple* $(Q_i, W_i)_{i=1,\cdots,4} \in U_{a,a,c;i,j,k}$*.*

Moreover, for every scheme R of type (c) and (d) there exists a finite covering of its base space in $G_1 \times G_2 \times G_3 \times G_4$ *by locally closed subschemes* T_l *that are invariant under the action of* $\mathbb{Z}_2$ *on* $G_2 \times G_3 = G_2 \times G_2$ *in* $G_1 \times G_2 \times G_3 \times G_4$*; in addition, there exist trivializations of the fibrations from R to* $U_{a,a,c;i,j,k}$

$$R|_{T_l} \xrightarrow{\sim} T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{c-1}$$

that are compatible with the natural action of $\mathbb{Z}_2$ *on* $T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{c-1}$*.*

Proposition 7.3.3. *Let us fix any triple* (n, d, k)*, a critical value* α_c *for it and a quadruple* $(n_i, d_i, k_i)_{i=1,\cdots,4}$ *compatible with* $(\alpha_c; n, d, k)$*. Let us assume that conditions (7.5), respectively (7.6), are satisfied. Moreover, let us assume that* $(n_2, k_2) = (n_3, k_3) \neq (n_4, k_4)$*. Let us denote by* G' *the set of all the* (E, V)*'s in* $G^+(\alpha_c; n, d, k)$*, respectively in* $G^-(\alpha_c; n, d, k)$*, such that:*

(i) they have graded at α_c *given by* $\oplus_{i=1}^4 (Q_i, W_i)$ *of type* $(n_i, d_i, k_i)_{i=1,\cdots,4}$*;*

(ii) $(Q_2, W_2) \simeq (Q_3, W_3)$*;*

(iii) their α_c*-canonical filtration is of type* $(1, 3)$*.*

Then there is finite family $\{R_{a,b;i,j}\}$ *for* $(a, b) \in \mathbb{N}^2$ *and* i, j *varying in finite sets (for* a, b *fixed) together with injective morphisms to* $G^+(\alpha_c; n, d, k)$*, respectively to* $G^-(\alpha_c; n, d, k)$*, such that the images form a disjoint locally closed covering of* G'*. Each* $R_{a,b;i,j}$ *comes with a sequence of 2 fibrations:*

$$R_{a,b;i,j} \xrightarrow{\phi_1} A_{a,b;i,j} \xrightarrow{\phi_2} U_{a,b;i,j}$$

such that:

- ϕ_1 *has fibers isomorphic to* $\mathbb{P}^{b-1}$ *and* ϕ_2 *is a grassmannian fibration* $Grass(2, Q_{a,b;i,j})$*, where* $Q_{a,b;i,j}$ *is a locally trivial fibration of rank a over* $U_{a,b;i,j}$*;*

- *$U_{a,b;i,j}$ is the fiber product of $U^2_{a;i}$ and $U^4_{b;j}$ over G_1, where the set $\{U^2_{a;i}\}_i$ is a disjoint locally closed covering of*

$$U^2_a := \{((Q_1,W_1),(Q_2,W_2)) \in G_1 \times G_2 \text{ s.t. } \dim \mathrm{Ext}^1((Q_2,W_2),(Q_1,W_1)) = a\}$$

and analogously for $U^4_{b;j} \subset G_1 \times G_4$. In particular, $\{U_{a,b;i,j}\}_{a,b;i,j}$ is a disjoint covering of $G_1 \times G_2 \times G_4$ by locally closed subschemes.

Proposition 7.3.4. *Let us fix any triple (n,d,k), a critical value α_c for it and a quadruple $(n_i,d_i,k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c;n,d,k)$. Let us assume that conditions (7.5), respectively (7.6), are satisfied. Moreover, let us suppose that $(n_2,k_2)=(n_3,k_3)=(n_4,k_4)$. Let us denote by G' the set of all the (E,V)'s in $G^+(\alpha_c;n,d,k)$, respectively in $G^-(\alpha_c;n,d,k)$, such that*

(i) they have graded at α_c given by $\oplus_{i=1}^4(Q_i,W_i)$ of type $(n_i,d_i,k_i)_{i=1,\cdots,4}$;

(ii) $(Q_i,W_i) \not\simeq (Q_j,W_j)$ for all $i \neq j \in \{2,3,4\}$;

(iii) their α_c-canonical filtration is of type $(1,3)$.

Then there is a finite family of schemes as follows:

(a) $R^1 = R_{a,b,c;i,j,k}$ for every $a,b,c \in \mathbb{N}_0, a<b<c$ and $(i,j,k) \in L^2_a \times L^3_b \times L^4_c$;

(b) $R^2 = R_{a,a,c;i,j,k}|_{G_1\times(G_2\times G_2\smallsetminus\Delta)\times G_2\smallsetminus U_{a,a,c;i,j,k}\cap U_{a,a,c;j,i,k}}$ for every $a,c \in \mathbb{N}_0$ with $a<c, i<j \in L^2_a$ and $k \in L^4_c$;

(c) $R^3 = (R_{a,a,c;i,j,k}|_{G_1\times(G_2\times G_2\smallsetminus\Delta)\times G_2\cap U_{a,a,c;i,j,k}\cap U_{a,a,c;j,i,k}})/\mathbb{Z}_2$ for every $a,c \in \mathbb{N}_0$ with $a<c, i<j \in L^2_a, k \in L^4_c$;

(d) $R^4 = (R_{a,a,c;i,i,k}|_{G_1\times(G_2\times G_2\smallsetminus\Delta)\times G_2})/\mathbb{Z}_2$ for every $a,c \in \mathbb{N}_0$ with $a<c, i \in L^2_a, k \in L^4_c$;

(e) $R^5 = R_{a,a,a;i,j,k}|_{G_1\times(G_2\times G_2\times G_2\smallsetminus\Delta)\smallsetminus(\cup_{\sigma\in S_3\smallsetminus\{id\}}U_{a,a,a;\sigma(i),\sigma(j),\sigma(k)})}$ for every $a \in \mathbb{N}_0, i,j,k \in L^2_a$ with $i<j<k$. Here every $\sigma \in S_3$ acts by permutations on the ordered set $\{i,j,k\}$;

(f) $R^6 = (R_{a,a,a;i,j,k}|_{G_1\times(G_2\times G_2\times G_2\smallsetminus\Delta)\smallsetminus(\cup_{\sigma\in S_3\smallsetminus\{id,(1\,2)\}}U_{a,a,a;\sigma(i),\sigma(j),\sigma(k)})})/\mathbb{Z}_2$ for every $a \in \mathbb{N}_0, i, j,k \in L^2_a$ with $i<j<k$;

(g) $R^7 = (R_{a,a,a;i,j,k}|_{G_1\times(G_2\times G_2\times G_2\smallsetminus\Delta)\smallsetminus(\cup_{\sigma\in S_3\smallsetminus\{id,(2\,2)\}}U_{a,a,a;\sigma(i),\sigma(j),\sigma(k)})})/\mathbb{Z}_2$ for every $a \in \mathbb{N}_0, i, j,k \in L^2_a$ with $i<j<k$;

(h) $R^8 = (R_{a,a,a;i,j,k}|_{G_1\times(G_2\times G_2\times G_2\smallsetminus\Delta)\smallsetminus(\cup_{\sigma\in S_3\smallsetminus\{id,(1\,3)\}}U_{a,a,a;\sigma(i),\sigma(j),\sigma(k)})})/\mathbb{Z}_2$ for every $a \in \mathbb{N}_0, i, j,k \in L^2_a$ with $i<j<k$;

(i) $R^9 = (R_{a,a,a;i,i,k}|_{G_1\times(G_2\times G_2\smallsetminus\Delta)\times G_2})/\mathbb{Z}_2$ for every $a \in \mathbb{N}_0, i,k \in L^2_a$ with $i<k$;

(j) $R^{10} = (R_{a,a,a;i,i,i}|_{G_1\times(G_2\times G_2\times G_2\smallsetminus\Delta)})/S_3$ for every $a \in \mathbb{N}_0, i \in L^2_a$.

Here Δ denotes both the diagonal of $G_2 \times G_2$ and the "big" diagonal of $G_2 \times G_2 \times G_2$ (i.e. the set of triples of objects such that at least 2 of them are isomorphic). All the schemes of the form $R_{a,b,c;i,j,k}$ are obtained exactly as in proposition 7.3.1, together with the same triples of projective fibrations to the corresponding base $U_{a,b,c;i,j,k}$. Each scheme of type (a)-(j) comes with an injective morphism to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, such that the images form a disjoint locally closed covering of G'.

The schemes (c),(d) and (f)-(i) come with actions of $\mathbb{Z}_2$ on the base space and on the fibers (compatible with the projective fibrations) as follows

- $(Q_i, W_i)_{i=1,2,3,4} \mapsto ((Q_1, W_1), (Q_{\sigma(i)}, W_{\sigma(i)})_{i=2,3,4})$;
- $(\mu_2, \mu_3, \mu_4) \mapsto (\mu_{\sigma(2)}, \mu_{\sigma(3)}, \mu_{\sigma(4)})$ *for every point* (μ_2, μ_3, μ_4) *in the fiber over a quadruple* $(Q_i, W_i)_{i=1,\cdots,4}$.

Here σ is the permutation $(1\,2)$ in cases (c),(d),(f) and (i), $\sigma = (2\,3)$ in case (g) and $\sigma = (1\,3)$ in case (h); in all the various cases σ acts by permutations on the ordered set $\{2, 3, 4\}$. For every scheme R of type (c),(d) and (f)-(i) there exists a finite covering of its base space in $G_1 \times G_2 \times G_2 \times G_2$ by locally closed subschemes T_l that are invariant under the action of $\mathbb{Z}_2$ on the i-th and j-th copies of G_2 in $G_1 \times G_2 \times G_2 \times G_2$ if $\sigma = (i\,j)$. In addition, there exist trivializations of the fibrations from R to $U_{a,a,c;i,j,k}$

$$R|_{T_l} \xrightarrow{\sim} T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{c-1}$$

that are compatible with the action of $\mathbb{Z}_2$ on $T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{c-1}$.

The schemes of the form (j) come with actions of S_3 on the base space and on the fibers (compatible with the projective fibrations) as follows: for every $\sigma \in S_3$ (considered as the group of permutations of $\{2, 3, 4\}$) we have:

- $(Q_i, W_i)_{i=1,2,3,4} \mapsto ((Q_1, W_1), (Q_{\sigma(i)}, W_{\sigma(i)})_{i=2,3,4})$ *for every point of the scheme* $U_{a,a,a;i,i,i}$;
- $(\mu_i)_{i=2,3,4} \mapsto (\mu_{\sigma(i)})_{i=2,3,4}$ *for every point* (μ_2, μ_3, μ_4) *in the fiber over a quadruple* $(Q_i, W_i)_{i=1,\cdots,4} \in U_{a,a,a;i,i,i}$.

Moreover, for every scheme R of type (j) there exists a finite covering of its base space in $G_1 \times G_2 \times G_2 \times G_2$ by locally closed subschemes T_l that are invariant under the action of S_3 on $G_1 \times G_2 \times G_2 \times G_2$; in addition, there exist trivializations of the fibrations from R to $U_{a,a,i;i,i,i}$

$$R|_{T_l} \xrightarrow{\sim} T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$$

that are compatible with the natural action of S_3 on $T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$.

Proposition 7.3.5. *Let us fix any triple (n, d, k), a critical value α_c for it and a quadruple $(n_i, d_i, k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c; n, d, k)$. Let us assume that conditions (7.5), respectively (7.6), are satisfied. Moreover, let us assume that $(n_2, k_2) = (n_3, k_3) = (n_4, k_4)$. Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in $G^-(\alpha_c; n, d, k)$, such that:*

(i) *they have graded at* α_c *given by* $\oplus_{i=1}^4(Q_i,W_i)$ *of type* $(n_i,d_i,k_i)_{i=1,\cdots,4}$*;*

(ii) $(Q_2,W_2)\simeq(Q_3,W_3)\not\simeq(Q_4,W_4)$*;*

(iii) *their* α_c*-canonical filtration is of type* $(1,3)$*.*

Then there is finite family $\{R_{a,b;i,j}\}$ *for* $(a,b)\in\mathbb{N}^2$ *and* i,j *varying in finite sets (for* a,b *fixed) together with injective morphisms to* $G^+(\alpha_c;n,d,k)$*, respectively to* $G^-(\alpha_c;n,d,k)$*, such that the images form a disjoint locally closed covering of* G'*. Each* $R_{a,b;i,j}$ *comes with a sequence of 2 morphisms:*

$$R_{a,b;i,j}\xrightarrow{\phi_1}B_{a,b;i,j}\xrightarrow{\phi_2}V_{a,b;i,j}$$

such that:

- *the first fibration has fibers isomorphic to* $\mathbb{P}^{b-1}$*;*
- *the second fibration is a grassmannian fibration* $Grass(2,Q_{a,b;i,j})$*, where* $Q_{a,b;i,j}$ *is a locally trivial fibration of rank* a *over* $V_{a,b;i,j}$*;*
- *for each* (a,b,i,j)*,* $V_{a,b;i,j}$ *is defined as the scheme*

$$V_{a,b;i,j}:=(U^2_{a;i}\times_{G_1}U^4_{b;j})\cap(G_1\times(G_2\times G_2\setminus\Delta)).$$

Here $\{U^2_{a;i}\}_i$ *is a finite disjoint locally closed covering of*

$$U^2_a:=\{((Q_1,W_1),(Q_2,W_2))\in G_1\times G_2\ s.t.\ dim\ Ext^1((Q_2,W_2),(Q_1,W_1))=a\}$$

and analogously for $U^4_{b;j}\subset G_1\times G_2$*. In particular,* $\{V_{a,b;i,j}\}_{a,b;i,j}$ *is a disjoint covering of* $G_1\times(G_2\times G_2\setminus\Delta)$ *by locally closed subschemes.*

Proposition 7.3.6. *Let us fix any triple* (n,d,k)*, a critical value* α_c *for it and a quadruple* $(n_i,d_i,k_i)_{i=1,\cdots,4}$ *compatible with* $(\alpha_c;n,d,k)$*. Let us assume that conditions (7.5), respectively (7.6), are satisfied. Moreover, let us assume that* $(n_2,k_2)=(n_3,k_3)=(n_4,k_4)$*. Let us denote by* G' *the set of all the* (E,V)*'s in* $G^+(\alpha_c;n,d,k)$*, respectively in* $G^-(\alpha_c;n,d,k)$*, such that:*

(i) *they have graded at* α_c *given by* $\oplus_{i=1}^4(Q_i,W_i)$ *of type* $(n_i,d_i,k_i)_{i=1,\cdots,4}$*;*

(ii) $(Q_2,W_2)\simeq(Q_3,W_3)\simeq(Q_4,W_4)$*,*

(iii) *their canonical filtration is of type* $(1,3)$*.*

Then there is finite family $\{R_{3;a;i}=Grass(3,R_{a;i})\}$ *for* $a\in\mathbb{N}$ *and* i *varying in a finite set (for* a *fixed), together with injective morphisms to* $G^+(\alpha_c;n,d,k)$*, respectively to* $G^-(\alpha_c;n,d,k)$*, such that the images form a disjoint locally closed covering of* G'*. Each* $R_{a;i}$ *is a locally trivial fibration over* $U_{a,i}$ *with fibers isomorphic to* $\mathbb{C}^a$ *and* $\{U_{a;i}\}_i$ *is a locally closed disjoint covering of*

$$U_a:=\{((Q_1,W_1),(Q_2,W_2))\in G_1\times G_2\ s.t.\ dim\ Ext^1((Q_2,W_2),(Q_1,W_1))=a\}.$$

7.4 Canonical filtration of type (3,1)

Proposition 7.4.1. *Let us fix any triple (n, d, k), a critical value α_c for it and a quadruple $(n_i, d_i, k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c; n, d, k)$. Let us assume that $(n_i, k_i) \neq (n_j, k_j)$ for $i \neq j \in \{1, 2, 3\}$ and that*

$$\frac{k_i}{n_i} < \frac{k}{n} \quad \forall\, i \in \{1, 2, 3\} \tag{7.7}$$

respectively that

$$\frac{k_i}{n_i} > \frac{k}{n} \quad \forall\, i \in \{1, 2, 3\}. \tag{7.8}$$

Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in $G^-(\alpha_c; n, d, k)$, such that

(i) they have graded at α_c given by $\oplus_{i=1}^4 (Q_i, W_i)$ of type $(n_i, d_i, k_i)_{i=1,\cdots,4}$;

(ii) their α_c-canonical filtration is of type $(3, 1)$.

Then there is a finite family $\{R_{a,b,c;i,j,k}\}$ of schemes for $(a, b, c) \in \mathbb{N}^3$ and i, j, k varying in finite sets (for a, b, c fixed), together with injective morphisms to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Every scheme $R_{a,b,c;i,j,k}$ comes with a sequence of 3 morphisms:

$$R_{a,b,c;i,j} \xrightarrow{\phi_1} C_{a,b,c;i,j,k} \xrightarrow{\phi_2} B_{a,b,c;i,j,k} \xrightarrow{\phi_3} U_{a,b,c;i,j,k}$$

such that:

- *ϕ_1 has fibers isomorphic to $\mathbb{P}^{c-1}$, ϕ_2 has fibers isomorphic to $\mathbb{P}^{b-1}$ and ϕ_3 has fibers isomorphic to $\mathbb{P}^{a-1}$;*
- *every $U_{a,b,c;i,j,k}$ is the fiber product*

$$U_{a,b,c;i,j,k} = U^1_{a;i} \times_{G_4} U^2_{b;j} \times_{G_4} U^3_{c;k},$$

where the set $\{U^1_{a;i}\}_i$ is a locally closed disjoint covering of

$$U^1_a := \{((Q_1, W_1), (Q_4, W_4)) \in G_1 \times G_4 \text{ s.t. } \dim \mathrm{Ext}^1((Q_4, W_4), (Q_1, W_1)) = a\}$$

and analogously for $U^2_{b;j} \subset G_2 \times G_4$ and $U^3_{c;k} \subset G_3 \times G_4$. In particular, $\{U_{a;b,c;i,j,k}\}_{a,b,c;i,j,k}$ is a disjoint covering of $G_1 \times G_2 \times G_3 \times G_4$ by locally closed subschemes.

Proposition 7.4.2. *Let us fix any triple (n, d, k), a critical value α_c for it and a quadruple $(n_i, d_i, k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c; n, d, k)$. Let us assume that conditions (7.7), respectively (7.8), are satisfied. Moreover, let us suppose that $(n_1, k_1) = (n_2, k_2) \neq (n_3, k_3)$. Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in $G^-(\alpha_c; n, d, k)$, such that* *(i) they have graded at α_c given by $\oplus_{i=1}^4 (Q_i, W_i)$ of type $(n_i, d_i, k_i)_{i=1,\cdots,4}$;*

(ii) $(Q_1, W_1) \not\simeq (Q_2, W_2)$;

(iii) their α_c-canonical filtration is of type $(3,1)$.

Then there is a finite family of schemes as follows:

(a) $R^1 = R_{a,b,c;i,j,k}$ for every $a, b, c \in \mathbb{N}_0, a < b$ and $(i,j,k) \in L_a^1 \times L_b^2 \times L_c^3$;

(b) $R^2 = R_{a,a,c;i,j,k}|_{(G_1 \times G_1 \smallsetminus \Delta_{12}) \times G_3 \times G_4 \smallsetminus U_{a,a,c;i,j,k} \cap U_{a,a,a;j,i,k}}$ for every $a, c \in \mathbb{N}_0, i < j \in L_a^1$ and $k \in L_c^3$;

(c) $R^3 = (R_{a,a,c;i,j,k}|_{(G_1 \times G_1 \smallsetminus \Delta_{12}) \times G_3 \times G_4 \cap U_{a,a,c;i,j,k}) \cap U_{a,a,c;j,i,k}})/\mathbb{Z}_2$ for every $a, c \in \mathbb{N}_0$ and $i < j \in L_a^1, k \in L_c^3$;

(d) $R^4 = (R_{a,a,c;i,i,k}|_{(G_1 \times G_2 \smallsetminus \Delta_{12}) \times G_3 \times G_4})/\mathbb{Z}_2$ for every $a, c \in \mathbb{N}_0$ and $i \in L_a^1, k \in L_c^3$;

where all the schemes of the form $R_{a,b,c;i,j,k}$ are obtained exactly as in proposition 7.4.1, together with the same triples of projective fibrations to the corresponding base $U_{a,b,c;i,j,k}$. Each scheme of type (a)-(d) comes with an injective morphism to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, such that the images form a disjoint locally closed covering of G'.

The last 2 types of schemes come with actions of $\mathbb{Z}_2$ on the base space and on the fibers (compatible with the projective fibrations) as follows:

- *$(Q_i, W_i)_{i=1,2,3,4} \mapsto (Q_i, W_i)_{i=2,1,3,4}$ for every point of $U_{a,a,c;i,j,k}$;*
- *$(\mu_1, \mu_2, \mu_3) \mapsto (\mu_2, \mu_1, \mu_3)$ for every point (μ_1, μ_2, μ_3) in the fiber over a quadruple $(Q_i, W_i)_{i=1,\cdots,4}$ in $U_{a,a,c;i,j,k}$.*

Moreover, for every scheme R of type (c) and (d) there exists a finite covering of its base space in $G_1 \times G_2 \times G_3 \times G_4$ by locally closed subschemes T_l that are invariant under the action of $\mathbb{Z}_2$ on $G_1 \times G_2 = G_1 \times G_2$ in $G_1 \times G_2 \times G_3 \times G_4$; in addition, there exist trivializations of the fibrations from R to $U_{a,a,c;i,j,k}$

$$R|_{T_l} \xrightarrow{\sim} T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{c-1}$$

that are compatible with the natural action of $\mathbb{Z}_2$ on $T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{c-1}$.

Proposition 7.4.3. *Let us fix any triple (n, d, k), a critical value α_c for it and a quadruple $(n_i, d_i, k_i)_{i=1,\cdots,4}$ that is compatible with $(\alpha_c; n, d, k)$. Let us assume that conditions (7.7), respectively (7.8), are satisfied. Moreover, let us assume that $(n_1, k_1) = (n_2, k_2) \neq (n_3, k_3)$. Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in $G^-(\alpha_c; n, d, k)$, such that:*

(i) they have graded at α_c given by $\oplus_{i=1}^4 (Q_i, W_i)$ of type $(n_i, d_i, k_i)_{i=1,\cdots,4}$;

(ii) $(Q_1, W_1) \simeq (Q_2, W_2)$;

(iii) *their* α_c*-canonical filtration is of type* $(3,1)$.

Then there is a finite family $\{R_{a,b;i,j}\}$ *for* $(a,b) \in \mathbb{N}^2$ *and* i, j *varying in finite sets (for* a, b *fixed) together with injective morphisms to* $G^+(\alpha_c; n, d, k)$, *respectively to* $G^-(\alpha_c; n, d, l)$, *such that the images form a disjoint locally closed covering of* G'. *Each* $R_{a,b;i,j}$ *comes with a sequence of 2 morphisms:*

$$R_{a,b;i,j} \xrightarrow{\phi_1} A_{a,b;i,j} \xrightarrow{\phi_2} U_{a,b;i,j}$$

such that:

- *the first fibration has fibers isomorphic to* $\mathbb{P}^{b-1}$;
- *the second fibration is a grassmannian fibration* $Grass(2, Q_{a,b;i,j})$, *where* $Q_{a,b;i,j}$ *is a locally trivial fibration of rank* a *over* $U_{a,b;i,j}$;
- $U_{a,b;i,j}$ *is the fiber product of* $U^1_{a;i}$ *and* $U^3_{b;j}$ *over* G_4, *where* $\{U^1_{a;i}\}_i$ *is a disjoint locally closed covering of*

$$U^1_a := \{((Q_1, W_1), (Q_4, W_4)) \in G_1 \times G_4 \ s.t. \ dim \ Ext^1((Q_4, W_4), (Q_1, W_1)) = a\}$$

  and analogously for $U^3_{b;j} \subset G_3 \times G_4$. *In particular,* $\{U_{a,b;i,j}\}_{a,b;i,j}$ *is a disjoint covering of* $G_1 \times G_3 \times G_4$ *by locally closed subschemes.*

Proposition 7.4.4. *Let us fix any triple* (n, d, k), *a critical value* α_c *for it and a quadruple* $(n_i, d_i, k_i)_{i=1,\cdots,4}$ *compatible with* $(\alpha_c; n, d, k)$. *Let us assume that conditions (7.7), respectively (7.8), are satisfied. Moreover, let us suppose that* $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$. *Let us denote by* G' *the set of all the* (E, V)*'s in* $G^+(\alpha_c; n, d, k)$, *respectively in* $G^-(\alpha_c; n, d, k)$, *such that:*

(i) *they have graded at* α_c *given by* $\oplus_{i=1}^4 (Q_i, W_i)$ *of type* $(n_i, d_i, k_i)_{i=1,\cdots,4}$;

(ii) $(Q_i, W_i) \not\simeq (Q_j, W_j)$ *for all* $i \neq j \in \{1, 2, 3\}$;

(iii) *their* α_c*-canonical filtration is of type* $(3,1)$.

Then there is a finite family of schemes as follows:

(a) $R^1 = R_{a,b,c;i,j,k}$ *for every* $a, b, c \in \mathbb{N}_0, a < b < c$ *and* $(i, j, k) \in L^1_a \times L^2_b \times L^3_c$;

(b) $R^2 = R_{a,a,c;i,j,k}|_{(G_1 \times G_1 \smallsetminus \Delta) \times G_1 \times G_4 \smallsetminus U_{a,a,c;i,j,k} \cap U_{a,a,c;j,i,k}}$ *for every* $a, c \in \mathbb{N}_0$ *with* $a < c, i < j \in L^1_a$ *and* $k \in L^3_c$;

(c) $R^3 = (R_{a,a,c;i,j,k}|_{(G_1 \times G_1 \smallsetminus \Delta) \times G_1 \times G_4 \cap U_{a,a,c;i,j,k} \cap U_{a,a,c;j,i,k}})/\mathbb{Z}_2$ *for every* $a, c \in \mathbb{N}_0$ *with* $a < c, i < j \in L^1_a, k \in L^3_c$;

(d) $R^4 = (R_{a,a,c;i,i,k}|_{(G_1\times G_1\smallsetminus\Delta)\times G_1\times G_4})/\mathbb{Z}_2$ *for every* $a, c \in \mathbb{N}_0$ *with* $a < c$, $i \in L^1_a$, $k \in L^3_c$*;*

(e) $R^5 = R_{a,a,a;i,j,k}|_{(G_1\times G_1\smallsetminus\Delta)\times G_1\times G_4\smallsetminus\cup_{\sigma\in S_3\smallsetminus\{id\}}U_{a,a,a;\sigma(i),\sigma(j),\sigma(k)}}$ *for every* $a \in \mathbb{N}_0$, $i, j, k \in L^1_a$ *with* $i < j < k$. *Here every* $\sigma \in S_3$ *acts by permutations on the ordered set* $\{i, j, k\}$*;*

(f) $R^6 = (R_{a,a,a;i,j,k}|_{(G_1\times G_1\smallsetminus\Delta)\times G_1\times G_4\smallsetminus\cup_{\sigma\in S_3\smallsetminus\{id,(1\,2)\}}U_{a,a,a;\sigma(i),\sigma(j),\sigma(k)}})/\mathbb{Z}_2$ *for every* $a \in \mathbb{N}_0$, i, $j, k \in L^1_a$ *with* $i < j < k$.*;*

(g) $R^7 = (R_{a,a,a;i,j,k}|_{(G_1\times G_1\smallsetminus\Delta)\times G_1\times G_4\smallsetminus\cup_{\sigma\in S_3\smallsetminus\{id,(2\,3)\}}U_{a,a,a;\sigma(i),\sigma(j),\sigma(k)}})/\mathbb{Z}_2$ *for every* $a \in \mathbb{N}_0$, i, $j, k \in L^1_a$ *with* $i < j < k$*;*

(h) $R^8 = (R_{a,a,a;i,j,k}|_{(G_1\times G_1\smallsetminus\Delta)\times G_1\times G_4\smallsetminus\cup_{\sigma\in S_3\smallsetminus\{id,(1\,3)\}}U_{a,a,a;\sigma(i),\sigma(j),\sigma(k)}})/\mathbb{Z}_2$ *for every* $a \in \mathbb{N}_0$, i, $j, k \in L^1_a$ *with* $i < j < k$*;*

(i) $R^9 = (R_{a,a,a;i,i,k}|_{(G_1\times G_1\smallsetminus\Delta)\times G_1\times G_4})/\mathbb{Z}_2$ *for every* $a \in \mathbb{N}_0$, $i, k \in L^2_a$ *with* $i < k$*;*

(j) $R^{10} = (R_{a,a,a;i,i,i}|_{(G_1\times G_1\times G_1\smallsetminus\Delta)\times G_4})/S_3$ *for every* $a \in \mathbb{N}_0$, $i \in L^1_a$.

Here Δ *denotes both the diagonal of* $G_1 \times G_1$ *and the "big" diagonal of* $G_1 \times G_1 \times G_1$. *All the schemes of the form* $R_{a,b,c;i,j,k}$ *are obtained exactly as in proposition 7.4.1, together with the same triples of projective fibrations to the corresponding base* $U_{a,b,c;i,j,k}$. *Each scheme of type (a)-(i) comes with an injective morphism to* $G^+(\alpha_c; n, d, k)$, *respectively to* $G^-(\alpha_c; n, d, k)$, *such that the images form a disjoint locally closed covering of* G'.

The schemes (c),(d) and (f)-(i) come with actions of $\mathbb{Z}_2$ *on the base space and on the fibers (compatible with the projective fibrations) as follows*

- $(Q_i, W_i)_{i=1,2,3,4} \mapsto ((Q_{\sigma(i)}, W_{\sigma(i)})_{i=1,2,3}, (Q_4, W_4))$
- $(\mu_1, \mu_2, \mu_3) \mapsto (\mu_{\sigma(1)}, \mu_{\sigma(2)}, \mu_{\sigma(3)})$ *for every point* (μ_1, μ_2, μ_3) *in the fiber over a quadruple* $(Q_i, W_i)_{i=1,\cdots,4}$.

Here σ *is the permutation* $(1\,2)$ *in cases (c), (d), (f) and (i),* $\sigma = (2\,3)$ *in case (g) and* $\sigma = (1\,3)$ *in case (h). For every scheme* R *of type (c),(d) and (f)-(l) there exists a finite covering of its base space in* $G_1 \times G_1 \times G_1 \times G_4$ *by locally closed subschemes* T_l *that are invariant under the action of* $\mathbb{Z}_2$ *on the* i*-th and* j*-th copies of* G_1 *in* $G_1 \times G_1 \times G_1 \times G_4$ *if* $\sigma = (i\,j)$*; in addition, there exist trivializations of the fibrations from* R *to* $U_{a,a,c;i,j,k}$*:*

$$R|_{T_l} \xrightarrow{\sim} T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{c-1}$$

that are compatible with the action of $\mathbb{Z}_2$ *on* $T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{c-1}$.

The schemes of the form (j) come with actions of S_3 *on the base space and on the fibers (compatible with the projective fibrations) as follows: for every* $\sigma \in S_3$ *we have:*

- $(Q_i, W_i)_{i=1,2,3,4} \mapsto ((Q_{\sigma(i)}, W_{\sigma(i)})_{i=1,2,3}, (Q_4, W_4))$ *for every point of the scheme* $U_{a,a,a;i,i,i}$*;*

- $(\mu_i)_{i=1,2,3} \mapsto (\mu_{\sigma(i)})_{i=1,2,3}$ *for every point* (μ_1, μ_2, μ_3) *in the fiber over a quadruple* $(Q_i, W_i)_{i=1,\cdots,4} \in U_{a,a,a;i,i,i}$.

Moreover, for every scheme R of type (j) there exists a finite covering of its base space in $G_1 \times G_1 \times G_1 \times G_4$ *by locally closed subschemes* T_l *that are invariant under the action of* S_3 *on* $G_1 \times G_1 \times G_1 \times G_4$*; in addition, there exist trivializations of the fibrations from R to* $U_{a,a,i;i,i,i}$

$$R|_{T_l} \xrightarrow{\sim} T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$$

that are compatible with the natural action of S_3 *on* $T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$.

Proposition 7.4.5. *Let us fix any triple* (n, d, k)*, a critical value* α_c *for it and a quadruple* $(n_i, d_i, k_i)_{i=1,\cdots,4}$ *compatible with* $(\alpha_c; n, d, k)$*. Let us assume that conditions (7.7), respectively (7.8), are satisfied. Moreover, let us assume that* $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$*. Let us denote by* G' *the set of all the* (E, V)*'s in* $G^+(\alpha_c; n, d, k)$*, respectively in* $G^-(\alpha_c; n, d, k)$*, such that:*

(i) they have graded at α_c *given by* $\oplus_{i=1}^4 (Q_i, W_i)$ *of type* $(n_i, d_i, k_i)_{i=1,\cdots,4}$*;*

(ii) $(Q_1, W_1) \simeq (Q_2, W_2) \not\simeq (Q_3, W_3)$*;*

(iii) their α_c*-canonical filtration is of type* $(3, 1)$.

Then there is finite family $\{R_{a,b;i,j}\}$ *for* $(a, b) \in \mathbb{N}^2$ *and* i, j *varying in finite sets (for* a, b *fixed) together with injective morphisms to* $G^+(\alpha_c; n, d, k)$*, respectively to* $G^-(\alpha_c; n, d, k)$*, such that the images form a disjoint locally closed covering of* G'*. Each* $R_{a,b;i,j}$ *comes with a sequence of 2 fibration:*

$$R_{a,b;i,j} \xrightarrow{\phi_1} B_{a,b;i,j} \xrightarrow{\phi_2} V_{a,b;i,j}$$

such that:

- ϕ_1 *has fibers isomorphic to* $\mathbb{P}^{b-1}$ *and* ϕ_2 *is a grassmannian fibration* $Grass(2, Q_{a,b;i,j})$ *where* $Q_{a,b;i,j}$ *is a locally trivial fibration of rank a over* $V_{a,b;i,j}$*;*
- *every* $V_{a,b;i,j}$ *is defined as the scheme*

$$V_{a,b;i,j} := (U^1_{a;i} \times_{G_4} U^3_{b;j}) \cap ((G_1 \times G_1 \smallsetminus \Delta) \times G_4).$$

Here $\{U^1_{a;i}\}_i$ *is a disjoint locally closed covering of*

$$U^1_a := \{((Q_1, W_1), (Q_4, W_4)) \in G_1 \times G_4 \text{ s.t. } \dim Ext^1((Q_4, W_4), (Q_1, W_1)) = a\}$$

and analogously for $U^3_{b;j} \subset G_1 \times G_4$*. In particular,* $\{V_{a,b;i,j}\}_{a,b;i,j}$ *is a disjoint covering of* $(G_1 \times G_1 \smallsetminus \Delta) \times G_4$ *by locally closed subschemes.*

Proposition 7.4.6. *Let us fix any triple (n, d, k), a critical value α_c for it and a quadruple $(n_i, d_i, k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c; n, d, k)$. Let us assume that conditions (7.7), respectively (7.8), are satisfied. Moreover, let us assume that $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$. Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in in $G^-(\alpha_c; n, d, k)$, such that:*

(i) they have graded at α_c given by $\oplus_{i=1}^4 (Q_i, W_i)$ of type $(n_i, d_i, k_i)_{i=1,\cdots,4}$;

(ii) $(Q_1, W_1) \simeq (Q_2, W_2) \simeq (Q_3, W_3)$;

(iii) their α_c-canonical filtration is of type $(3, 1)$.

Then there is finite family $\{R_{3;a;i} = Grass(3, R_{a;i})\}$ for $a \in \mathbb{N}$ and i varying in finite sets, together with injective morphisms to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d,)$, such that the images form a disjoint locally closed covering of G'. Each $R_{a;i}$ is a locally trivial fibration over $U_{a,i}$ with fibers isomorphic to $\mathbb{C}^a$ and $\{U_{a;i}\}_i$ is a locally closed disjoint covering of

$$U_a := \{((Q_1, W_1), (Q_4, W_4)) \in G_1 \times G_4 \ s.t.\ dim\ Ext^1((Q_4, W_4), (Q_1, W_1)) = a\}.$$

7.5 Canonical filtration of type (2,1,1)

In this and in the next 2 sections we state for simplicity only the results that we will need in the case $(n, d, k) = (4, d, 1)$. In this case we will always have that either the first 3 objects of the graded will be of the same type or the last 3 objects of the graded will be of the same type. To be more precise, we will only need to consider the first possibility in the case of canonical filtrations of type (2,1,1), while we will only need to consider the second possibility in the case of canonical filtrations of type (2,1,1). In case (1,2,1) both possibilities will have to be taken into account.

Proposition 7.5.1. *Let us fix any triple (n, d, k), a critical value α_c for it and a quadruple $(n_i, d_i, k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c; n, d, k)$. Let us assume that $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$ and that*

$$\frac{k_1}{n_1} < \frac{k}{n} \tag{7.9}$$

respectively that

$$\frac{k_1}{n_1} > \frac{k}{n}. \tag{7.10}$$

Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in $G^-(\alpha_c; n, d, k)$, such that

(i) they have graded at α_c given by $\oplus_{i=1}^4 (Q_i, W_i)$ of type $(n_i, d_i, k_i)_{i=1,\cdots,4}$;

(ii) $(Q_1, W_1) \not\simeq (Q_2, W_2) \simeq (Q_3, W_3)$;

(iii) their α_c-canonical filtration is of type $(2,1,1)$.

Then there is a finite family of schemes $\{R^l_{a,b,c,d;i,j}\}_{l=1,2,3}$ for $(a,b,c,d) \in \mathbb{N}^4$ and i,j varying in finite sets (for a,b,c,d fixed), together with injective morphisms to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Every scheme $R^l_{a,b,c,d;i,j}$ comes with a triple of fibrations:

$$R^l_{a,b,c,d;i,j} \xrightarrow{\phi^l} A^l_{a,b,c,d;i,j} \xrightarrow{\theta^l} U_{a,b,c,d;i,j} \subset G_1 \times R_{a;i},$$
$$\varphi_{a;i} : R_{a;i} \longrightarrow U_{a;i} \subset G_3 \times G_4$$

such that:

- *ϕ^1 has fibers isomorphic to $\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}$, θ^1 has fibers isomorphic to $\mathbb{P}^{d-1} \smallsetminus \mathbb{P}^{a-2}$;*
- *ϕ^2 has fibers isomorphic to $\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}$, θ^2 has fibers isomorphic to $\mathbb{P}^{a-2}$;*
- *ϕ^3 has fibers isomorphic to $\mathbb{P}^{c-1}$, θ^3 has fibers isomorphic to $\mathbb{P}^{d-1} \smallsetminus \mathbb{P}^{a-2}$;*
- *$\varphi_{a;i}$ has fibers isomorphic to $\mathbb{P}^{a-1}$;*
- *$U_{a;i}$ is a finite locally closed disjoint covering of*

  $$U_a := \{((Q_3,W_3),(Q_4,W_4)) \in G_3 \times G_4 \text{ s.t. } \dim \mathrm{Ext}^1((Q_4,W_4),(Q_3,W_3)) = a\};$$

  every U_a is a locally closed subscheme of $G_3 \times G_4$ and so are all the $U_{a;i}$'s;
- *$\{U_{a,b,c,d;i,j}\}_j$ is a finite locally closed disjoint covering of*

  $$U_{a,b,c,d;i} := \{((Q_1,W_1),(E'',V'')) \in G_1 \times R_{a;i} \text{ s.t. } \dim \mathrm{Ext}^1((E'',V''),(Q_1,W_1)) = b,$$
  $$\dim \mathrm{Ext}^1(\widetilde{\varphi}_{a;i}(E'',V''),(Q_1,W_1)) = c, \quad \dim \mathrm{Ext}^1((E'',V''),\overline{\varphi}_{a;i}(E'',V'')) = d,$$
  $$\overline{\varphi}_{a;i}(E'',V'') \not\simeq (Q_1,W_1)\},$$

  where $\widetilde{\varphi}_{a;i}$ is the composition of $\varphi_{a;i}$ with the projection to G_4 and $\overline{\varphi}_{a;i}$ is the composition of $\varphi_{a;i}$ with the projection to G_3. Every $U_{a,b,c,d;i}$ is locally closed in $G_1 \times R_{a;i}$ and so are all the $U_{a,b,c,d;i,j}$'s.

Proposition 7.5.2. *Let us fix any triple (n,d,k), a critical value α_c for it and a quadruple $(n_i,d_i,k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c;n,d,k)$. Let us assume that $(n_1,k_1) = (n_2,k_2) = (n_3,k_3)$ and that conditions (7.9), respectively (7.10), are satisfied. Let us denote by G' the set of all the (E,V)'s in $G^+(\alpha_c;n,d,k)$, respectively in $G^-(\alpha_c;n,d,k)$, such that*

(i) they have graded at α_c given by $\oplus_{i=1}^4(Q_i,W_i)$ of type $(n_i,d_i,k_i)_{i=1,\cdots,4}$;

(ii) $(Q_i, W_i) \not\simeq (Q_j, W_j)$ *for all* $i \neq j \in \{1, 2, 3\}$*;*

(iii) their α_c*-canonical filtration is of type* $(2, 1, 1)$*.*

Then there is a finite family of schemes as follows:

(a) $R^1_{a,b,c,d,e;i,j}$ *for* $(a, b, c, d, e) \in \mathbb{N}^5$*,* $(b, c) < (d, e)$ *(with lexicographic order) and* i, j *varying in finite sets (for* a, b, c, d, e *fixed);*

(b) $R^1_{a,b,c,b,c;i,j}/\mathbb{Z}_2$ *for* $(a, b, c) \in \mathbb{N}^3$ *and* i, j *varying in finite sets (for* a, b, c *fixed);*

(c) $R^2_{a,b,c,d,e;i,j}$ *for* $(a, b, c, d, e) \in \mathbb{N}^5$ *and* i, j *varying in finite sets (for* a, b, c, d, e *fixed).*

Each such scheme comes with an injective morphisms to $G^+(\alpha_c; n, d, k)$*, respectively to* $G^-(\alpha_c; n, d, k)$*, such that the images form a disjoint covering of* G' *by locally closed subschemes. Every scheme* $R^l_{a,b,c,d,e;i,j}$ *comes with a triple of fibrations:*

$$R^l_{a,b,c,d,e;i,j} \xrightarrow{\phi^l} A^l_{a,b,c,d,e;i,j} \xrightarrow{\theta^l} U_{a,b,c,d,e;i,j} \subset G_1 \times G_2 \times R_{a;i},$$
$$\varphi_{a;i} : R_{a;i} \longrightarrow U_{a;i} \subset G_3 \times G_4$$

such that:

- ϕ^1 *has fibers isomorphic to* $\mathbb{P}^{b-1} \setminus \mathbb{P}^{c-1}$*,* θ^1 *has fibers isomorphic to* $\mathbb{P}^{d-1} \setminus \mathbb{P}^{e-1}$*;*
- ϕ^2 *has fibers isomorphic to* $\mathbb{P}^{b-1} \setminus \mathbb{P}^{c-1}$*,* θ^2 *has fibers isomorphic to* $\mathbb{P}^{e-1}$*;*
- $\varphi_{a;i}$ *has fibers isomorphic to* $\mathbb{P}^{a-1}$*;*
- $U_{a;i}$ *is a finite locally closed disjoint covering of*

$$U_a := \{((Q_3, W_3), (Q_4, W_4)) \in G_3 \times G_4 \text{ s.t. } \dim Ext^1((Q_4, W_4), (Q_3, W_3)) = a\};$$

  every U_a *is a locally closed subscheme of* $G_3 \times G_4$ *and so are all the* $U_{a;i}$*'s;*

- $\{U_{a,b,c,d,e;i,j}\}_j$ *is a finite locally closed disjoint covering of*

$$\begin{gathered} U_{a,b,c,d,e;i} := \{((Q_1, W_1), (Q_2, W_2), (E'', V'')) \in G_1 \times G_2 \times R_{a;i} \text{ s.t.} \\ \dim Ext^1((E'', V''), (Q_1, W_1)) = b, \quad \dim Ext^1(\widetilde{\varphi}_{a;i}(E'', V''), (Q_1, W_1)) = c, \\ \dim Ext^1((E'', V''), (Q_2, W_2)) = d, \quad \dim Ext^1(\widetilde{\varphi}_{a;i}(E'', V''), (Q_2, W_2)) = e, \\ (Q_l, W_l) \not\simeq \overline{\varphi}_{a;i}(E'', V'') \, \forall \, l = 1, 2, \quad (Q_1, W_1) \not\simeq (Q_2, W_2)\}, \end{gathered}$$

  where $\widetilde{\varphi}_{a;i}$ *is the composition of* $\varphi_{a;i}$ *with the projection to* G_4 *and* $\overline{\varphi}_{a;i}$ *is the composition of* $\varphi_{a;i}$ *with the projection to* G_3*. Every* $U_{a,b,c,d,e;i}$ *is locally closed in* $G_1 \times G_2 \times R_{a;i}$ *and so are all the* $U_{a,b,c,d,e;i,j}$*'s.*

There is an action of $\mathbb{Z}_2$ both on $R^1_{a,b,c,b,c;i,j}$ and on its base $U_{a,b,c,b,c;i,j}$ given by

- $((Q_1, W_1), (Q_2, W_2), (E'', V'')) \mapsto ((Q_2, W_2), (Q_1, W_1), (E'', V''))$;
- $([\nu_1], [\nu_2]) \mapsto ([\nu_2], [\nu_1])$.

Moreover, there exists a finite locally closed disjoint $\mathbb{Z}_2$-invariant covering $\{T_l\}_l$ of $U_{a,b,c,b,c}$ and trivializations

$$R^1_{a,b,c,b,c;i,j}|_{T_l} \xrightarrow{\sim} T_l \times (\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}) \times (\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1})$$

that are compatible with the natural action of $\mathbb{Z}_2$ on

$$T_l \times (\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}) \times (\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}).$$

Proposition 7.5.3. *Let us fix any triple (n, d, k), a critical value α_c for it and a quadruple $(n_i, d_i, k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c; n, d, k)$. Let us assume that $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$ and that conditions (7.9), respectively (7.10), are satisfied. Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in $G^-(\alpha_c; n, d, k)$, such that*

(i) they have graded at α_c given by $\oplus_{i=1}^4 (Q_i, W_i)$ of type $(n_i, d_i, k_i)_{i=1,\cdots,4}$;

(ii) $(Q_1, W_1) \simeq (Q_2, W_2) \not\simeq (Q_3, W_3)$;

(iii) their α_c-canonical filtration is of type $(2, 1, 1)$;

Then there is a finite family $\{R_{a,b,c;i,j}\}$ of schemes for $(a, b, c) \in \mathbb{N}^3$ and i, j varying in finite sets (for a, b, c fixed), together with injective morphisms to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Each scheme $R_{a,b,c;i,j}$ comes with a pair of fibrations:

$$\varphi_{a,b,c;i,j} : R_{a,b,c;i,j} \longrightarrow U_{a,b,c;i,j} \subset G_1 \times R_{a;i},$$

$$\varphi_{a;i} : R_{a;i} \longrightarrow U_{a;i} \subset G_3 \times G_4,$$

such that:

- *$\varphi_{a,b,c;i,j}$ has fibers isomorphic to $Grass(2, b) \smallsetminus Grass(2, c)$ and $\varphi_{a;i}$ has fibers isomorphic to $\mathbb{P}^{a-1}$;*
- *$U_{a;i}$ is a finite locally closed disjoint covering of*

  $$U_a := \{((Q_3, W_3), (Q_4, W_4)) \in G_3 \times G_4 \ s.t. \ dim \ Ext^1((Q_4, W_4), (Q_3, W_3)) = a\};$$

  every U_a is a locally closed subscheme of $G_3 \times G_4$ and so are all the $U_{a;i}$'s;

- $\{U_{a,b,c;i,j}\}_j$ *is a finite locally closed disjoint covering of*

$$U_{a,b,c;i} := \{((Q_1,W_1),(E'',V'')) \in G_1 \times R_{a;i} \text{ s.t. } dim\ Ext^1((E'',V''),(Q_1,W_1)) = b,$$
$$dim\ Ext^1(\widetilde{\varphi}_{a;i}(E'',V''),(Q_1,W_1)) = c,\ \overline{\varphi}_{a;i}(E'',V'') \not\simeq (Q_1,W_1)\},$$

where $\widetilde{\varphi}_{a;i}$ is the composition of $\varphi_{a;i}$ with the projection to G_4 and $\overline{\varphi}_{a;i}$ is the composition of $\varphi_{a;i}$ with the projection to G_3. Every $U_{a,b,c;i}$ is locally closed in $G_1 \times R_{a;i}$ and so are all the $U_{a,b,c;i,j}$'s.

Proposition 7.5.4. *Let us fix any triple (n,d,k), a critical value α_c for it and a quadruple $(n_i,d_i,k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c;n,d,k)$. Let us assume that conditions (7.9), respectively (7.10), are satisfied. Let us denote by G' the set of all the (E,V)'s in $G^+(\alpha_c;n,d,k)$, respectively in $G^-(\alpha_c;n,d,k)$, such that*

(i) they have graded at α_c given by $\oplus_{i=1}^4(Q_i,W_i)$ of type $(n_i,d_i,k_i)_{i=1,\cdots,4}$;

(ii) $(Q_1,W_1) \simeq (Q_2,W_2) \simeq (Q_3,W_3)$;

(iii) their α_c-canonical filtration is of type $(2,1,1)$;

Then there is a finite family $\{R_{a,b;i,j}\}$ of schemes for $(a,b) \in \mathbb{N}^2$ and i,j varying in finite sets (for a,b fixed), together with injective morphisms to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,\ d,k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Each scheme $R_{a,b;i,j}$ comes with a pair of fibrations:

$$\varphi_{a,b;i,j} : R_{a,b;i,j} \longrightarrow U_{a,b;i,j} \subset R_{a;i},$$
$$\varphi_{a;i} : R_{a;i} \longrightarrow U_{a;i} \subset G_3 \times G_4,$$

such that:

- $\varphi_{a,b;i,j}$ *has fibers isomorphic to $Grass(2,b) \setminus Grass(2,a-1)$ and $\varphi_{a;i}$ has fibers isomorphic to $\mathbb{P}^{a-1}$;*

- $U_{a;i}$ *is a finite locally closed disjoint covering of*

$$U_a := \{((Q_3,W_3),(Q_4,W_4)) \in G_3 \times G_4 \text{ s.t. } dim\ Ext^1((Q_4,W_4),(Q_3,W_3)) = a\};$$

every U_a is a locally closed subscheme of $G_3 \times G_4$ and so are all the $U_{a;i}$'s;

- $\{U_{a,b;i,j}\}_j$ *is a finite locally closed disjoint covering of*

$$U_{a,b;i} := \{(E'',V'') \in R_{a;i} \text{ s.t. } dim\ Ext^1((E'',V''),\overline{\varphi}_{a;i}(E'',V'')) = b\},$$

where $\overline{\varphi}_{a;i}$ is the composition of $\varphi_{a;i}$ with the projection to G_3.

7.6 Canonical filtration of type (1,2,1)

Remark 7.6.1. In order to get a complete description of the Hodge-Deligne polynomials for the moduli spaces $G(\alpha; 4, d, 1)$ we should need to compute also 8 polynomials associated to various subcases for canonical filtrations of type (1,2,1). Regrettably, we are able to give only a geometric description of 4 subcases, as stated below. The 4 subcases that don't appear here are still an open problem; to be more precise, we are able to give a point-wise description having fixed a graded (see §12.2 for the details), but we are not able to get a global (or local) description out of it.

Proposition 7.6.1. *Let us fix any triple (n, d, k), a critical value α_c for it and a quadruple $(n_i, d_i, k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c; n, d, k)$. Let us assume that $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$ and that*

$$\frac{k_1}{n_1} < \frac{k}{n} \tag{7.11}$$

respectively that

$$\frac{k_1}{n_1} > \frac{k}{n}. \tag{7.12}$$

Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in $G^-(\alpha_c; n, d, k)$, such that

(i) they have graded at α_c given by $\oplus_{i=1}^4 (Q_i, W_i)$ of type $(n_i, d_i, k_i)_{i=1,\cdots,4}$;

(ii) $(Q_1, W_1) \simeq (Q_2, W_2) \not\simeq (Q_3, W_3)$;

(iii) their α_c-canonical filtration is of type $(1, 2, 1)$.

Then there is a finite family of schemes $\{R_{a,b,c,d,e;i,j,k}\}$ for $(a, b, c, d, e) \in \mathbb{N}^5$ and i, j, k varying in finite sets (for a, b, c, d, e fixed), together with injective morphisms to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Every scheme $R_{a,b,c,d,e;i,j,k}$ comes with a fibration

$$\varphi_{a,b,c,d,e;i,j,k} : R_{a,b,c,d,e;i,j,k} \longrightarrow U_{a,b,c,d,e;i,j,k} \subset R_{a,b;i,j}$$

with fibers isomorphic to $\mathbb{P}^{c-1} \smallsetminus \mathbb{P}^{d+e-a-1}$. Every scheme $R_{a,b;i,j}$ is obtained as follows. First of all, we consider a cartesian diagram:

$$\begin{tikzcd}
Q_{a,b;i,j} \arrow[r, "\theta^3_{b;j}"] \arrow[d, "\theta^2_{a;i}"] \arrow[dr, phantom, "\square"] & Q^2_{a;i} \arrow[r, "s^3_{b;j}"] \arrow[d, "\psi^2_{a;i}"] \arrow[dr, phantom, "\square"] & R^2_{a;i} \arrow[d, "\varphi^2_{a;i}"] \\
Q^3_{b;j} \arrow[r, "\psi^3_{b;j}"] \arrow[d, "s^2_{a;i}"] \arrow[dr, phantom, "\square"] & U_{a,b;i,j} \arrow[r, "r^3_{b;j}"] \arrow[d, "r^2_{a;i}"] \arrow[dr, phantom, "\square"] & U^2_{a;i} \subset G_2 \times G_4 \arrow[d, "q^2_{a;i}"] \\
R^3_{b;j} \arrow[r, "\varphi^3_{b;j}"] & U^3_{b;j} \subset G_3 \times G_4 \arrow[r, "q^3_{b;j}"] & G_4,
\end{tikzcd} \tag{7.13}$$

where:

- $\varphi^2_{a;i}$ *is a projective fibration with fibers isomorphic to* $\mathbb{P}^{a-1}$ *and* $\{U^2_{a;i}\}_i$ *is a finite locally closed disjoint covering of*

$$U^2_a := \{((Q_2, W_2), (Q_4, W_4)) \in G_2 \times G_4 \ s.t. \ dim \ Ext^1((Q_4, W_4), (Q_2, W_2)) = a\};$$

  every U^2_a *is a locally closed subscheme of* $G_2 \times G_4$ *and so are all the* $U^2_{a;i}$*'s; we denote by* $p^2_{a;i}$ *and* $q^2_{a;i}$ *the projections from* $U^2_{a;i}$ *to* G_2 *and* G_4 *respectively;*

- *analogously* $\varphi^3_{b;j}$ *is a projective fibration with fibers isomorphic to* $\mathbb{P}^{b-1}$ *and* $\{U^3_{b;j}\}_j$ *is a finite locally closed disjoint covering of*

$$U^3_b := \{((Q_3, W_3), (Q_4, W_4)) \in G_3 \times G_4 \ s.t. \ dim \ Ext^1((Q_4, W_4), (Q_3, W_3)) = b\};$$

  every U^3_b *is a locally closed subscheme of* $G_3 \times G_4$ *and so are all the* $U^3_{b;j}$*'s; we denote by* $p^3_{b;j}$ *and* $q^3_{b;j}$ *the projections from* $U^3_{b;j}$ *to* G_3 *and* G_4 *respectively.*

Then we define

$$V_{a,b;i,j} := \{((Q_2, W_2), (Q_3, W_3), (Q_4, W_4)) \in U_{a,b;i,j} \ s.t. \ (Q_2, W_2) \not\simeq (Q_3, W_3)\}$$

and

$$R_{a,b;i,j} := Q_{a,b;i,j}|_{V_{a,b;i,j}}.$$

Finally, $\{U_{a,b,c,d,e;i,j,k}\}_k$ *is a finite locally closed disjoint covering of*

$$U_{a,b,c,d,e;i,j} := \{(E'', V'') \in R_{a,b;i,j} \ s.t. \ dim \ Ext^1((E'', V''), p^2_{a;i} \circ \varphi^2_{a;i} \circ s^3_{b;j} \circ \theta^3_{b;j}(E'', V'')) = c,$$

$$\textit{dim } Ext^1(s^3_{b;j} \circ \theta^3_{b;j}(E'', V''), p^2_{a;i} \circ \varphi^2_{a;i} \circ s^3_{b;j} \circ \theta^3_{b;j}(E'', V'')) = d,$$
$$\textit{dim } Ext^1(s^2_{a;i} \circ \theta^2_{a;i}(E'', V''), p^2_{a;i} \circ \varphi^2_{a;i} \circ s^3_{b;j} \circ \theta^3_{b;j}(E'', V'')) = e\}.$$

Every $U_{a,b,c,d,e;i,j}$ is locally closed in $R_{a,b;i,j}$ and so are all the $U_{a,b,c,d,e;i,j,k}$'s.

Proposition 7.6.2. *Let us fix any triple (n, d, k), a critical value α_c for it and a quadruple $(n_i, d_i, k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c; n, d, k)$. Let us assume that $(n_2, k_2) = (n_3, k_3) = (n_4, k_4)$ and that*

$$\frac{k_2}{n_2} > \frac{k}{n}, \tag{7.14}$$

respectively that

$$\frac{k_2}{n_2} < \frac{k}{n}. \tag{7.15}$$

Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in $G^-(\alpha_c; n, d, k)$, such that

(i) they have graded at α_c given by $\oplus_{i=1}^4 (Q_i, W_i)$ of type $(n_i, d_i, k_i)_{i=1,\cdots,4}$;

(ii) $(Q_2, W_2) \not\simeq (Q_3, W_3) \simeq (Q_4, W_4)$;

(iii) their α_c-canonical filtration is of type $(1, 2, 1)$.

Then there is a finite family of schemes $\{R_{a,b,c;i,j,k}\}$ for $(a, b, c) \in \mathbb{N}^3$ and i, j, k varying in finite sets (for a, b, c fixed), together with injective morphisms to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Every scheme $R_{a,b,c;i,j,k}$ comes with a fibration

$$\varphi_{a,b,c;i,j,k} : R_{a,b,c;i,j,k} \longrightarrow U_{a,b,c;i,j,k} \subset R_{a,b;i,j}$$

with fibers isomorphic to $\mathbb{P}^{c-1} \smallsetminus \mathbb{P}^{b-2}$. Every scheme $R_{a,b;i,j}$ is obtained as follows. First of all, we consider a cartesian diagram:

$$\begin{tikzcd}
Q_{a,b;i,j} \arrow[r, "\theta^3_{b;j}"] \arrow[d, "\theta^2_{a;i}"] \arrow[dr, phantom, "\square"] & Q^2_{a;i} \arrow[r, "s^3_{b;j}"] \arrow[d, "\psi^2_{a;i}"] \arrow[dr, phantom, "\square"] & R^2_{a;i} \arrow[d, "\varphi^2_{a;i}"] \\
Q^3_{b;j} \arrow[r, "\psi^3_{b;j}"] \arrow[d, "s^2_{a;i}"] \arrow[dr, phantom, "\square"] & U_{a,b;i,j} \arrow[r, "r^3_{b;j}"] \arrow[d, "r^2_{a;i}"] \arrow[dr, phantom, "\square"] & U^2_{a;i} \subset G_1 \times G_2 \arrow[d, "q^2_{a;i}"] \\
R^3_{b;j} \arrow[r, "\varphi^3_{b;j}"] & U^3_{b;j} \subset G_1 \times G_3 \arrow[r, "q^3_{b;j}"] & G_1,
\end{tikzcd} \tag{7.16}$$

where:

- $\varphi^2_{a;i}$ *is a projective fibration with fibers isomorphic to* $\mathbb{P}^{a-1}$ *and* $\{U^2_{a;i}\}_i$ *is a finite locally closed disjoint covering of*

$$U^2_a := \{((Q_1, W_1), (Q_2, W_2)) \in G_1 \times G_2 \ s.t. \ dim \ Ext^1((Q_2, W_2), (Q_1, W_1)) = a\};$$

  every U^2_a *is a locally closed subscheme of* $G_1 \times G_2$ *and so are all the* $U^2_{a;i}$*'s; we denote by* $p^2_{a;i}$ *and* $q^2_{a;i}$ *the projections from* $U^2_{a;i}$ *to* G_2 *and* G_1 *respectively;*

- *analogously* $\varphi^3_{b;j}$ *is a projective fibration with fibers isomorphic to* $\mathbb{P}^{b-1}$ *and* $\{U^3_{b;j}\}_j$ *is a finite locally closed disjoint covering of*

$$U^3_b := \{((Q_1, W_1), (Q_3, W_3)) \in G_1 \times G_3 \ s.t. \ dim \ Ext^1((Q_3, W_3), (Q_1, W_1)) = b\};$$

  every U^3_b *is a locally closed subscheme of* $G_1 \times G_3$ *and so are all the* $U^3_{b;j}$*'s; we denote by* $p^3_{b;j}$ *and* $q^3_{b;j}$ *the projections from* $U^3_{b;j}$ *to* G_3 *and* G_1 *respectively.*

Then we define

$$V_{a,b;i,j} := \{((Q_1, W_1), (Q_2, W_2), (Q_3, W_3)) \in U_{a,b;i,j} \ s.t. \ (Q_2, W_2) \not\simeq (Q_3, W_3)\}$$

and

$$R_{a,b;i,j} := Q_{a,b;i,j}|_{V_{a,b;i,j}}.$$

Finally, $\{U_{a,b,c;i,j,k}\}_k$ *is a finite locally closed disjoint covering of*

$$U_{a,b,c;i,j} := \{(E_2, V_2) \in R_{a,b;i,j} \ s.t. \ dim \ Ext^1(p^3_{b;j} \circ \varphi^3_{b;j} \circ s^2_{a;i} \circ \theta^2_{a;i}(E_2, V_2), (E_2, V_2)) = c\}.$$

Every $U_{a,b,c;i,j}$ *is locally closed in* $R_{a,b;i,j}$ *and so are all the* $U_{a,b,c;i,j,k}$*'s.*

Proposition 7.6.3. *Let us fix any triple* (n, d, k)*, a critical value* α_c *for it and a quadruple* $(n_i, d_i, k_i)_{i=1,\cdots,4}$ *compatible with* $(\alpha_c; n, d, k)$*. Let us assume that* $(n_2, k_2) = (n_3, k_3) = (n_4, k_4)$ *and that conditions (7.14), respectively (7.15), are satisfied. Let us denote by* G' *the set of all the* (E, V)*'s in* $G^+(\alpha_c; n, d, k)$*, respectively in* $G^-(\alpha_c; n, d, k)$*, such that*

(i) they have graded at α_c *given by* $\oplus_{i=1}^4 (Q_i, W_i)$ *of type* $(n_i, d_i, k_i)_{i=1,\cdots,4}$*;*

(ii) $(Q_2, W_2) \simeq (Q_3, W_3) \not\simeq (Q_4, W_4)$*;*

(iii) their α_c*-canonical filtration is of type* $(1, 2, 1)$*;*

Then there is a finite family $\{R_{a,b,c;i,j}\}$ of schemes for $(a,b,c) \in \mathbb{N}^3$ and i,j varying in finite sets (for a,b,c fixed), together with injective morphisms to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Each scheme $R_{a,b,c;i,j}$ comes with a pair of fibrations:

$$\varphi_{a,b,c;i,j} : R_{a,b,c;i,j} \longrightarrow U_{a,b,c;i,j} \subset R_{a;i} \times G_4,$$
$$\varphi_{a;i} : R_{a;i} \longrightarrow U_{a;i} \subset G_1 \times G_2,$$

such that:

- *$\varphi_{a,b,c;i,j}$ has fibers isomorphic to $\mathbb{P}^{b-1} \setminus \mathbb{P}^{c-1}$ and $\varphi_{a;i}$ is a grassmannian fibration with fibers isomorphic to $Grass(2,a)$;*
- *$U_{a;i}$ is a finite locally closed disjoint covering of*

$$U_a := \{((Q_1,W_1),(Q_2,W_2)) \in G_1 \times G_2 \ s.t. \ dim\ Ext^1((Q_2,W_2),(Q_1,W_1)) = a\};$$

  every U_a is a locally closed subscheme of $G_1 \times G_2$ and so are all the $U_{a;i}$'s;
- *$\{U_{a,b,c;i,j}\}_j$ is a finite locally closed disjoint covering of*

$$U_{a,b,c;i} := \{((E_2,V_2),(Q_4,W_4)) \in R_{a;i} \times G_4 \ s.t. \ dim\ Ext^1((Q_4,W_4),(E_2,V_2)) = b,$$
$$dim\ Ext^1((Q_4,W_4),\overline{\varphi}_{a;i}(E_2,V_2)) = c, \ \widetilde{\varphi}_{a;i}(E_2,V_2) \not\simeq (Q_4,W_4)\},$$

  where $\overline{\varphi}_{a;i}$ is the composition of $\varphi_{a;i}$ with the projection to G_1 and $\widetilde{\varphi}_{a;i}$ is the composition of $\varphi_{a;i}$ with the projection to G_2. Every $U_{a,b,c;i}$ is locally closed in $R_{a;i} \times G_4$ and so are all the $U_{a,b,c;i,j}$'s.

Proposition 7.6.4. *Let us fix any triple (n,d,k), a critical value α_c for it and a quadruple $(n_i,d_i,k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c;n,d,k)$. Let us assume that $(n_2,k_2) = (n_3,k_3) = (n_4,k_4)$ and that conditions (7.14), respectively (7.15), are satisfied. Let us denote by G' the set of all the (E,V)'s in $G^+(\alpha_c;n,d,k)$, respectively in $G^-(\alpha_c;n,d,k)$, such that*

(i) they have graded at α_c given by $\oplus_{i=1}^4 (Q_i,W_i)$ of type $(n_i,d_i,k_i)_{i=1,\cdots,4}$;

(ii) $(Q_2,W_2) \simeq (Q_3,W_3) \simeq (Q_4,W_4)$;

(iii) their α_c-canonical filtration is of type $(1,2,1)$;

Then there is a finite family $\{R_{a,b;i,j}\}$ of schemes for $(a,b) \in \mathbb{N}^2$ and i,j varying in finite sets (for a,b fixed), together with injective morphisms to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Each scheme $R_{a,b;i,j}$ comes with a pair of fibrations:

$$\varphi_{a,b;i,j} : R_{a,b;i,j} \longrightarrow U_{a,b;i,j} \subset R_{a;i},$$

$$\varphi_{a;i} : R_{a;i} \longrightarrow U_{a;i} \subset G_1 \times G_2,$$

such that:

- $\varphi_{a,b;i,j}$ *has fibers isomorphic to* $\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{a-3}$ *and* $\varphi_{a;i}$ *is a grassmannian fibration with fibers isomorphic to* $Grass(2,a)$*;*

- $U_{a;i}$ *is a finite locally closed disjoint covering of*

$$U_a := \{((Q_1,W_1),(Q_2,W_2)) \in G_1 \times G_2 \ s.t. \ dim \ Ext^1((Q_2,W_2),(Q_1,W_1)) = a\};$$

  every U_a *is a locally closed subscheme of* $G_1 \times G_2$ *and so are all the* $U_{a;i}$*'s;*

- $\{U_{a,b;i,j}\}_j$ *is a finite locally closed disjoint covering of*

$$U_{a,b;i} := \{(E_2,V_2) \in R_{a;i} \ s.t. \ dim \ Ext^1(\widetilde{\varphi}_{a;i}(E_2,V_2),(E_2,V_2)) = b\},$$

  where $\widetilde{\varphi}_{a;i}$ *is the composition of* $\varphi_{a;i}$ *with the projection to* G_2*. Every* $U_{a,b;i}$ *is locally closed in* $R_{a;i}$ *and so are all the* $U_{a,b;i,j}$*'s.*

7.7 Canonical filtration of type (1,1,2)

Proposition 7.7.1. *Let us fix any triple* (n,d,k)*, a critical value* α_c *for it and a quadruple* $(n_i,d_i,k_i)_{i=1,\cdots,4}$ *compatible with* $(\alpha_c;n,d,k)$*. Let us assume that* $(n_2,k_2)=(n_3,k_3)=(n_4,k_4)$ *and that*

$$\frac{k_2}{n_2} > \frac{k}{n}, \tag{7.17}$$

respectively that

$$\frac{k_2}{n_2} < \frac{k}{n}. \tag{7.18}$$

Let us denote by G' *the set of all the* (E,V)*'s in* $G^+(\alpha_c;n,d,k)$*, respectively in* $G^-(\alpha_c;n,d,k)$*, such that*

(i) they have graded at α_c *given by* $\oplus_{i=1}^4(Q_i,W_i)$ *of type* $(n_i,d_i,k_i)_{i=1,\cdots,4}$*;*

(ii) $(Q_2,W_2) \simeq (Q_3,W_3) \not\simeq (Q_4,W_4)$*;*

(iii) their α_c*-canonical filtration is of type* $(1,1,2)$*.*

Then there is a finite family $\{R_{a,b,c,d;i,j}\}$ *of schemes for* $(a,b,c,d) \in \mathbb{N}^4$ *and* i,j *varying in finite sets (for* a,b,c,d *fixed), together with injective morphisms to* $G^+(\alpha_c;n,d,k)$*, respectively to* $G^-(\alpha_c;n,d,k)$*, such that the images form a disjoint covering of* G' *by locally closed subschemes. Every scheme* $R_{a,b,c,d;i,j}$ *comes with a triple of fibrations:*

$$R_{a,b,c,d;i,j} \xrightarrow{\phi_1} A_{a,b,c,d;i,j} \xrightarrow{\phi_2} U_{a,b,c,d;i,j} \subset R_{a;i} \times G_4,$$
$$\varphi_{a;i} : R_{a;i} \longrightarrow U_{a;i} \subset G_1 \times G_2$$

such that:

- ϕ_1 *has fibers isomorphic to* $\mathbb{P}^{d-1} \smallsetminus \mathbb{P}^{a-2}$*,* ϕ_2 *has fibers isomorphic to* $\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}$*;*
- $\varphi_{a;i}$ *has fibers isomorphic to* $\mathbb{P}^{a-1}$*;*
- $U_{a;i}$ *is a finite locally closed disjoint covering of*

$$U_a := \{((Q_1,W_1),(Q_2,W_2)) \in G_1 \times G_2 \text{ s.t. } \dim Ext^1((Q_2,W_2),(Q_1,W_1)) = a\};$$

  every U_a *is a locally closed subscheme of* $G_1 \times G_2$ *and so are all the* $U_{a;i}$*'s;*

- $\{U_{a,b,c,d;i,j}\}_j$ *is a finite locally closed disjoint covering of*

$$U_{a,b,c,d;i} := \{((E_2,V_2),(Q_4,W_4)) \in R_{a;i} \times G_4 \text{ s.t. } \dim Ext^1((Q_4,W_4),(E_2,V_2)) = b,$$
$$\dim Ext^1((Q_4,W_4),\widetilde{\varphi}_{a;i}(E_2,V_2)) = c, \quad \dim Ext^1(\overline{\varphi}_{a;i}(E_2,V_2),(E_2,V_2)) = d,$$
$$\overline{\varphi}_{a;i}(E_2,V_2) \not\simeq (Q_4,W_4)\},$$

  where $\widetilde{\varphi}_{a;i}$ *is the composition of* $\varphi_{a;i}$ *with the projection to* G_1 *and* $\overline{\varphi}_{a;i}$ *is the composition of* $\varphi_{a;i}$ *with the projection to* G_2*. Every* $U_{a,b,c,d;i}$ *is locally closed in* $R_{a;i} \times G_4$ *and so are all the* $U_{a,b,c,d;i,j}$*'s.*

Proposition 7.7.2. *Let us fix any triple* (n,d,k)*, a critical value* α_c *for it and a quadruple* $(n_i,d_i,k_i)_{i=1,\cdots,4}$ *compatible with* $(\alpha_c;n,d,k)$*. Let us assume that* $(n_2,k_2) = (n_3,k_3) = (n_4,k_4)$ *and that conditions (7.17), respectively (7.18), are satisfied. Let us denote by* G' *the set of all the* (E,V)*'s in* $G^+(\alpha_c;n,d,k)$*, respectively in* $G^-(\alpha_c;n,d,k)$*, such that*

(i) they have graded at α_c *given by* $\oplus_{i=1}^4(Q_i,W_i)$ *of type* $(n_i,d_i,k_i)_{i=1,\cdots,4}$*;*

(ii) $(Q_i,W_i) \not\simeq (Q_j,W_j)$ *for all* $i \neq j \in \{2,3,4\}$*;*

(iii) their α_c*-canonical filtration is of type* $(1,1,2)$*.*

Then there is a finite family of schemes as follows.

(a) $R_{a,b,c,d,e;i,j}$ *for* $(a,b,c,d,e) \in \mathbb{N}^5$ *with* $(b,c) < (d,e)$ *(with lexicographic order) and* i,j *varying in finite sets (for* a,b,c,d,e *fixed);*

(b) $R_{a,b,c,b,c;i,j}/\mathbb{Z}_2$ *for* $(a,b,c) \in \mathbb{N}^3$ *for and* i,j *varying in finite sets (for* a,b,c *fixed).*

Each such scheme comes with an injective morphisms to $G^+(\alpha_c;n,d,k)$*, respectively to* $G^-(\alpha_c;n,d,k)$*, such that the images form a disjoint covering of* G' *by locally closed subschemes. Every scheme* $R_{a,b,c,d,e;i,j}$ *comes with a triple of fibrations:*

$$R_{a,b,c,d,e;i,j} \xrightarrow{\phi_1} A_{a,b,c,d,e;i,j} \xrightarrow{\phi_2} U_{a,b,c,d,e;i,j} \subset R_{a;i} \times G_3 \times G_4,$$

$$\varphi_{a;i} : R_{a;i} \longrightarrow U_{a;i} \subset G_1 \times G_2$$

such that:

- ϕ_1 *has fibers isomorphic to* $\mathbb{P}^{d-1} \smallsetminus \mathbb{P}^{e-1}$*,* ϕ_2 *has fibers isomorphic to* $\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}$*;*
- $\varphi_{a;i}$ *has fibers isomorphic to* $\mathbb{P}^{a-1}$*;*
- $U_{a;i}$ *is a finite locally closed disjoint covering of*

$$U_a := \{((Q_1,W_1),(Q_2,W_2)) \in G_1 \times G_2 \ s.t.\ \dim \mathrm{Ext}^1((Q_2,W_2),(Q_1,W_1)) = a\};$$

every U_a *is a locally closed subscheme of* $G_1 \times G_2$ *and so are all the* $U_{a;i}$*'s;*

- $\{U_{a,b,c,d,e;i,j}\}_j$ *is a finite locally closed disjoint covering of*

$$\begin{gathered} U_{a,b,c,d,e;i} := \{((E_2,V_2),(Q_3,W_3),(Q_4,W_4)) \in R_{a;i} \times G_3 \times G_4 \ s.t. \\ \dim \mathrm{Ext}^1((Q_4,W_4),(E_2,V_2)) = b, \quad \dim \mathrm{Ext}^1((Q_4,W_4),\widetilde{\varphi}_{a;i}(E_2,V_2)) = c, \\ \dim \mathrm{Ext}^1((Q_3,W_3),(E_2,V_2)) = d, \quad \dim \mathrm{Ext}^1((Q_3,W_3),\widetilde{\varphi}_{a;i}(E_2,V_2)) = e, \\ (Q_l,W_l) \not\simeq \overline{\varphi}_{a;i}(E_2,V_2) \text{ for } l = 3,4, \quad (Q_3,W_3) \not\simeq (Q_4,W_4)\}, \end{gathered}$$

where $\widetilde{\varphi}_{a;i}$ *is the composition of* $\varphi_{a;i}$ *with the projection to* G_1 *and* $\overline{\varphi}_{a;i}$ *is the composition of* $\varphi_{a;i}$ *with the projection to* G_2*. Every* $U_{a,b,c,d,e;i}$ *is locally closed in* $R_{a;i} \times G_3 \times G_4$ *and so are all the* $U_{a,b,c,d,e;i,j}$*'s.*

There is an action of $\mathbb{Z}_2$ *both on* $R_{a,b,c,b,c;i,j}$ *and on its base* $U_{a,b,c,b,c;i,j}$ *given by*

- $((E_2,V_2),(Q_3,W_3),(Q_4,W_4)) \mapsto ((E_2,V_2),(Q_4,W_4),(Q_3,W_3))$*;*
- $([\nu_3],[\nu_4]) \mapsto ([\nu_4],[\nu_3])$*.*

Moreover, there exists a finite locally closed disjoint $\mathbb{Z}_2$-invariant covering $\{T_l\}_l$ of $U_{a,b,c,b,c}$ and trivializations

$$R_{a,b,c,b,c}|_{T_l} \xrightarrow{\sim} T_l \times (\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}) \times (\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1})$$

that are compatible with the natural action of $\mathbb{Z}_2$ on

$$T_l \times (\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}) \times (\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}).$$

Proposition 7.7.3. *Let us fix any triple (n, d, k), a critical value α_c for it and a quadruple $(n_i, d_i, k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c; n, d, k)$. Let us assume that $(n_2, k_2) = (n_3, k_3) = (n_4, k_4)$ and that conditions (7.17), respectively (7.18), are satisfied. Let us denote by G' the set of all the (E, V)'s in $G^+(\alpha_c; n, d, k)$, respectively in $G^-(\alpha_c; n, d, k)$, such that*

(i) they have graded at α_c given by $\oplus_{i=1}^4 (Q_i, W_i)$ of type $(n_i, d_i, k_i)_{i=1,\cdots,4}$;

(ii) $(Q_2, W_2) \not\simeq (Q_3, W_3) \simeq (Q_4, W_4)$;

(iii) their α_c-canonical filtration is of type $(1, 1, 2)$.

Then there is a finite family $\{R_{a,b,c;i,j}\}$ of schemes for $(a, b, c) \in \mathbb{N}^3$ and i, j varying in finite sets (for a, b, c fixed), together with injective morphisms to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Each scheme $R_{a,b,c;i,j}$ comes with a pair of fibrations:

$$\varphi_{a,b,c;i,j} : R_{a,b,c;i,j} \longrightarrow U_{a,b,c;i,j} \subset R_{a;i} \times G_3,$$
$$\varphi_{a;i} : R_{a;i} \longrightarrow U_{a;i} \subset G_1 \times G_2,$$

such that:

- *$\varphi_{a,b,c;i,j}$ has fibers isomorphic to $\mathbb{C}^{2c} \times Grass(2, b-c)$ and $\varphi_{a;i}$ has fibers isomorphic to $\mathbb{P}^{a-1}$;*
- *$U_{a;i}$ is a finite locally closed disjoint covering of*

  $$U_a := \{((Q_1, W_1), (Q_2, W_2)) \in G_1 \times G_2 \text{ s.t. } \dim Ext^1((Q_2, W_2), (Q_1, W_1)) = a\};$$

  every U_a is a locally closed subscheme of $G_1 \times G_2$ and so are all the $U_{a;i}$'s;
- *$\{U_{a,b,c;i,j}\}_j$ is a finite locally closed disjoint covering of*

  $$U_{a,b,c;i} := \{((E_2, V_2), (Q_3, W_3)) \in R_{a;i} \times G_3 \text{ s.t. } \dim Ext^1((Q_3, W_3), (E_2, V_2)) = b,$$
  $$\dim Ext^1((Q_3, W_3), \widetilde{\varphi}_{a;i}(E_2, V_2)) = c,\ (Q_3, W_3) \not\simeq \overline{\varphi}_{a;i}(E_2, V_2)\},$$

  where $\widetilde{\varphi}_{a;i}$ is the composition of $\varphi_{a;i}$ with the projection to G_1 and $\overline{\varphi}_{a;i}$ is the composition of $\varphi_{a;i}$ with the projection to G_2. Every $U_{a,b,c;i}$ is locally closed in $R_{a;i} \times G_3$ and so are all the $U_{a,b,c;i,j}$'s.

Proposition 7.7.4. *Let us fix any triple (n,d,k), a critical value α_c for it and a quadruple $(n_i,d_i,k_i)_{i=1,\cdots,4}$ compatible with $(\alpha_c;n,d,k)$. Let us assume that $(n_2,k_2)=(n_3,k_3)=(n_4,k_4)$ and that conditions (7.17), respectively (7.18), are satisfied. Let us denote by G' the set of all the (E,V)'s in $G^+(\alpha_c;n,d,k)$, respectively in $G^-(\alpha_c;n,d,k)$, such that*

(i) they have graded at α_c given by $\oplus_{i=1}^4(Q_i,W_i)$ of type $(n_i,d_i,k_i)_{i=1,\cdots,4}$;

(ii) $(Q_2,W_2)\simeq(Q_3,W_3)\simeq(Q_4,W_4)$;

(iii) their α_c-canonical filtration is of type $(1,1,2)$.

Then there is a finite family $\{R_{a,b;i,j}\}$ of schemes for $(a,b)\in\mathbb{N}^2$ and i,j varying in finite sets (for a,b fixed), together with injective morphisms to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, such that the images form a disjoint covering of G' by locally closed subschemes. Each scheme $R_{a,b;i,j}$ comes with a pair of fibrations:

$$\varphi_{a,b;i,j}:R_{a,b;i,j}\longrightarrow U_{a,b;i,j}\subset R_{a;i},$$
$$\varphi_{a;i}:R_{a;i}\longrightarrow U_{a;i}\subset G_1\times G_2,$$

such that:

- *$\varphi_{a,b;i,j}$ has fibers isomorphic to $\mathbb{C}^{2a-2}\times Grass(2,b-a+1)$ and $\varphi_{a;i}$ has fibers isomorphic to $\mathbb{P}^{a-1}$;*
- *$U_{a;i}$ is a finite locally closed disjoint covering of*

$$U_a:=\{((Q_1,W_1),(Q_2,W_2))\in G_1\times G_2 \ s.t.\ dim\ Ext^1((Q_2,W_2),(Q_1,W_1))=a\};$$

  every U_a is a locally closed subscheme of $G_1\times G_2$ and so are all the $U_{a;i}$'s;
- *$\{U_{a,b;i,j}\}_j$ is a finite locally closed disjoint covering of*

$$U_{a,b;i}:=\{(E_2,V_2)\in R_{a;i}\ s.t.\ dim\ Ext^1(\overline{\varphi}_{a;i}(E_2,V_2),(E_2,V_2))=b\},$$

  where $\overline{\varphi}_{a;i}$ is the composition of $\varphi_{a;i}$ with the projection to G_2. Every $U_{a,b;i}$ is locally closed in $R_{a;i}$ and so are all the $U_{a,b;i,j}$'s.

Chapter 8

Hodge-Deligne polynomials

We use Deligne's extension of Hodge theory which applies to varieties which are not necessarily compact, projective or smooth (see [D1], [D2] and [D3]). We start by giving a review of the notions of pure Hodge structure, mixed Hodge structure, Hodge–Deligne and Hodge–Poincaré polynomials.

Definition 8.0.1. A *pure Hodge structure of weight* m is given by a finite dimensional $\mathbb{Q}$–vector space $H_{\mathbb{Q}}$ and a finite decreasing filtration F^p of $H = H_{\mathbb{Q}} \otimes \mathbb{C}$

$$H \supset \ldots \supset F^p \supset \ldots \supset (0),$$

called *the Hodge filtration*, such that $H = F^p \oplus \overline{F^{m-p+1}}$ for all p. When $p+q=m$, if we set $H^{p,q} = F^p \cap \overline{F^q}$, the condition $H = F^p \oplus \overline{F^{m-p+1}}$ for all p implies an equivalent definition for a pure Hodge structure, that is, a decomposition

$$H = \bigoplus_{p+q=m} H^{p,q}$$

such that $H^{p,q} = \overline{H^{q,p}}$. The relation between the two equivalent definitions is the following. Given a filtration $\{F^p\}_p$, we obtain a decomposition by considering $H^{p,q} = F^p \cap \overline{F^q}$. Conversely, given a decomposition $\{H^{p,q}\}_{p,q}$, this defines a filtration as above by $F^p = \bigoplus_{i \geq p} H^{i,m-i}$.

The n-th cohomology group of a smooth projective variety $H^n(X)$ carries a pure Hodge structure of weight n. If $\Omega_X^\bullet$ denotes the complex of holomorphic differential forms, and $(\Omega_X^\bullet)^{\geq p}$ is the subcomplex of forms of degree greater than or equal to p, then $H^n(X, \mathbb{C}) = \mathbb{H}(X, \Omega_X^\bullet)$. The role of the Hodge filtration is played here by the following filtration:

$$F^p = \mathrm{Im}(\mathbb{H}^n(X, (\Omega_X^\bullet)^{\geq p}) \to \mathbb{H}^n(X, \Omega_X^\bullet)).$$

A *morphism of Hodge structures* is a map $f_{\mathbb{Q}} : H_{\mathbb{Q}} \to H'_{\mathbb{Q}}$ such that $f_{\mathbb{C}}(F^pH) \subset F^pH'$ for all p, where $f_{\mathbb{C}} = f_{\mathbb{Q}} \otimes \mathbb{C}$ and F^pH is the p-th element in the Hodge filtration of H. When the Hodge structures have the same weight, $f_{\mathbb{Q}}$ strictly preserves the filtration, that is

$$f_{\mathbb{C}}(F^pH) = \mathrm{Im}(f_{\mathbb{C}}) \cap F^pH'.$$

Definition 8.0.2. A *mixed Hodge structure* consists of a finite dimensional $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$, an increasing filtration W_l of $H_{\mathbb{Q}}$, called *the weight filtration*

$$\ldots \subset W_l \subset \ldots \subset H_{\mathbb{Q}},$$

and a Hodge filtration F^p of $H = H_{\mathbb{Q}} \otimes \mathbb{C}$, such that the filtrations $F^p Gr_l^W$ induced by F^p on

$$Gr_l^W = (W_l H_{\mathbb{Q}} / W_{l-1} H_{\mathbb{Q}}) \otimes \mathbb{C} = W_l H / W_{l-1} H$$

give a pure Hodge structure of weight l to that object. Here $F^p Gr_l^W$ is given by

$$(W_l H \cap F^p + W_{l-1} H)/W_{l-1} H.$$

A *morphism of type* (r, r) between mixed Hodge structures, $H_{\mathbb{Q}}$ with filtrations W_m and F^p, and $H'_{\mathbb{Q}}$ with W'_l and F'^q, is given by a linear map

$$L : H_{\mathbb{Q}} \to H'_{\mathbb{Q}}$$

satisfying $L(W_m) \subset W'_{m+2r}$ and $L(F^p) \subset F'^{p+r}$. Any such morphism is then strict in the sense that $L(F^p) = \mathrm{Im}(L) \cap F'^{p+r}$, and the same for the weight filtration.

Definition 8.0.3. A morphism of type $(0, 0)$ between mixed Hodge structures, is called *a morphism of mixed Hodge structures.*

Deligne proved that for every complex variety X (not necessarily irreducible, smooth or projective) the cohomology groups $H^k(X, \mathbb{Q})$ and the cohomology groups with compact support $H_c^k(X, \mathbb{Q})$ carry natural mixed Hodge structures (see [D1], [D2] and [D3]). Associated to the Hodge filtration and the weight filtration we can consider the quotients $Gr_l^W = W_l/W_{l-1}$ and $Gr_F^p Gr_l^W = F^p Gr_l^W / F^{p+1} Gr_l^W$, and analogously for the cohomology groups with compact support. Then we can define the Hodge-Deligne numbers of X as follows.

Definition 8.0.4. For a complex algebraic variety X, not necessarily smooth, compact or irreducible, we define its *Hodge-Deligne numbers* as

$$h^{p,q}(H_c^k(X)) := \dim Gr_F^p Gr_{p+q}^W H_c^k(X).$$

Then we introduce the following Euler characteristic:

$$\chi_{p,q}^c(X) := \sum_k (-1)^k h^{p,q}(H_c^k(X)). \tag{8.1}$$

Analogously, we write $\chi_{p,q}(X)$ for the Euler characteristic (8.1) of $H^\bullet(X)$. If X is smooth of dimension n, then Poincaré duality tells us that

$$\chi_{p,q}^c(X) = \chi_{n-p,n-q}(X).$$

We are now ready to define the Hodge–Deligne polynomial.

Definition 8.0.5 ([DK])**.** For any complex algebraic variety X, we define its *Hodge-Deligne polynomial* (or virtual Hodge polynomial) as:

$$\mathcal{HD}(X)(u,v) := \sum_{p,q} (-1)^{p+q} \chi^c_{p,q}(X) u^p v^q \in \mathbb{Z}[u,v].$$

Danilov and Khovanskiǐ ([DK]) observed that $\mathcal{HD}(X)(u,v)$ coincides with the classical Hodge polynomial when X is smooth and projective. Indeed, under these hypotheses, the mixed Hodge structure on $H^k_c(X)$ is pure of weight k, so

$$Gr^W_m H^k_c(X) = \begin{cases} H^k(X) & \text{if } m = k \\ 0 & \text{if } m \neq k. \end{cases}$$

Then

$$\mathcal{HD}(X)(u,v) = \sum_{p,q} h^{p,q}(X) u^p v^q, \tag{8.2}$$

where $h^{p,q}(X) = h^{p,q}(H^{p+q}(X))$ are the classical Hodge numbers of X and (8.2) the classical Hodge polynomial.

We may define another polynomial using the Euler characteristic $\chi_{p,q}(X)$ for rational cohomology groups without compact support. As we have already said Deligne proved that these groups carry a natural mixed Hodge structure.

Definition 8.0.6. For a complex algebraic variety X, not necessarily smooth, compact or irreducible, we define its *Hodge-Poincaré numbers* as

$$h^{p,q}(H^k(X)) := \dim\, Gr^p_F Gr^W_{p+q} H^k(X).$$

We are ready now to define the Hodge-Poincaré polynomial.

Definition 8.0.7. For any complex algebraic variety X, we define its *Hodge–Poincaré polynomial* as

$$HP(X)(u,v) := \sum_{p,q} (-1)^{p+q} \chi_{p,q}(X) u^p v^q = \sum_{p,q,k} (-1)^{p+q+k} h^{p,q}(H^k(X)) u^p v^q.$$

Remark 8.0.1. When our algebraic variety X is smooth, Poincaré duality gives us the following functional identity relating Hodge-Deligne and Hodge-Poincaré polynomials

$$\mathcal{HD}(X)(u,v) = (uv)^{\dim_{\mathbb{C}} X} \cdot HP(X)(u^{-1}, v^{-1}) \tag{8.3}$$

where $\dim_{\mathbb{C}} X$ denotes the complex dimension of X. If X is not only smooth, but also projective, then its Hodge-Deligne polynomial coincides with the Hodge-Poincaré polynomial, so the in this case Poincaré duality can be stated as:

$$\mathcal{HD}(X)(u,v) = (uv)^{\dim_{\mathbb{C}} X} \cdot \mathcal{HD}(X)(u^{-1}, v^{-1}) \tag{8.4}$$

Let $b^k(X) = \dim H^k(X)$ be the k-th Betti number of the variety X and let $P_X(t) = \sum_k b^k(X)t^k$ be its Poincaré polynomial. If X is not only smooth, but also projective, the Betti numbers of X satisfy

$$b^k(X) = \sum_{p+q=k} h^{p,q}(H^k(X)), \tag{8.5}$$

so that

$$P_X(t) = \sum_k b^k(X)t^k = \mathcal{HD}(X)(t,t) = HP(X)(t,t). \tag{8.6}$$

We list below some very useful properties of Hodge-Deligne polynomials that we will need in the following chapter.

Theorem 8.0.5. *([D3, proposition 8.3.9], see also [MOVG, theorem 2.2]) Let X be a complex variety. Let us suppose that X is a finite disjoint union $X = \amalg_i X_i$, where the X_i's are locally closed subvarieties. Then*

$$\mathcal{HD}(X)(u,v) = \sum_i \mathcal{HD}(X_i)(u,v).$$

We recall some known formulae; if not otherwise stated, a reference for them is [M, §2].

(a) For the complex projective space $\mathbb{P}^{n-1}$ we have

$$\mathcal{HD}(\mathbb{P}^{n-1}) = 1 + uv + (uv)^2 + \cdots + (uv)^{n-1} = \frac{1-(uv)^n}{1-uv}.$$

(b) For the affine space $\mathbb{C}^n$ we have

$$\mathcal{HD}(\mathbb{C}^n) = (uv)^n.$$

(c) If we denote by J^dC the d-th jacobian of any smooth projective irreducible complex curve C of genus g, then

$$\mathcal{HD}(J^dC) = (1+u)^g(1+v)^g.$$

(d) The Hodge-Deligne polynomial of the Grassmannian $\mathrm{Gr}(k,N)$ is given by

$$\mathcal{HD}(\mathrm{Gr}(k,N)) = \frac{(1-(uv)^{N-k+1})\cdot\dots\cdot(1-(uv)^{N-1})(1-(uv)^N)}{(1-uv)\cdot\dots\cdot(1-(uv)^{k-1})(1-(uv)^k)}.$$

This formula is still correct when $N < k$, since in this case the Grassmannian is empty and the previous expression is equal to zero, that is the Hodge-Deligne polynomial of the empty scheme.

(e) For any pair of integers k, N, let us define

$$F(k,N) := \{(v_1, \cdots, v_k) \in \mathbb{C}^N \text{ s.t. the } v_i\text{'s are linearly independent in } \mathbb{C}^N\}.$$

By the proof of [MOVG2, lemma 2.5], we get that

$$\begin{aligned}\mathcal{HD}(F(k,N)) &= ((uv)^N - (uv)^{k-1}) \cdots ((uv)^N - uv)((uv)^N - 1) = \\ &= (uv)^{k(k-1)/2}((uv)^{N-k+1} - 1) \cdots ((uv)^{N-1} - 1)((uv)^N - 1).\end{aligned}$$

(f) Let us suppose that $\pi : Z \to Y$ is an algebraic fiber bundle with fiber F which is locally trivial in the Zariski topology, then

$$\mathcal{HD}(Z) = \mathcal{HD}(F) \cdot \mathcal{HD}(Y).$$

In particular this is true for $Z = F \times Y$.

(g) Let us suppose that $\pi : Z \to Y$ is a map between quasi-projective varieties which is a locally trivial fiber bundle in the usual topology, with fibers given by projective spaces $F = \mathbb{P}^N$ for some $N > 0$. Then:

$$\mathcal{HD}(Z) = \mathcal{HD}(F) \cdot \mathcal{HD}(Y).$$

(h) Let M be a smooth projective variety. Consider the algebraic variety $Z = (M \times M)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts as $(x, y) \mapsto (y, x)$. Then by [MOVG2, lemma 2.6] the Hodge-Deligne polynomial of Z is:

$$\mathcal{HD}(Z)(u,v) = \frac{1}{2}\Big((\mathcal{HD}(M)(u,v))^2 + \mathcal{HD}(M)(-u^2, -v^2)\Big).$$

We also state and prove a lemma on the same lines as [MOVG2, lemma 2.6] in the case when we have an action of the symmetric group S_3.

Lemma 8.0.6. *Let M be a smooth projective variety. Let us consider the algebraic variety $Z = (M \times M \times M)/S_3$, where S_3 acts by permutations. Then the Hodge-Deligne polynomial of Z is given by:*

$$\frac{1}{6}(\mathcal{HD}(M)(u,v))^3 + \frac{1}{2}\mathcal{HD}(M)(-u^2, -v^2) \cdot \mathcal{HD}(M)(u,v) + \frac{1}{3}\mathcal{HD}(M)(u^3, v^3).$$

Since we are assuming that M is smooth and projective, then $h^{p,q}(H^k(M)) = 0$ if $p+q \neq k$. So the same is true for $Z = (M \times M \times M)/S_3$; hence

$$\mathcal{HD}(M)(t,t) = \frac{1}{6}(P_Z(t))^3 + \frac{1}{2}P_Z(-t^2) \cdot P_Z(t) + \frac{1}{3}P_Z(t^3)$$

is equal to the Poincaré polynomial of Z, in agreement with the formula given at the end of [Mac].

Proof. The cohomology of Z is given by

$$H^*(Z) = H^*(M \times M \times M)^{S_3} = (H^*(M) \otimes H^*(M) \otimes H^*(M))^{S_3}.$$

This is an equality of Hodge structures. Since M is smooth and projective, the Hodge structure of M is of pure type, so also the Hodge structure of Z is of pure type. Moreover, for all $(p,q) \in \mathbb{N}_0^2$:

$$H^{p,q}(Z) = \left(\bigoplus_{\substack{p_1+p_2+p_3=p \\ q_1+q_2+q_3=q}} H^{p_1,q_1}(M) \otimes H^{p_2,q_2}(M) \otimes H^{p_3,q_3}(M) \right)^{S_3}.$$

Now let us describe what is the action of S_3 on such a space. Let us fix any triple $(p_i, q_i)_{i=1,2,3}$ and let us denote by α^i any object in $H^{p_i,q_i}(M)$. Then

$$(1\,2)\alpha^1 \otimes \alpha^2 \otimes \alpha^3 = (-1)^{(p_1+q_1)(p_2+q_2)} \alpha^2 \otimes \alpha^1 \otimes \alpha^3$$

and

$$(2\,3)\alpha^1 \otimes \alpha^2 \otimes \alpha^3 = (-1)^{(p_2+q_2)(p_3+q_3)} \alpha^1 \otimes \alpha^3 \otimes \alpha^2.$$

Since the cycle $(1\,3)$ can be obtained as the composition $(1\,2) \circ (2\,3) \circ (1\,2)$, we have:

$$\begin{aligned}(1\,3)\alpha^1 \otimes \alpha^2 \otimes \alpha^3 &= (1\,2) \circ (2\,3)(-1)^{(p_1+q_1)(p_2+q_2)} \alpha^2 \otimes \alpha^1 \otimes \alpha^3 = \\ &= (1\,2)(-1)^{(p_1+q_1)(p_2+q_2)+(p_1+q_1)(p_3+q_3)} \alpha^2 \otimes \alpha^3 \otimes \alpha^1 = \\ &= (-1)^{(p_1+q_1)(p_2+q_2)+(p_1+q_1)(p_3+q_3)+(p_2+q_2)(p_3+q_3)} \alpha^3 \otimes \alpha^2 \otimes \alpha^1.\end{aligned}$$

Analogously, the cycle $(1\,2\,3)$ can be obtained as the composition $(2\,3) \circ (1\,2)$, so we have:

$$\begin{aligned}(1\,2\,3)\alpha^1 \otimes \alpha^2 \otimes \alpha^3 &= (2\,3)(-1)^{(p_1+q_1)(p_2+q_2)} \alpha^2 \otimes \alpha^1 \otimes \alpha^3 = \\ &= (-1)^{(p_1+q_1)(p_2+q_2)+(p_1+q_1)(p_3+q_3)} \alpha^2 \otimes \alpha^3 \otimes \alpha^1.\end{aligned}$$

Moreover, the cycle $(1\,3\,2)$ can be obtained as the composition $(1\,2) \circ (2\,3)$, so we have:

$$\begin{aligned}(1\,3\,2)\alpha^1 \otimes \alpha^2 \otimes \alpha^3 &= (1\,2)(-1)^{(p_2+q_2)(p_3+q_3)} \alpha^1 \otimes \alpha^3 \otimes \alpha^2 = \\ &= (-1)^{(p_2+q_2)(p_3+q_3)+(p_1+q_1)(p_3+q_3)} \alpha^3 \otimes \alpha^1 \otimes \alpha^2.\end{aligned}$$

Now for every triple $(p_i, q_i)_{i=1,2,3}$ and for every $\sigma \in S_3$ let us define $\operatorname{sgn}(\sigma, p_1, q_1, p_2, q_2, p_3, q_3)$ as $+1$ or -1 according to the previous description. For $\sigma = \mathrm{id}$ we set $\operatorname{sgn}(\sigma, \cdots) = 1$ Then for every triple α_i for $i = 1,2,3$ as before, we get that the object

$$f(\alpha^1, \alpha^2, \alpha^3) := \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma, p_1, q_1, p_2, q_2, p_3, q_3) \alpha^{\sigma(1)} \otimes \alpha^{\sigma(2)} \otimes \alpha^{\sigma(3)} \tag{8.7}$$

is invariant under the action of S_3. If $(p_i, q_i) \neq (p_j, q_j)$ for every $i \neq j \in \{1,2,3\}$ and if every α^i is non-zero, then this object is also non-zero. If some of the (p_i, q_i)'s are equal, we will give a more precise description below.

Now let us define the following spaces.

- If we fix any *unordered* triple $(p_i, q_i)_{i=1,2,3}$ such that $(p_i, q_i) \neq (p_j, q_j)$ for all $i \neq j \in \{1,2,3\}$, then we set

$$H_{p_1,q_1,p_2,q_2,p_3,q_3} := \bigoplus_{\sigma \in S_3} H^{p_{\sigma(1)},q_{\sigma(1)}}(M) \otimes H^{p_{\sigma(2)},q_{\sigma(2)}}(M) \otimes H^{p_{\sigma(3)},q_{\sigma(3)}}(M).$$

- For every pair (p, q) we define

$$H_1^{p,q} := \bigoplus_{\substack{(p_1,q_1)+(p_2,q_2)+(p_3,q_3)=(p,q) \\ (p_1,q_1)<(p_2,q_2)<(p_3,q_3)}} H_{p_1,q_1,p_2,q_2,p_3,q_3},$$

  where we use $<$ to denote the strict lexicographic order.

- For every ordered pair $(p_i, q_i)_{i=1,3}$ such that $(p_1, q_1) \neq (p_3, q_3)$ we set

$$H_{p_1,q_1,p_3,q_3} := H^{p_1,q_1}(M) \otimes H^{p_1,q_1}(M) \otimes H^{p_3,q_3}(M) \oplus$$
$$\oplus H^{p_1,q_1}(M) \otimes H^{p_3,q_3}(M) \otimes H^{p_1,q_1}(M) \oplus H^{p_3,q_3}(M) \otimes H^{p_1,q_1}(M) \otimes H^{p_1,q_1}(M).$$

- For every (p, q) we define

$$H_2^{p,q} := \bigoplus_{\substack{(2p_1,2q_1)+(p_3,q_3)=(p,q) \\ (p_1,q_1)\neq(p_3,q_3)}} H_{p_1,q_1,p_3,q_3}.$$

- For every (p_1, q_1) we set

$$H_{p_1,q_1} := H^{p_1,q_1}(M) \otimes H^{p_1,q_1}(M) \otimes H^{p_1,q_1}(M)$$

- For every (p, q) such there exists (p_1, q_1) with $(3p_1, 3q_1) = (p, q)$, we set $H_3^{p,q} := H_{p_1,q_1}$; in the opposite case we set $H_3^{p,q} := \{0\}$.

Then we get that

$$H^{p,q}(Z) = (H_1^{p,q} \oplus H_2^{p,q} \oplus H_3^{p,q})^{S_3} =$$
$$= \bigoplus_{\substack{(p_1,q_1)+(p_2,q_2)+(p_3,q_3)=(p,q) \\ (p_1,q_1)<(p_2,q_2)<(p_3,q_3)}} (H_{p_1,q_1,p_2,q_2,p_3,q_3})^{S_3} \bigoplus \bigoplus_{\substack{(2p_1,2q_1)+(p_3,q_3)=(p,q) \\ (p_1,q_1)<(p_3,q_3)}} (H_{p_1,q_1,p_3,q_3})^{S_3} \bigoplus (H_3^{p,q})^{S_3}.$$

Now let us describe a basis for the S_3-invariant parts of all these spaces.

- Let us suppose that $(p_i, q_i) \neq (p_j, q_j)$ for $i \neq j \in \{1, 2, 3\}$ and let us denote by α^i any non-zero object in $H^{p_i,q_i}(M)$. Then every object of the form (8.7) is invariant under the action of S_3 and non-zero. Therefore, if we write $h^{p_i,q_i} := \dim H^{p_i,q_i}(M)$ for $i = 1, 2, 3$, then any base for $(H_{p_1,q_1,p_2,q_2,p_3,q_3})^{S_3}$ has cardinality

$$h^{p_1,q_1} h^{p_2,q_2} h^{p_3,q_3}.$$

Since $H_1^{p,q}$ is defined as the direct sum over all $H_{p_1,q_1,p_2,q_2,p_3,q_3}$ such that $(p_1, q_1) < (p_2, q_2) < (p_3, q_3)$, then we have that

$$\dim(H_1^{p,q})^{S_3} = \sum_{\substack{(p_1,q_1)+(p_2,q_2)+(p_3,q_3)=(p,q)\\(p_1,q_1)<(p_2,q_2)<(p_3,q_3)}} h^{p_1,q_1} h^{p_2,q_2} h^{p_3,q_3} =$$

$$= \frac{1}{6} \sum_{\substack{(p_1,q_1)+(p_2,q_2)+(p_3,q_3)=(p,q)\\(p_i,q_i)\neq(p_j,q_j) \text{ for } i\neq j}} h^{p_1,q_1} h^{p_2,q_2} h^{p_3,q_3}.$$

- Let us suppose that $(p_1, q_1) = (p_2, q_2) \neq (p_3, q_3)$ and let us assume that $\{\alpha_i^1\}_{i=1,\cdots,h^{p_1,q_1}}$ is a basis for $H^{p_1,q_1}(M) = H^{p_2,q_2}(M)$ and that $\{\alpha_k^3\}_{k=1,\cdots,h^{p_3,q_3}}$ is a basis for H^{p_3,q_3}. Then the family

$$\{f(\alpha_i^1, \alpha_j^1, \alpha_k^3)\}_{\substack{i\neq j\in\{1,\cdots,h^{p_1,q_1}\}\\k\in\{1,\cdots,h^{p_3,q_3}\}}}$$

is a partial basis for $(H_{p_1,q_1,p_3,q_3})^{S_3}$. Since the pair (i, j) (with $i \neq j$) is an unordered pair, the cardinality of such a set is

$$\frac{h^{p_1,q_1}(h^{p_1,q_1} - 1)}{2} h^{p_3,q_3}.$$

Then we need also to consider what happens when $i = j$: if we set $\alpha^1 = \alpha^2 := \alpha_i^1$ and $\alpha^3 =: \alpha_k^3$ for some i, k, then the previous identities give the following results.

$$(1\,2)\alpha_i^1 \otimes \alpha_i^1 \otimes \alpha_k^3 = (-1)^{(p_1+q_1)}\alpha_i^1 \otimes \alpha_i^1 \otimes \alpha_k^3,$$

$$(2\,3)\alpha_i^1 \otimes \alpha_i^1 \otimes \alpha_k^3 = (-1)^{(p_1+q_1)(p_3+q_3)}\alpha_i^1 \otimes \alpha_k^3 \otimes \alpha_i^1,$$

$$(1\,3)\alpha_i^1 \otimes \alpha_i^1 \otimes \alpha_k^3 = (-1)^{(p_1+q_1)(p_1+q_1)+(p_1+q_1)(p_3+q_3)+(p_1+q_1)(p_3+q_3)}\alpha_k^3 \otimes \alpha_i^1 \otimes \alpha_i^1 =$$

$$= (-1)^{(p_1+q_1)}\alpha_k^3 \otimes \alpha_i^1 \otimes \alpha_i^1.$$

$$(1\,2\,3)\alpha_i^1 \otimes \alpha_i^1 \otimes \alpha_k^3 = (-1)^{(p_1+q_1)(p_1+q_1)+(p_1+q_1)(p_3+q_3)}\alpha_i^1 \otimes \alpha_i^3 \otimes \alpha_i^3.$$

$$(1\,3\,2)\alpha_i^1 \otimes \alpha_i^1 \otimes \alpha_k^3 = (-1)^{(p_1+q_1)(p_3+q_3)+(p_1+q_1)(p_3+q_3)}\alpha_k^3 \otimes \alpha_i^1 \otimes \alpha_i^1 = \alpha_k^3 \otimes \alpha_i^1 \otimes \alpha_i^1.$$

So in order to get a non-zero invariant form out of α_i^1 and α_k^3 the only possibility is to assume that (p_1+q_1) is even. In that case the object

$$\alpha_i^1 \otimes \alpha_i^1 \otimes \alpha_k^3 + \alpha_i^1 \otimes \alpha_k^3 \otimes \alpha_i^1 + \alpha_k^3 \otimes \alpha_i^1 \otimes \alpha_i^1$$

is non-zero and S_3-invariant. So we conclude that a basis for $(H_{p_1,q_1,p_3,q_3})^{S_3}$ has cardinality

$$\frac{h^{p_1,q_1}(h^{p_1,q_1}-1)}{2}h^{p_3,q_3}$$

if p_1+q_1 is odd and

$$\frac{h^{p_1,q_1}(h^{p_1,q_1}-1)}{2}h^{p_3,q_3} + h^{p_1,q_1}h^{p_3,q_3}$$

if p_1+q_1 is even. A formula that takes into account both cases at the same time is given by

$$\frac{(h^{p_1,q_1})^2 h^{p_3,q_3}}{2} + (-1)^{p_1+q_1}\frac{h^{p_1,q_1}h^{p_3,q_3}}{2}.$$

Since $H_2^{p,q}$ is defined as the direct sum over all H_{p_1,q_1,p_3,q_3} such that $(p_1,q_1) \neq (p_3,q_3)$, then we have that

$$\dim(H_2^{p,q})^{S_3} = \sum_{\substack{(2p_1,2q_1)+(p_3,q_3)=(p,q)\\(p_1,q_1)\neq(p_3,q_3)}} \left(\frac{(h^{p_1,q_1})^2 h^{p_3,q_3}}{2} + (-1)^{p_1+q_1}\frac{h^{p_1,q_1}h^{p_3,q_3}}{2}\right) =$$

$$= \frac{1}{6}\sum_{\substack{(p_1,q_1)+(p_2,q_2)+(p_3,q_3)=(p,q)\\(p_i,q_i)=(p_j,q_j)\neq(p_k,q_k)}} h^{p_1,q_1}h^{p_2,q_2}h^{p_3,q_3} + \frac{1}{2}\sum_{\substack{(2p_1,2q_1)+(p_3,q_3)=(p,q)\\(p_1,q_1)\neq(p_3,q_3)}} (-1)^{p_1+q_1}h^{p_1,q_1}h^{p_3,q_3}.$$

- Let us assume that $(p_1,q_1)=(p_2,q_2)=(p_3,q_3)$ and that $\{\alpha_i^1\}_{i=1,\cdots,h^{p_1,q_1}}$ is a basis for $H^{p_1,q_1}(M) = H^{p_2,q_2} = H^{p_3,q_3}(M)$. Then the family

$$\{f(\alpha_i^1,\alpha_j^1,\alpha_k^1)\}_{i\neq j\neq k, i\neq k\in\{1,\cdots,h^{p_1,q_1}\}}$$

is a partial basis for $(H_{p_1,q_1})^{S_3} = (H_3^{3p_1,3q_1})^{S_3}$; since the order of i,j,k is not important, the cardinality of such a set is equal to

$$\frac{h^{p_1,q_1}(h^{p_1,q_1}-1)(h^{p_1,q_1}-2)}{6} = \frac{(h^{p_1,q_1})^3 - 3(h^{p_1,q_1})^2 + 2h^{p_1,q_1}}{6}.$$

If some (possibly all) indices i, j, k are equal, again we can get a non-zero invariant if and only if $p_1 + q_1$ is even. In that case we will have to take into account objects of the form

$$\alpha_i^1 \otimes \alpha_i^1 \otimes \alpha_k^1 + \alpha_i^1 \otimes \alpha_k^1 \otimes \alpha_i^1 + \alpha_k^1 \otimes \alpha_i^1 \otimes \alpha_i^1 \tag{8.8}$$

for all $i \neq k \in \{1, \cdots, h^{p_1,q_1}\}$ and

$$\alpha_i^1 \otimes \alpha_i^1 \otimes \alpha_i^1 \tag{8.9}$$

for all possible values of i. Since i and k don't play the same role in (8.8), then the number of objects of type (8.8) or (8.9) is equal to

$$h^{p_1,q_1}(h^{p_1,q_1} - 1) + h^{p_1,q_1} = (h^{p_1,q_1})^2$$

So if $p_1 + q_1$ is odd, then the dimension of $(H_{p_1,q_1})^{S_3} = (H_3^{3p_1,3q_1})^{S_3}$ is equal to

$$\frac{(h^{p_1,q_1})^3 - 3(h^{p_1,q_1})^2 + 2h^{p_1,q_1}}{6}.$$

If $p_1 + q_1$ is even, then the dimension is given by:

$$\frac{(h^{p_1,q_1})^3 - 3(h^{p_1,q_1})^2 + 2h^{p_1,q_1}}{6} + (h^{p_1,q_1})^2.$$

A common formula for both values is the following

$$\dim(H_3^{3p_1,3q_1})^{S_3} = \frac{(h^{p_1,q_1})^3}{6} + (-1)^{p_1+q_1}\frac{(h^{p_1,q_1})^2}{2} + \frac{h^{p_1,q_1}}{3}.$$

By summing everything we get:

$$\mathcal{HD}(Z)(u,v) = \sum_{(p,q)} h^{p,q}(Z)u^p v^q =$$

$$= \frac{1}{6}\sum_{(p_1,q_1),(p_2,q_2),(p_3,q_3)} h^{p_1,q_1}h^{p_2,q_2}h^{p_3,q_3}u^{p_1+p_2+p_3}v^{q_1+q_2+q_3} +$$

$$+ \sum_{(p_1,q_1),(p_3,q_3)} (-1)^{p_1+q_1}\frac{h^{p_1,q_1}h^{p_3,q_3}}{2}u^{2p_1+p_3}v^{2q_1+q_3} + \frac{1}{3}\sum_{(p_1,q_1)} h^{p_1,q_1}u^{3p_1}v^{3q_1} =$$

$$= \frac{1}{6}(\mathcal{HD}(M)(u,v))^3 + \frac{1}{2}\mathcal{HD}(M)(-u^2,-v^2)\cdot\mathcal{HD}(u,v) + \frac{1}{3}\mathcal{HD}(u^3,v^3).$$

□

If d is odd, then the Hodge-Deligne polynomial of the moduli space $M(2,d) = M^s(2,d) = M^{ss}(2,d)$ of stable vector bundles of rank 2 and degree d can be found, for example, in [MOVG, proposition 8.1]:

$$\mathcal{HD}(M(2,d)) = \frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^{2g}(1+v)^{2g}}{(1-uv)(1-(uv)^2)}. \tag{8.10}$$

If d is even then the Hodge-Deligne polynomial of $M^s(2,d)$ can be found in [MOVG2, theorem A]:

$$\mathcal{HD}(M^s(2,d)) = \frac{1}{2(1-uv)(1-(uv)^2)}\Big(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g+$$
$$-(1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1}-u^2v^2)-(1-u^2)^g(1-v^2)^g(1-uv)^2\Big). \tag{8.11}$$

Since these polynomials do not depend on d, but only the parity of d, we denote them by $\mathcal{HD}(M(2,odd))$ and $\mathcal{HD}(M(2,even))$ respectively.

If $d \not\equiv 0 \bmod 3$, then the formula for the Hodge-Deligne polynomial of $M(3,d) = M^s(3,d)$ can be obtained by combining lemma 3 and corollary 5(b) in [EK] and is given as follows.

$$\begin{aligned}\mathcal{HD}(M(3,d)) = (1+u)^g(1+v)^g&\frac{1}{(1-uv)(1-u^2v^2)^2(1-u^3v^3)}\cdot\\ \cdot\big((1+u^2v^3)^g&(1+u^3v^2)^g(1+uv^2)^g(1+u^2v)^g+\\ -u^{2g-1}v^{2g-1}&(1+uv)^2(1+u)^g(1+v)^g(1+uv^2)^g(1+u^2v)^g+\\ +u^{3g-1}&v^{3g-1}(1+uv+u^2v^2)(1+u)^{2g}(1+v)^{2g}\big).\end{aligned} \tag{8.12}$$

There is a similar formula in [M, theorem 1.2] but some of the signs are different. The author confirms that those signs are wrong and that the version in [EK] should be the correct one.

We recall also 2 results about moduli spaces of stable pairs (holomorphic triples). We recall that for every coherent system (E,V) of type $(n,d,1)$ we can associate an holomorphic triple (E_1,E_2,ϕ) of type $(n,1,d,0)$ with $E_2 = \mathcal{O}_C \otimes V$. Whenever (E,V) is stable for a certain value of α, the corresponding triple is stable for an associated value of the stability parameter σ, and conversely. The values of the stability parameter σ for which the moduli spaces $\mathcal{N}_\sigma$ of σ-stable holomorphic triples are non-empty consist of an interval $[\sigma_m, \sigma_M]$, and we have the following results.

Proposition 8.0.7. *([MOVG, theorem 6.2]) Let C be a smooth projective curve of genus $g \geq 2$ and let us consider the moduli space $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2,1,d_1,d_2)$, for a non-critical value $\sigma > \sigma_m$. Set $d_0 = \left\lceil \frac{1}{3}(\sigma + d_1 + d_2)\right\rceil$. Then the Hodge-Deligne polynomial of $\mathcal{N}_\sigma$ is*

$$\mathcal{HD}(\mathcal{N}_\sigma) = \operatorname*{coeff}_{x^0}\left[\frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2-d_0}}\cdot\left(\frac{(uv)^{d_1-d_2-d_0}}{1-(uv)^{-1}x} - \frac{(uv)^{-d_1+g-1+2d_0}}{1-(uv)^2x}\right)\right].$$

Proposition 8.0.8. *([M, theorem 6.5]) Let $\sigma > \sigma_m$ be a non-critical value. Set $n_0 = \lceil \frac{\sigma+d_1+d_2}{2} \rceil$ and $\bar{n}_0 = 2\lfloor \frac{n_0+1}{2} \rfloor$. Then the Hodge polynomial of $\mathcal{N}_\sigma = \mathcal{N}_\sigma(3,1,d_1,d_2)$ is*

$$\mathcal{HD}(\mathcal{N}_\sigma) = (1+u)^{2g}(1+v)^{2g} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^{d_1-d_2}} \cdot$$

$$\cdot \Bigg[\left(\frac{(uv)^{2d_1-2d_2-2n_0}x^{n_0}}{1-(uv)^{-2}x} - \frac{(uv)^{2g-2-2d_1+3n_0}x^{n_0}}{1-(uv)^3x} \right) \cdot \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} +$$

$$+ \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)} \left(\frac{(uv)^{2d_1-2d_2-2\bar{n}_0+1}x^{\bar{n}_0}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)} + \frac{(uv)^{2g-2-2d_1+3\bar{n}_0}x^{\bar{n}_0}}{(1-(uv)^3x)(1-(uv)^2x)} + \right.$$

$$\left. - \frac{(1+uv)(uv)^{g-1-d_2+\bar{n}_0/2}x^{\bar{n}_0}}{(1-(uv)^2x)(1-(uv)^{-1}x)} \right) \Bigg].$$

We will need also some results from [GM] about some moduli spaces $G_L(n,d,k)$ of α-stable coherent systems for α large.

Theorem 8.0.9. *[GM, theorem 8.19] The Hodge-Deligne polynomial of the moduli space $G_L(n,d,k)$ for $(n-k,d) = (2,d) = 1$ is*

$$\mathcal{HD}(G_L(n,d,k)) = (1+u)^g(1+v)^g \cdot \frac{(1+u^2v)^g(1+uv^2)^g - u^gv^g(1+u)^g(1+v)^g}{(1-uv)(1-u^2v^2)} \cdot$$

$$\cdot \frac{(1-(uv)^{2(g-1)+d-k+1}) \cdot \ldots \cdot (1-(uv)^{2(g-1)+d})}{(1-uv) \cdot \ldots \cdot (1-(uv)^k)}.$$

A formula for $G_L(n,d,k)$ for $(n-k,d) = (2,d) \neq 1$ appears in [GM, theorem 8.20], but it seems that it is incorrect. A corrected (and still unpublished) version of that formula by the same author is known in the case $n = 3$. We are not going to use this formula, we will only compare it with our approach in a special case, described below.

Theorem 8.0.10. *(Cristian Gonzalez-Martinez) The Hodge-Deligne polynomial of the moduli space $G_L(3,d,1)$ for d even and $g \geq \frac{3-d}{2}$ is:*

$$\mathcal{HD}(G_L(3,d,1)) = \frac{(1+u)^g(1+v)^g}{u^3v^3(uv-1)^3(uv+1)} \Big((uv)^{4+g}(1+u)^g(1+v)^g +$$

$$-(uv)^{d+3g+2}(1+u)^g(1+v)^g + (uv)^{d+2g}(-1+(uv)^g - (uv)^g(1+u)^g(1+v)^g) +$$

$$+(uv)^{2+d+2g}(1-(uv)^g + (uv)^g(1+u)^g(1+v)^g) + (-(uv)^{d/2+2g+3} + (uv)^{d/2+3g+3} +$$

$$-(uv)^{d/2+2g+3}(1+u)^g(1+v)^g - (1+u^2v)^g(1+uv^2)^g) +$$

$$-(uv)^{d/2+2g+1}(-1+(uv)^g - (1+u)^g(1+v)^g - (uv)^{d/2}(1+u^2v)^g(1+uv^2)^g) \Big).$$

As a corollary of this theorem, the following explicit formula is known.

Corollary 8.0.11. *(Cristian Gonzalez-Martinez) The Hodge-Deligne polynomial of the moduli space $G_L(3,2,1)$ for $g=2$ is given by*

$$
\begin{aligned}
&= 1+2u+2v+u^2+v^2+6uv+8u^2v+8uv^2+6u^3v+6uv^3+21u^2v^2+2u^4v+2uv^4+\\
&+26u^3v^2+26u^2v^3+50u^3v^3+17u^4v^2+17u^2v^4+52u^4v^3+52u^3v^4+6u^5v^2+6u^2v^5+\\
&+u^6v^2+28u^5v^3+74u^4v^4+28u^3v^5+u^2v^6+6u^6v^3+6u^3v^6+52u^5v^4+52u^4v^5+\\
&+17u^6v^4+17u^4v^6+50u^5v^5+26u^6v^5+26u^5v^6+2u^7v^4+2u^4v^7+21u^6v^6+6u^7v^5+\\
&+6u^5v^7+8u^7v^6+8u^6v^7+6u^7v^7+u^8v^6+u^6v^8+2u^8v^7+2u^7v^8+u^8v^8
\end{aligned}
\tag{8.13}
$$

and therefore the Poincaré polynomial of the moduli space $G_L(3,2,1)$ for $g=2$ is given by

$$
\begin{aligned}
P_{G_L(3,2,1)}(t) &= 1+4t+8t^2+16t^3+33t^4+56t^5+85t^6+116t^7+\\
&+132t^8+116t^9+84t^{10}+56t^{11}+33t^{12}+16t^{13}+8t^{14}+4t^{15}+t^{16}.
\end{aligned}
$$

We will need also to use the Hodge-Deligne polynomials of the moduli spaces $G(\alpha;1,d,1)$ for $d\geq 0$ and any $\alpha\in\mathbb{R}_{\geq 0}$. Since there are no critical values for $(1,d,1)$), those spaces are usually simply denoted by $G(1,d,1)$. If d is bigger than zero, we use the following lemma:

Lemma 8.0.12. *For every positive integer d and for every positive real number α, we can identify $G(\alpha;1,d,1)=G(1,d,1)$ with the symmetric product $C^{(d)}$.*

If the genus g is equal to 0, this is proposition 2.1 in [LN].

Proof. First of all, we recall that for every positive integer d there exists a well-known morphism, the Abel-Jacobi map:

$$AJ : C^{(d)} \longrightarrow J^dC$$

that associates to every effective divisor $D=P_1+\cdots+P_d$ of degree d the line bundle $\mathcal{O}(D)$. Then for every line bundle L of degree d, $AJ^{-1}(L)$ is the set of effective divisors D such that $L\simeq\mathcal{O}(D)$. Now every such D is in bijection with a section $s\neq 0$ of L, up to scalar multiples, i.e. it is in bijection with a subvector space $V\subseteq H^0(L)$ of dimension 1. Therefore, we can identify the points of $C^{(d)}$ with coherent systems (L,V) of type $(1,d,1)$. Now each such coherent system is automatically α-stable for every positive real number α, so we get the required identification. □

Now we recall that by [M, §2], for every smooth projective irreducible curve C and for every $d\geq 1$, the Hodge-Deligne polynomial of the symmetric product $C^{(d)}=\mathrm{Sym}^dC$ is given by

$$\mathcal{HD}(C^{(d)}) = \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-d}}{(1-x)(1-uvx)}. \tag{8.14}$$

Then this formula computes the Hodge-Deligne polynomial of $G(1,d,1)$ whenever $d \geq 1$. When $d = 0$, then the only coherent system (E,V) of type $(1,0,1)$ is $(\mathcal{O}, H^0(\mathcal{O}) = \mathbb{C})$ since a line bundle of degree zero has sections if and only if it is the trivial line bundle. Such a coherent system is α-stable for every $\alpha \in \mathbb{R}_{\geq 0}$, so $G(\alpha;1,0,1)$ consists of a single point. So

$$\mathcal{HD}(G(\alpha;1,0,1)) = \mathcal{HD}(\mathbb{C}^0) = (uv)^0 = 1.$$

Now let us consider the right hand side of (8.14) for $d = 0$. In this case the function

$$f(x) := \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}$$

is holomorphic near $x = 0$, therefore

$$\operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} = f(0) = 1 = \mathcal{HD}(G(\alpha;1,0,1)).$$

So using the previous lemma for $d \geq 1$ and this remark for $d = 0$, we get that:

Lemma 8.0.13. *For every $\alpha \in \mathbb{R}_{\geq 0}$ and for every non-negative integer d, the Hodge-Deligne polynomial of $G(\alpha;1,d,1) = G(1,d,1)$ is given by:*

$$\mathcal{HD}(G(\alpha;1,d,1)) = \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-d}}{(1-x)(1-uvx)}.$$

Chapter 9

Results on Hodge-Deligne polynomials

In this chapter we summarize the results that we were able to obtain about the Hodge-Deligne polynomials of some moduli spaces of coherent systems, listed below. For the proof of each result, see part II of this work. Unless otherwise stated all the results of this chapter hold for moduli spaces of (semi-)stable coherent systems over a smooth projective irreducible curve C of genus $g \geq 2$ over $\mathbb{C}$.

9.1 The Hodge-Deligne polynomials of $G(\alpha; 2, d, 1)$

Theorem 9.1.1. *(theorem 13.3.1, corollary 13.3.2 and corollary 13.4.2). Let us fix any positive integer d; then the non-zero actual critical values for the triple $(2, d, 1)$ are all of the form $\alpha(k) = d - 2k$ for $0 \leq k < d/2$. For any such k we have that*

$$\mathcal{HD}(G(\alpha(k)^-; 2, d, 1)) =$$

$$= \frac{(1+u)^g(1+v)^g}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} \left[\frac{(uv)^k x^{-k}}{1-x(uv)^{-1}} - \frac{(uv)^{g+d-1-2k} x^{-k}}{1-x(uv)^2}\right].$$

Moreover, the Hodge-Deligne polynomial of the stable locus at any critical value

$$G^s(\alpha(k); 2, d, 1) \simeq G(\alpha(k)^+; 2, d, 1) \smallsetminus G^+(\alpha(k); 2, d, 1) \simeq$$

$$\simeq G(\alpha(k)^-; 2, d, 1) \smallsetminus G^-(\alpha(k); 2, d, 1)$$

is given by:

$$\mathcal{HD}(G^s(\alpha(k); 2, d, 1)) = \frac{(1+u)^g(1+v)^g}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} \cdot$$

$$\cdot \left[\frac{(uv)^k x^{-k}}{1-x(uv)^{-1}} - \frac{(uv)^{g+d+1-2k} x^{1-k}}{1-x(uv)^2} - x^{-k}\right].$$

In addition, if

$$d > 4g-4 \quad \textit{and} \quad 2g-2 \leq k < d/2$$

then

$$\mathcal{HD}(G(\alpha(k)^-;2,d,1) =$$
$$= \frac{(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\left\{\Big[(1+uv)((uv)^{d-k}-(uv)^{k+1})+(uv)^{2k+3-g}+\right.$$
$$\left.-(uv)^{g+d-1-2k}\Big]\cdot(1+u)^g(1+v)^g+[1-(uv)^{d+2-2g}](1+u^2v)^g(1+uv^2)^g\right\}.$$

Remark 9.1.1. The first formula agrees with that given in [M, proposition 5.4] up to a multiplicative term $(1+u)^g(1+v)^g$, once we set in that formula $d_1 := d$, $d_2 := 0$, $\bar{d}_M := d-k$. This makes sense because in [M] the moduli spaces of triples are studied and such moduli spaces can be considered as fibrations over corresponding moduli spaces of coherent systems, with fibers isomorphic to a Jacobian. The Hodge-Deligne polynomial of the Jacobian (for any degree) is exactly equal to $(1+u)^g(1+v)^g$.

The last formula of the previous theorem gives also the following results for the moduli space with stability parameter small.

Corollary 9.1.2. *(corollaries 13.4.3 and 13.4.4) If d is odd and $d > 4g-4$, then*

$$\mathcal{HD}(G_0(2,d,1)) = \frac{(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\left\{\Big[(uv)^{d+2-g}-(uv)^g\Big]\cdot\right.$$
$$\left.\cdot(1+u)^g(1+v)^g+[1-(uv)^{d+2-2g}](1+u^2v)^g(1+uv^2)^g\right\}.$$

If d is even and $d > 4g-4$, then

$$\mathcal{HD}(G_0(2,d,1)) = \frac{(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\Big\{[1-(uv)^{d+2-2g}](1+u^2v)^g(1+uv^2)^g+$$
$$-[(uv)^{\frac{d}{2}}(1-(uv)^2)+(uv)^{g+1}(1-(uv)^{d-2g})](1+u)^g(1+v)^g\Big\}.$$

Remark 9.1.2. The first polynomial coincides with

$$\mathcal{HD}(M(2,d))\cdot\mathcal{HD}(\mathbb{P}^{d+1-2g}),$$

so this agrees with the known fact that if d is odd and $d > n(2g-2) = 4g-4$, then $G_0(2,d,1)$ is a grassmannian fibration over the moduli space $M(2,d)$ of stable rank 2 bundles of degree d, with fiber over any vector bundle E given by

$$\mathrm{Grass}(1,\chi(E)) = \mathrm{Grass}(1,H^0(E)) = \mathrm{Grass}(1,d+2(1-g)) = \mathbb{P}^{d+1-2g}.$$

Here the first identity comes from the fact that $H^1(E) = 0$ for $d > 4g-4$, while the second one is Riemann-Roch. For the Hodge-Deligne polynomial of $M(2,d)$, see (8.10). We also remark that the leading term of the second polynomial coincides with the leading term of (8.11) times $\mathcal{HD}(\mathbb{P}^{d+1-2g})$.

Remark 9.1.3. The previous results (that a priori are valid only under the hypothesis $g \geq 2$) agree with the known literature for the cases $g = 0$ and $g = 1$. When $g = 0$, the Hodge-Deligne polynomials of the moduli spaces $G(\alpha;n,d,1)$ for all n and for all α non-critical for $(n,d,1)$ were computed in [LN]. Moreover, the Hodge-Deligne polynomials of the moduli spaces $G(\alpha;2+ad,d,1)$ (for any non-negative integer a) for $g = 1$ were computed in [LN2]. In section 13.5 we will prove that the formulae of the previous theorem are valid also in the cases when $g = 0$ and $g = 1$, by comparing them with the formulae of [LN] and [LN2].

9.2 The Hodge-Deligne polynomials of $G(\alpha;3,d,1)$

Let us fix any positive integer d; then the non-zero actual critical values for the triple $(3,d,1)$ are all of the form $\alpha(k) = (d-3k)/2$ for $0 \leq k < d/3$. For any such k we have 2 different expressions for the Hodge-Deligne polynomials of $G(\alpha(k);3,d,1)$ depending on the parity of $d-k$. For the last moduli space we have the following expressions:

Proposition 9.2.1. *(propositions 14.3.3 and 14.3.4) If d is odd, then:*

$$\mathcal{HD}(G(\alpha(0)^-;3,d,1)) = \Big(1-(uv)^{2g-2+d}\Big)\cdot$$
$$\cdot\frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^{2g}(1+v)^{2g}}{(1-uv)^2(1-(uv)^2)}.$$

If d is even, then

$$\mathcal{HD}(G(\alpha(0)^-;3,d,1)) = (1+u)^g(1+v)^g\Big\{[1-(uv)^{2g-2+d}]\cdot$$
$$\cdot\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}+$$
$$+\frac{(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot\Big(1+(uv)^{2g-3+d}-(uv)^{g-2+d/2}-(uv)^{g+d/2-1}\Big)\Big\}.$$

Moreover,

Theorem 9.2.2. *(theorem 14.3.5) For every $d > 0$ and for every critical value*

$$\alpha(k) = (d-3k)/2, \quad 0 \leq k < d/3$$

the following formula holds:

$$\mathcal{HD}(G(\alpha(k)^-;3,d,1)) = (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot$$
$$\cdot\Big\{\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\cdot\Big(\frac{(uv)^{2k}x^{-k}}{1-(uv)^{-2}x}+$$
$$-\frac{(uv)^{2g-2+d-3k}x^{-k}}{1-(uv)^3x}\Big)+\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot\Big(\frac{(uv)^{2d-4l_0+1}x^{2l_0-d}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)}+$$

$$+\frac{(uv)^{2g-2-2d+6l_0}x^{2l_0-d}}{(1-(uv)^3x)(1-(uv)^2x)}-\frac{(1+uv)(uv)^{g-1+l_0}x^{2l_0-d}}{(1-(uv)^{-1}x)(1-(uv)^2x)}\Bigg)\Bigg\},$$

where $l_0 := \lceil (d-k)/2 \rceil$.

Moreover, we have the following results for the stable locus at every critical value.

Proposition 9.2.3. *(corollaries 14.3.6 and 14.3.7). For every critical value* $\alpha(k)$ *such that* $d-k$ *is odd, the following formula holds:*

$$\mathcal{HD}(G^s(\alpha(k);3,d,1)) = (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot$$

$$\cdot\Bigg\{\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\cdot\Bigg(\frac{(uv)^{2k}x^{-k}}{1-(uv)^{-2}x}+$$

$$-\frac{(uv)^{2g+1+d-3k}x^{1-k}}{1-(uv)^3x}-x^{-k}\Bigg)+\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot$$

$$\cdot\Bigg(\frac{(uv)^{2k-1}x^{1-k}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)}+\frac{(uv)^{2g+d+1-3k}x^{1-k}}{(1-(uv)^3x)(1-(uv)^2x)}+$$

$$-\frac{(1+uv)(uv)^{g-1+(d-k+1)/2}x^{1-k}}{(1-(uv)^{-1}x)(1-(uv)^2x)}\Bigg)\Bigg\}.$$

For every critical value $\alpha(k)$ *such that* $d-k$ *is even, the following formula holds:*

$$\mathcal{HD}(G^s(\alpha(k);3,d,1)) =$$

$$= (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot\Bigg\{\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\cdot$$

$$\cdot\Bigg(\frac{(uv)^{2k}x^{-k}}{1-(uv)^{-2}x}-\frac{(uv)^{2g+1+d-3k}x^{1-k}}{1-(uv)^3x}-x^{-k}\Bigg)+\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot$$

$$\cdot\Bigg(\frac{(uv)^{2k+1}x^{-k}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)}+\frac{(uv)^{2g+d+4-3k}x^{2-k}}{(1-(uv)^3x)(1-(uv)^2x)}-\frac{(1+uv)(uv)^{g+1+(d-k)/2}x^{1-k}}{(1-(uv)^{-1}x)(1-(uv)^2x)}+$$

$$-uvx^{-k}\Bigg)-\frac{(1+u)^g(1+v)^g}{(1-uv)^2}\left[\frac{(uv)^kx^{-k}}{1-(uv)^{-1}x}-\frac{(uv)^{g+1+(d-3k)/2}x^{1-k}}{1-(uv)^2x}-x^{-k}\right]\Bigg\}.$$

Remark 9.2.1. The formula presented in theorem 9.2.2 agrees with that presented in [M, theorem 6.5] for the moduli spaces of triples, up to a multiplicative term $(1+u)^g(1+v)^g$ (see remark 9.1.1) once we set in that formula $d_1 := d$, $d_2 := 0$, $n_0 := d-k$, so that

$$\overline{n_0} = 2\lfloor (n_0+1)/2 \rfloor = 2\lceil n_0/2 \rceil = 2\lceil (d-k)/2 \rceil = 2l_0.$$

Remark 9.2.2. The formula of proposition 9.2.1 for d even coincides with that given in (8.13) in the case when $d=2$ and $g=2$; see the computations after proposition 14.3.4 for the details.

9.3 Some results on the polynomials for $G(\alpha;4,d,1)$

As we said in the introduction, currently we are not able to get a formula that holds in full generality. Indeed, in order to cross some of the critical values for $(4,d,1)$ we need to compute 50 polynomials associated to various subloci; at the moment we are able to compute only 42 of them (we don't list all of them here, we refer directly to the detailed computations of chapter 15). For large values of α (and $d\not\equiv_3 0$), this is enough to get explicit results, but in general we are not able to conclude. The non-zero actual critical values are all of the form

$$\alpha(j)=\frac{d-2j}{3},\quad 0\leq j<d/2,\quad [j]\in\{0,2,4,2d+3\}_{\text{mod }6}=\{2d,2d+2,2d+3,2d+4\}_{\text{mod }6}.$$

We have complete results whenever we cross actual critical values $\alpha(j)$ with $[j]\in\{2d+2,2d+3,2d+4\}_{\text{mod }6}$; we don't have a complete formula for the case when $j\equiv_6 2d$. To be more precise,

Lemma 9.3.1. *(formulae (15.10) and (15.11)) If j is equivalent to $2d+3$ modulo 6, then*

$$\begin{aligned}&\mathcal{HD}(G^-(\alpha(j);4,d,1))-\mathcal{HD}(G^+(\alpha(j);4,d,1))=\\&=\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^3(1-u^2v^2)}\cdot\\&\cdot(1+u)^{2g}(1+v)^{2g}\cdot\left[(uv)^{2g-2+j}-(uv)^{4g-4+(2d-4j)/3}\right]\cdot\\&\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot\left[\frac{(uv)^{(j-1)/2}x^{(1-j)/2}}{1-x(uv)^{-1}}-\frac{(uv)^{g+(d-2j)/3}x^{(1-j)/2}}{1-x(uv)^2}\right].\end{aligned}$$

If j is equivalent to $2d+2$ or to $2d+4$ modulo 6, then

$$\begin{aligned}&\mathcal{HD}(G^-(\alpha(j);4,d,1))-\mathcal{HD}(G^+(\alpha(j);4,d,1))=\\&=(1+u)^g(1+v)^g\frac{(uv)^{3j/2}-(uv)^{3g-3+d-2j}}{(1-uv)^2(1-u^2v^2)^2(1-u^3v^3)}\cdot\\&\cdot\big[(1+u^2v^3)^g(1+u^3v^2)^g(1+uv^2)^g(1+u^2v)^g+\\&-u^{2g-1}v^{2g-1}(1+uv)^2(1+u)^g(1+v)^g(1+uv^2)^g(1+u^2v)^g+\\&+u^{3g-1}v^{3g-1}(1+uv+u^2v^2)(1+u)^{2g}(1+v)^{2g}\big]\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^gx^{-j/2}}{(1-x)(1-uvx)}.\end{aligned}$$

The last non-empty moduli space $G_L(4,d,1)$ is the one for $\alpha=d/3-\varepsilon=\alpha(0)^-$. As a corollary of the previous lemma, we have:

Corollary 9.3.2. *(corollary 15.4.1) Let us suppose that $d\not\equiv_3 0$. Then the Hodge-Deligne polynomial of $G(\alpha(0)^-;4,d,1)=G(d/3-\varepsilon,4,d,1)=G_L(4,d,1)$ is given by*

$$\mathcal{HD}(G(\alpha(0)^-;4,d,1))=(1+u)^g(1+v)^g\frac{1-(uv)^{3g-3+d}}{(1-uv)^2(1-u^2v^2)^2(1-u^3v^3)}.$$

$$
\begin{aligned}
&\cdot \big[(1+u^2v^3)^g(1+u^3v^2)^g(1+uv^2)^g(1+u^2v)^g+ \\
&-u^{2g-1}v^{2g-1}(1+uv)^2(1+u)^g(1+v)^g(1+uv^2)^g(1+u^2v)^g+ \\
&+u^{3g-1}v^{3g-1}(1+uv+u^2v^2)(1+u)^{2g}(1+v)^{2g}\big]\,.
\end{aligned} \tag{9.1}
$$

Moreover, we have:

Corollary 9.3.3. *(corollary 15.5.2) If $d \equiv_3 1$, then $\alpha(1)$ is not an actual critical value, so $G(\alpha(1)^-;4,d,1) = G(\alpha(0)^-;4,d,1)$; therefore formula (9.1) gives also the Hodge-Deligne polynomial of $G(\alpha(1)^-;4,d,1)$. If $d \equiv_3 2$, then $\alpha(1)$ is an actual critical value and*

$$
\begin{aligned}
\mathcal{HD}(G(\alpha(1)^-;4,d,1)) = &\frac{(1+u)^g(1+v)^g}{(1-uv)^3(1-u^2v^2)}\left\{\frac{1-(uv)^{3g-3+d}}{(1+uv)(1-u^3v^3)}\cdot\right. \\
&\cdot \big[(1+u^2v^3)^g(1+u^3v^2)^g(1+uv^2)^g(1+u^2v)^g+ \\
&-u^{2g-1}v^{2g-1}(1+uv)^2(1+u)^g(1+v)^g(1+uv^2)^g(1+u^2v)^g+ \\
&+u^{3g-1}v^{3g-1}(1+uv+u^2v^2)(1+u)^{2g}(1+v)^{2g}\big]+ \\
&+\big[(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g\big]\cdot(1+u)^g(1+v)^g\cdot \\
&\left.\cdot\left[(uv)^{2g-1}-(uv)^{4g-5+(2d-1)/3}\right]\cdot\left[1-(uv)^{g+(d-2)/3}\right]\right\}.
\end{aligned}
$$

Also when $d \equiv_3 0$ the value $\alpha(1)$ is not an actual critical value, so $G(\alpha(1)^-;4,d,1) = G(\alpha(0)^-;4,d,1)$, but we don't have an explicit formula for that space since corollary 9.3.2 does not hold for $d \equiv_3 0$.

We are able to get explicit computations also for $G(\alpha(2)^-;4,d,1)$ only when $d \equiv_3 2$; to be more precise,

Corollary 9.3.4. *(corollaries 15.6.2 and 15.7.2) If $d \equiv_3 2$, then $\alpha(2)$ is an actual critical value and*

$$
\begin{aligned}
&\mathcal{HD}(G^-(\alpha(2);4,d,1)) = \\
&= \frac{(1+u)^g(1+v)^g}{(1-uv)^3(1-u^2v^2)}\left\{\frac{1-(uv)^{3g-3+d}+[(uv)^3-(uv)^{3g-7+d}]\cdot[1+g(u+v)+uv]}{(1+uv)(1-u^3v^3)}\cdot\right. \\
&\cdot \big[(1+u^2v^3)^g(1+u^3v^2)^g(1+uv^2)^g(1+u^2v)^g+ \\
&-u^{2g-1}v^{2g-1}(1+uv)^2(1+u)^g(1+v)^g(1+uv^2)^g(1+u^2v)^g+ \\
&+u^{3g-1}v^{3g-1}(1+uv+u^2v^2)(1+u)^{2g}(1+v)^{2g}\big]+ \\
&+\big[(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g\big]\cdot(1+u)^g(1+v)^g\cdot \\
&\left.\cdot\left[(uv)^{2g-1}-(uv)^{4g-5+(2d-1)/3}\right]\cdot\left[1-(uv)^{g+(d-2)/3}\right]\right\}.
\end{aligned}
$$

Moreover if $d \equiv_3 1,2$, then $\alpha(3)$ is not an actual critical value for $(4,d,1)$. Therefore the previous formula gives also the Hodge-Deligne polynomial for $G^-(\alpha(3);4,d,1)$ whenever $d \equiv_3 2$.

9.4 Some results on the polynomials for $G(\alpha;2,d,2)$ on a Petri curve

Theorem 9.4.1. *(theorem 16.3.1) Let us suppose that C is a Petri curve. Then for every $d \geq 4g-4$ and for every actual critical value*

$$\alpha(k) = \frac{d-2k}{2}, \quad \frac{g}{2}+1 \leq k < \frac{d}{2}$$

the following formula holds:

$$\mathcal{HD}(G(\alpha(k)^-;2,d,2)) = \frac{(1+u)^g(1+v)^g}{1-uv} \sum_{j=g/2+1}^{k} \left(((uv)^{d-g+1} - (uv)^{g+d-1-2j})\mathcal{HD}(G(1,j,2))\right).$$

Part II

Chapter 10

Parametrization of objects with canonical filtration of type $(1,2)$ and $(2,1)$

In this and in the next chapters we will see in details how to parametrize coherent systems with α_c-canonical filtration of type (1,2), (2,1), (1,3), (3,1), (2,1,1), (1,2,1) and (1,1,2). For all but (1,2,1) we are able to give both a pointwise description (having fixed a graded) and a global description (letting vary the graded and fixing only its type). For the case (1,2,1), we are able to give always pointwise descriptions but we are able to globalize them only in 4 subcases among the 8 subcases we would like to get (see the details in §12.2).

Having fixed any triple (n,d,k) and a critical value α_c for it, in this chapter we want to describe how to parametrize those (E,V)'s that have α_c-canonical filtration of type (1,2) or (2,1) and that belong to $G^+(\alpha_c;n,d,k)$ or to $G^-(\alpha_c;n,d,k)$.

10.1 Canonical filtration of type (1,2)

Let us fix any object $\oplus_{i=1}^3(Q_i,W_i)$, with all the (Q_i,W_i)'s α_c-stable coherent systems with the same α_c-slope μ; let us suppose that (E,V) has such a graded at α_c and that it has α_c-canonical filtration of type $(1,2)$. Then every (E,V) that we want to parametrize sits in an exact sequence of the form:

$$0 \to (Q_1,W_1) \xrightarrow{\alpha} (E,V) \xrightarrow{\beta} (Q_2,W_2)\oplus(Q_3,W_3) \to 0. \tag{10.1}$$

If (E,V) has α_c-canonical filtration of type (1,2), then the only α_c-semistable proper subobjects of (E,V) with α_c-slope equal to μ are the following:

- the only α_c-stable one is (Q_1,W_1) (if any other (Q_i,W_i) is a suboject, then the α_c-canonical filtration is no more of type (1,2));
- for all $i=2,3$, an extension (E_{i1},V_{i1}) of (Q_i,W_i) by (Q_1,W_1).

So given any (E,V) with α_c-canonical filtration of type (1,2), then (E,V) belongs to $G^+(\alpha_c;n,d,k)$ if and only if the following numerical conditions are satisfied:

$$\frac{k_1}{n_1}<\frac{k}{n},\quad \frac{k_1+k_i}{n_1+n_i}<\frac{k}{n}\quad \forall\, i=2,3. \tag{10.2}$$

If we use the fact that $\mu_{\alpha_c}(E,V)=\mu_{\alpha_c}(E_{i,1},V_{i,1})$ for $i=1,2$, then the second condition is equivalent to

$$\frac{k}{n}<\frac{k_i}{n_i}\quad \forall\, i=2,3. \tag{10.3}$$

Actually, (10.3) implies that $\frac{k_2+k_3}{n_2+n_3}>\frac{k}{n}$, that implies the first condition of (10.2). So if (E,V) has α_c-canonical filtration of type (1,2), then (E,V) belongs to $G^+(\alpha_c;n,d,k)$ if and only if (10.3) holds. Analogously, given any (E,V) with α_c-canonical filtration of type (2,1), then (E,V) belongs to $G^-(\alpha_c;n,d,k)$ if and only if:

$$\frac{k}{n}>\frac{k_i}{n_i}\quad \forall\, i=2,3. \tag{10.4}$$

Now if we denote by μ any extension like (10.1), we get that we can identify μ with a pair

$$(\mu_2,\mu_3)\in\bigoplus_{i=2}^{3}\mathrm{Ext}^1\Big((Q_i,W_i),(Q_1,W_1)\Big).$$

This identification gives a diagram of the following form for $i=2,3$:

$$\begin{tikzcd}
0 \arrow[r] & (Q_1,W_1) \arrow[r, "\alpha"] \arrow[d, equal] \arrow[dr, phantom, "\curvearrowright"] & (E,V) \arrow[r, "\beta"] \arrow[dr, phantom, "\curvearrowright"] & \bigoplus_{l=2}^{3}(Q_l,W_l) \arrow[r] & 0 & \mu \arrow[d, "\overline{\varepsilon_i}"] \\
0 \arrow[r] & (Q_1,W_1) \arrow[r, "\alpha_i"] & (E_{i1},V_{i1}) \arrow[r, "\beta_i"] \arrow[u, hook, "\delta_i"] & (Q_i,W_i) \arrow[r] \arrow[u, hook, "\varepsilon_i"'] & 0 & \mu_i
\end{tikzcd} \tag{10.5}$$

where ε_i is the embedding of (Q_i,W_i) in $(Q_2,W_2)\oplus(Q_3,W_3)$ for $i=2,3$. Then we have the following results.

Lemma 10.1.1. *Let us fix any triple $(Q_i,W_i)_{i=1,2,3}\in\prod_{i=1}^{3}G_i$ such that conditions (10.3), respectively (10.4) are satisfied (this automatically implies that $(n_1,k_1)\neq(n_i,k_i)$ for $i=2,3$) and let us suppose that $(Q_2,W_2)\not\simeq(Q_3,W_3)$. Then the (E,V)'s that belong to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, that have α_c-canonical filtration of type $(1,2)$ and graded $\oplus_{i=1}^{3}$ (Q_i,W_i) are parametrized by $\mathbb{P}(H_2)\times\mathbb{P}(H_3)$, where $H_i:=\mathrm{Ext}^1\left((Q_i,W_i),(Q_1,W_1)\right)$ for $i=2,3$.*

Proof. For every extension $\mu=(\mu_2,\mu_3)$ with representative (10.1), we have that (E,V) has a filtration of the form

$$0=(E_0,V_0)\subset(E_1,V_1):=(Q_1,W_1)\subset(E_2,V_2)=(E,V). \tag{10.6}$$

Here $(E_1,V_1)/(E_0,V_0)=(Q_1,W_1)$ is α_c-stable and $(E_2,V_2)/(E_1,V_1)=(Q_2,W_2)\oplus(Q_3,W_3)$ is α_c-polystable. Then by proposition 2.1.3 we get that (10.6) is the α_c-canonical filtration of (E,V) (and so (E,V) has α_c-canonical filtration of type $(1,2)$), if and only if condition (c) of that proposition is satisfied. In our case the index t is equal to 2, so (E,V) has α_c-canonical filtration of type $(1,2)$ if and only if for all $i=1,2,3$ and for all non-zero morphisms $\gamma_i:(Q_i,W_i)\to(E,V)$ we have $\beta\circ\gamma_i=0$. Now by hypothesis $(Q_1,W_1)\not\simeq(Q_i,W_i)$ for all $i=2,3$. Since all the (Q_i,W_i)'s for $i=1,2,3$ are α_c-stable of the same slope, then for all $\gamma_1:(Q_1,W_1)\to(E,V)$ we have that $\beta\circ\gamma_1=0$. Then we conclude that for every (E,V) as in (10.1) the following conditions are equivalent:

(a) (10.6) is the α_c-canonical filtration of (E,V);

(b) for all $i=2,3$ and for all morphisms $\gamma_i:(Q_i,W_i)\to(E,V)$ we have $\beta\circ\gamma_i=0$.

Since $(Q_2,W_2)\not\simeq(Q_3,W_3)$, we conclude by lemma 3.3.2 that this is equivalent to

(c) $\mu_i\neq 0$ for $i=2,3$.

Now if we look at the sequence (10.1) we get that $\mathrm{Aut}(Q_1,W_1)=\mathbb{C}^*$ and $\mathrm{Aut}((Q_2,W_2)\oplus(Q_3,W_3))=\mathbb{C}^*\times\mathbb{C}^*$ (because $(Q_2,W_2)\not\simeq(Q_3,W_3)$), so this proves the claim. □

Lemma 10.1.2. *Let us fix any triple $(Q_i,W_i)_{i=1,2,3}\in\prod_{i=1}^3 G_i$ such that conditions (10.3), respectively (10.4), are satisfied (this automatically implies that $(n_1,k_1)\neq(n_i,k_i)$ for $i=2,3$) and let us suppose that $(Q_2,W_2)\simeq(Q_3,W_3)$ (so this implies that $(n_2,k_2)=(n_3,k_3)$). Then the (E,V)'s that belong to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, that have α_c-canonical filtration of type $(1,2)$ and graded $\oplus_{i=1}^3(Q_i,W_i)$ are parametrized by Grass$(2,H_2)$, where $H_2:=$ Ext$^1((Q_2,W_2),(Q_1,W_1))$.*

Proof. As in the previous proof, we get that for every (E,V) that sits in a sequence (10.1) with $(Q_1,W_1)\not\simeq(Q_i,W_i)$ for $i=2,3$, the following facts are equivalent

(a) (10.6) is the α_c-canonical filtration of (E,V);

(b) for all morphisms $\gamma_2:(Q_2,W_2)\to(E,V)$ we have that $\beta\circ\gamma=0$„

Since $(Q_2,W_2)\simeq(Q_3,W_3)$, we conclude by lemma 3.3.2 that this is equivalent to

(c) μ_2,μ_3 linearly independent in H_2.

Now if we look at the sequence (10.1) with $(Q_2,W_2)\simeq(Q_3,W_3)$, then we get that $\mathrm{Aut}(Q_1,W_1)=\mathbb{C}^*$, while $\mathrm{Aut}((Q_2,W_2)\oplus(Q_2,W_2))=\mathrm{GL}(2,\mathbb{C})$, so we conclude. □

Now we give a global parametrization of the objects described before, i.e. we describe families of schemes that parametrize various types of (E,V)'s when the graded $\oplus_{i=1}^3(Q_i,W_i)$ varies over $\prod_{i=1}^3 G_i$ and the α_c-canonical filtration is of type (1,2). If we fix a graded $\oplus_{i=1}^3(Q_i,W_i)$ and we suppose that $(Q_2,W_2)\not\simeq(Q_3,W_3)$ (and suitable numerical conditions are satisfied), then we know that the corresponding (E,V)'s with α_c-canonical filtration of type (1,2) that belong to $G^+(\alpha_c;n,d,k)$ or to $G^-(\alpha_c;n,d,k)$ are parametrized by pairs (μ_2,μ_3) with

$$\mu_2 \in \mathbb{P}\left(\mathrm{Ext}^1\left((Q_2,W_2),(Q_1,W_1)\right)\right) \quad \text{and} \quad \mu_3 \in \mathbb{P}\left(\mathrm{Ext}^1\left((Q_3,W_3),(Q_1,W_1)\right)\right).$$

Then we need to distinguish the following subcases:

(1) the type of (Q_2,W_2) is different from the type of (Q_3,W_3): in this case every (E,V) is in bijection with an *ordered* pair (μ_2,μ_3) (it suffices to decide which is the type of (Q_2,W_2));

(2) the type of (Q_2,W_2) is equal to the type of (Q_3,W_3) and $(Q_2,W_2) \not\simeq (Q_3,W_3)$: in this case every (E,V) is in bijection with an *unordered* pair (μ_2,μ_3), therefore the good schemes to look at will be a quotient under the action of $\mathbb{Z}_2$ of some schemes constructed as in (1).

Moreover, we will have to describe the case:

(3) $(Q_2,W_2) \simeq (Q_3,W_3)$ (so in particular $(n_2,k_2)=(n_3,k_3)$): in this case if $(Q_1,W_1),(Q_2,W_2)$ are fixed, then the corresponding $(E,V) \in G^+(\alpha_c;n,d,k)$, respectively in $G^-(\alpha_c;n,d,k)$, are parametrized by $\mathrm{Grass}(2,H_2)$, where

$$H_2 := \mathrm{Ext}^1((Q_2,W_2),(Q_1,W_1)).$$

The 3 cases are taken into account by propositions 7.1.1, 7.1.2 and 7.1.3 respectively. We give below the proofs of those 3 results.

Proof of proposition 7.1.1. First of all, we consider a set of data $\mathscr{D}^2_a$ given by:

- $r=2$, i.e. we are considering a tree with only 2 leaves and an internal node;
- the invariants (n_1,k_1) and (n_2,k_2) associated to the first leaf, respectively to the second leaf;
- any non-negative integer a such that there exists $((Q_1,W_1),(Q_2,W_2)) \in G_1 \times G_2$ with

$$\dim \mathrm{Ext}^1((Q_2,W_2),(Q_1,W_1)) = a.$$

The numerical conditions (10.3), respectively (10.4) prove that $k_1/n_1 \neq k_2/n_2$, so $(k_1,n_1) \neq (k_2,n_2)$. Therefore by lemma 1.0.4 for every pair of points $(Q_1,W_1) \in G_1$ and $(Q_2,W_2) \in G_2$ we have

$$\mathrm{Hom}((Q_2,W_2),(Q_1,W_1)) = 0.$$

Then by proposition 5.0.5 for $r=2$ we get the following objects:

- a finite set of indices L^2_a;

- a covering of $\hat{G}(\alpha_c; n_1, d_1, k_1) \times \hat{G}(\alpha_c; n_2, d_2, k_2) = \hat{G}_1 \times \hat{G}_2$ by integral locally closed subschemes $\hat{U}^2_{a;i}$ with $i \in L^2_a$; we denote by $\hat{p}^2_{a;i}$, $\hat{q}^2_{a;i}$ and $\hat{\pi}^2_{a;i}$ the various projections composed with the corresponding locally closed embeddings; so for example:

$$\hat{p}^2_{a;i} : \hat{U}^2_{a;i} \hookrightarrow \hat{G}_1 \times \hat{G}_2 \to \hat{G}_1;$$

- for every $i \in L^2_a$, a locally free sheaf on $\hat{U}^2_{a;i}$:

$$\hat{\mathcal{H}}^2_{a;i} := \mathcal{E}xt^1_{\hat{\pi}^2_{a;i}} \Big((\hat{q}^{2\prime}_{a;i}, \hat{q}^2_{a;i})^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2), (\hat{p}^{2\prime}_{a;i}, \hat{p}^2_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \Big)^\vee,$$

  where $(\hat{\mathcal{Q}}_i, \hat{\mathcal{W}}_i)$ is the local universal family parametrized by $\hat{G}_1$;

- projective fibrations for every $i \in L^2_a$:

$$\hat{\varphi}^2_{a;i} : \hat{R}^2_{a;i} := \mathbb{P}(\hat{\mathcal{H}}^2_{a;i}) \longrightarrow \hat{U}^2_{a;i}$$

  with fibers isomorphic to $\mathbb{P}^{a-1}$;

- extensions for every $i \in L^2_a$, parametrized by $\hat{R}^2_{a;i}$:

$$\begin{aligned} 0 &\to (\hat{\varphi}^{2\prime}_{a;i}, \hat{\varphi}^2_{a;i})^*(\hat{p}^{2\prime}_{a;i}, \hat{p}^2_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \otimes_{\hat{R}^2_{a;i}} \mathcal{O}^2_{a;i}(1) \to \\ &\to (\hat{\mathcal{E}}^2_{a;i}, \hat{\mathcal{V}}^2_{a;i}) \to (\hat{\varphi}^{2\prime}_{a;i}, \hat{\varphi}^2_{a;i})^*(\hat{q}^{2\prime}_{a;i}, \hat{q}^2_{a;i})^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2) \to 0 \end{aligned} \tag{10.7}$$

  that are universal in the sense of corollary 4.3.3. Here $\mathcal{O}^2_{a;i}(1)$ is the tautological bundle of $\mathbb{P}(\hat{\mathcal{H}}^2_{a;i}) = \hat{R}^2_{a;i}$.

Then we tensor this exact sequence by $\mathcal{O}^2_{a;i}(-1)$. By lemma 3.2.2 we get again a short exact sequence:

$$\begin{aligned} 0 &\to (\hat{\varphi}^{2\prime}_{a;i}, \hat{\varphi}^2_{a;i})^*(\hat{p}^{2\prime}_{a;i}, \hat{p}^2_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \to (\hat{\mathcal{E}}^2_{a;i}, \hat{\mathcal{V}}^2_{a;i}) \otimes_{\hat{R}^2_{a;i}} \mathcal{O}^2_{a;i}(-1) \to \\ &\to (\hat{\varphi}^{2\prime}_{a;i}, \hat{\varphi}^2_{a;i})^*(\hat{q}^{2\prime}_{a;i}, \hat{q}^2_{a;i})^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2) \otimes_{\hat{R}^2_{a;i}} \mathcal{O}^2_{a;i}(-1) \to 0. \end{aligned} \tag{10.8}$$

Analogously, let us fix a set of data $\mathscr{D}^3_b$ as follows:

- $r = 2$, i.e. we are again considering a tree with only 2 leaves and an internal node;
- the invariants (n_1, k_1) and (n_3, k_3) associated to the first leaf, respectively to the second leaf;
- any non-negative integer b such that there exists $((Q_1, W_1), (Q_3, W_3)) \in G_1 \times G_3$ with

$$\dim \operatorname{Ext}^1((Q_3, W_3), (Q_1, W_1)) = b.$$

Then by proposition 5.0.5 we get the following objects:

- a finite set of indices L^3_b;
- a covering of $\hat{G}_1 \times \hat{G}_3$ by integral locally closed subschemes $\hat{U}^3_{b;j}$ with $j \in L^3_b$; we denote by $\hat{p}^3_{b;j}$, $\hat{q}^3_{b;j}$ and $\hat{\pi}^3_{b;j}$ the various projections composed with the corresponding locally closed embeddings;
- for every $j \in L^3_b$, a locally free sheaf on $\hat{U}^3_{b;j}$:

$$\hat{\mathcal{H}}^3_{b;j} := \mathcal{E}xt^1_{\hat{\pi}^3_{b;j}}\Big((\hat{q}^{3'}_{b;j}, \hat{q}^3_{b;j})^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3), (\hat{p}^{3'}_{b;j}, \hat{p}^3_{b;j})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \Big)^{\vee};$$

- projective fibrations for every $j \in L^3_b$:

$$\hat{\varphi}^3_{b;j} : \hat{R}^3_{b;j} := \mathbb{P}(\hat{\mathcal{H}}^3_{b;j}) \longrightarrow \hat{U}^3_{b;j}$$

  with fibers isomorphic to $\mathbb{P}^{b-1}$;
- extensions for every $j \in L^3_b$, parametrized by $\hat{R}^3_{b;j}$:

$$\begin{aligned} 0 &\to (\hat{\varphi}^{3'}_{b;j}, \hat{\varphi}^3_{b;j})^*(\hat{p}^{3'}_{b;j}, \hat{p}^3_{b;j})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \otimes_{\hat{R}^3_{b;j}} \mathcal{O}^3_{b;j}(1) \to \\ &\to (\hat{\mathcal{E}}^3_{b;j}, \hat{\mathcal{V}}^3_{b;j}) \to (\hat{\varphi}^{3'}_{b;j}, \hat{\varphi}^3_{b;j})^*(\hat{q}^{3'}_{b;j}, \hat{q}^3_{b;j})^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3) \to 0 \end{aligned} \tag{10.9}$$

  that are universal in the sense of corollary 4.3.3.

Then by tensoring by $\mathcal{O}^3_{b;j}(-1)$ we get an exact sequence:

$$\begin{aligned} 0 &\to (\hat{\varphi}^{3'}_{b;j}, \hat{\varphi}^3_{b;j})^*(\hat{p}^{3'}_{b;j}, \hat{p}^3_{b;j})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \to (\hat{\mathcal{E}}^3_{b;j}, \hat{\mathcal{V}}^3_{b;j}) \otimes_{\hat{R}^3_{b;j}} \mathcal{O}^3_{b;j}(-1) \to \\ &\to (\hat{\varphi}^{3'}_{b;j}, \hat{\varphi}^3_{b;j})^*(\hat{q}^{3'}_{b;j}, \hat{q}^3_{b;j})^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3) \otimes_{\hat{R}^3_{b;j}} \mathcal{O}^3_{b;j}(-1) \to 0. \end{aligned} \tag{10.10}$$

Now we let us fix any $(a,b;i,j)$ and let us consider the following cartesian diagram constructed in several steps, starting from (a):

$$\begin{tikzcd}
\hat{R}_{a,b;i;j} \arrow[r, "\hat{\theta}^3_{b;j}"] \arrow[d, "\hat{\theta}^2_{a;i}"'] \arrow[dr, phantom, "\square(d)"] & \hat{A}_{a,b;i,j} \arrow[r, "\hat{s}^3_{b;j}"] \arrow[d, "\hat{\psi}^2_{a;i}"] \arrow[dr, phantom, "\square(b)"] & \hat{R}^2_{a;i} \arrow[d, "\hat{\varphi}^2_{a;i}"] \\
\hat{R}'^3_{b;j} \arrow[r, "\hat{\psi}^3_{b;j}"] \arrow[d, "\hat{s}^2_{a;i}"'] \arrow[dr, phantom, "\square(c)"] & \hat{U}_{a,b;i,j} \arrow[r, "\hat{r}^3_{b;j}"] \arrow[d, "\hat{r}^2_{a;i}"] \arrow[dr, phantom, "\square(a)"] & \hat{U}^2_{a;i} \arrow[d, "\hat{p}^2_{a;i}"] \\
\hat{R}^3_{b;j} \arrow[r, "\hat{\varphi}^3_{b;j}"'] & \hat{U}^3_{b;j} \arrow[r, "\hat{p}^3_{b;j}"'] & \hat{G}_1.
\end{tikzcd} \tag{10.11}$$

By the commutativity of this diagram we get canonical isomorphisms:

$$\begin{aligned}
&(\hat{\theta}^{3'}_{b;j}, \hat{\theta}^{3}_{b;j})^*(\hat{s}^{3'}_{b;j}, \hat{s}^{3}_{b;j})^*(\hat{\varphi}^{2'}_{a;i}, \hat{\varphi}^{2}_{a;i})^*(\hat{p}^{2'}_{a;i}, \hat{p}^{2}_{a;i})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \simeq \\
&\simeq (\hat{\theta}^{2'}_{a;i}, \hat{\theta}^{2}_{a;i})^*(\hat{s}^{2'}_{a;i}, \hat{s}^{2}_{a;i})^*(\hat{\varphi}^{3'}_{b;j}, \hat{\varphi}^{3}_{b;j})^*(\hat{p}^{3'}_{b;j}, \hat{p}^{3}_{b;j})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \simeq \\
&\simeq (\hat{p}^{2'}_{a;i} \circ \hat{\varphi}^{2'}_{a;i} \circ \hat{s}^{3'}_{b;j} \circ \hat{\theta}^{3'}_{b;j}, \hat{p}^{2}_{a;i} \circ \hat{\varphi}^{2}_{a;i} \circ \hat{s}^{3}_{b;j} \circ \hat{\theta}^{3}_{b;j})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1),
\end{aligned} \tag{10.12}$$

so we will identity these families and we will write $(\overline{\mathcal{Q}}_1, \overline{\mathcal{W}}_1)$ for any of them.

By pullback from $\hat{R}^2_{a;i}$ and $\hat{R}^3_{b;j}$ (see lemma 3.2.1), the sequences (10.8) and (10.10) give rise to 2 short exact sequences of coherent systems parametrized by $\hat{R}_{a,b;i,j}$:

$$0 \to (\overline{\mathcal{Q}}_1, \overline{\mathcal{W}}_1) \to (\overline{\mathcal{E}}^2_{a;i}, \overline{\mathcal{V}}^2_{a;i}) \to (\overline{\mathcal{Q}}_2, \overline{\mathcal{W}}_2) \to 0, \tag{10.13}$$

$$0 \to (\overline{\mathcal{Q}}_1, \overline{\mathcal{W}}_1) \to (\overline{\mathcal{E}}^3_{b;j}, \overline{\mathcal{V}}^3_{b;j}) \to (\overline{\mathcal{Q}}_3, \overline{\mathcal{W}}_3) \to 0, \tag{10.14}$$

where for simplicity we use the following notation:

$$\begin{aligned}
(\overline{\mathcal{E}}^2_{a;i}, \overline{\mathcal{V}}^2_{a;i}) &:= \left(\hat{s}^{3'}_{b;j} \circ \hat{\theta}^{3'}_{b;j}, \hat{s}^{3}_{b;j} \circ \hat{\theta}^{3}_{b;j}\right)^* (\hat{\mathcal{E}}^2_{a;i}, \hat{\mathcal{V}}^2_{a;i}) \otimes_{\hat{R}_{a,b;i,j}} \left(\hat{s}^{3}_{b;j} \circ \hat{\theta}^{3}_{b;j}\right)^* \mathcal{O}^2_{a;i}(-1), \\
(\overline{\mathcal{Q}}_2, \overline{\mathcal{W}}_2) &:= \left(\hat{s}^{3'}_{b;j} \circ \hat{\theta}^{3'}_{b;j}, \hat{s}^{3}_{b;j} \circ \hat{\theta}^{3}_{b;j}\right)^* \left((\hat{\varphi}^{2'}_{a;i}, \hat{\varphi}^{2}_{a;i})^*(\hat{q}^{2'}_{a;i}, \hat{q}^{2}_{a;i})^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2) \otimes_{\hat{R}^2_{a;i}} \mathcal{O}^2_{a;i}(-1)\right), \\
(\overline{\mathcal{E}}^3_{b;j}, \overline{\mathcal{V}}^3_{b;j}) &:= \left(\hat{s}^{2'}_{a;i} \circ \hat{\theta}^{2'}_{a;i}, \hat{s}^{2}_{a;i} \circ \hat{\theta}^{2}_{a;i}\right)^* (\hat{\mathcal{E}}^3_{b;j}, \hat{\mathcal{V}}^3_{b;j}) \otimes_{\hat{R}_{a,b;i,j}} \left(\hat{s}^{2}_{a;i} \circ \hat{\theta}^{2}_{a;i}\right)^* \mathcal{O}^3_{b;j}(-1), \\
(\overline{\mathcal{Q}}_3, \overline{\mathcal{W}}_3) &:= \left(\hat{s}^{2'}_{a;i} \circ \hat{\theta}^{2'}_{a;i}, \hat{s}^{2}_{a;i} \circ \hat{\theta}^{2}_{a;i}\right)^* \left((\hat{\varphi}^{3'}_{b;j}, \hat{\varphi}^{3}_{b;j})^*(\hat{q}^{3'}_{b;j}, \hat{q}^{3}_{b;j})^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3) \otimes_{\hat{R}^3_{b;j}} \mathcal{O}^3_{b;j}(-1)\right).
\end{aligned}$$

Now we sum the sequences (10.13) and (10.14) in order to get an extension of the form:

$$0 \to (\overline{\mathcal{Q}}_1, \overline{\mathcal{W}}_1) \to (\hat{\mathcal{E}}_{a,b;i,j}, \hat{\mathcal{V}}_{a,b;i,j}) \to (\overline{\mathcal{Q}}_2, \overline{\mathcal{W}}_2) \oplus (\overline{\mathcal{Q}}_3, \overline{\mathcal{W}}_3) \to 0. \tag{10.15}$$

Let us fix any point (t_1, t_2, t_3) in $\hat{U}_{a,b;i,j} \subset \hat{G}_1 \times \hat{G}_2 \times \hat{G}_3$ and let r, r' be two points in $\hat{R}_{a,b;i,j}$ over that point. For $i = 1, 2, 3$ let us denote by (Q_i, W_i) the image of t_i in the moduli space G_i. Then we can interpret r as a pair $([\alpha], [\beta])$ where α is the class of a non-split extension of (Q_2, W_2) by (Q_1, W_1) and β is the class of a non-split extension of (Q_3, W_3) by (Q_1, W_1); analogously we can interpret r' as pair $([\alpha'], [\beta'])$. If $r \neq r'$, this means that $([\alpha], [\beta]) \neq ([\alpha'], [\beta'])$. Now by construction of $(\hat{\mathcal{E}}_{a,b;i,j}, \hat{\mathcal{V}}_{a,b;i,j})$, if we restrict (10.15) to $\{r\} \times C$ we get an extension

$$0 \to (Q_1, W_1) \to (E, V) \to (Q_2, W_2) \oplus (Q_3, W_3) \to 0 \tag{10.16}$$

associated to the pair $([\alpha], [\beta])$ and analogously for r'. Therefore, for every pair of different points of $\hat{R}_{a,b;i,j}$ in the same fiber over $\hat{U}_{a,b;i,j}$, the sequence (10.15) restricts to different pairs of classes of extensions of a point of $\hat{G}_2 \times \hat{G}_3$ by a point of $\hat{G}_1$.

Now let us assume that conditions (10.3) are satisfied and let us fix any point $r \in \hat{R}_{a,b;i,j}$. Then lemma 10.1.1 proves that the corresponding coherent system $(E,V) := (\hat{\mathcal{E}}_{a,b;i,j}, \hat{\mathcal{V}}_{a,b;i,j})|_r$ in (10.16) has α_c-canonical filtration of type $(1,2)$ and it is α_c^+-stable. Using the universal property of the scheme $G(\alpha_c^+;n,d,k)$, we get an induced morphism

$$\hat{\omega}_{a,b;i,j} : \hat{R}_{a,b;i,j} \to G(\alpha_c^+;n,d,k).$$

Now there is a free action of $PGL(N_1) \times PGL(N_2) \times PGL(N_3)$ on $\hat{R}_{a,b;i,j}$, $\hat{A}_{a,b;i,j}$ and on $\hat{U}_{a,b;i,j}$. This gives rise to geometric quotients

$$\hat{R}_{a,b;i,j} \twoheadrightarrow R_{a,b;i,j}, \quad \hat{A}_{a,b;i,j} \twoheadrightarrow A_{a,b;i,j}, \quad \hat{U}_{a,b;i,j} \twoheadrightarrow U_{a,b;i,j}$$

and to induced fibrations

$$\phi_1 : R_{a,b;i,j} \to A_{a,b,i,j}, \quad \phi_2 : A_{a,b;i,j} \to U_{a,b;i,j}.$$

Now the morphism $\hat{\omega}_{a,b;i,j}$ is invariant under the action of $PGL(N_1) \times PGL(N_2) \times PGL(N_3)$, so it induces a morphism

$$\omega_{a,b;i,j} : R_{a,b;i,j} \to G(\alpha_c^+;n,d,k).$$

The previous remark proves that such a morphism is injective. Moreover, for every point (E,V) with properties (i)-(ii) there is exactly one pair (a,b) such that (E,V) is in the image of some $\omega_{a,b;i,j}$. Then all the other properties stated in the claim of the proposition are simple consequences of the proof of proposition 5.0.5.

If we assume conditions (10.4), we conclude in a similar way. □

Proof of proposition 7.1.2. There is almost nothing to prove in cases (a) and (b), since in those cases there is no action of $\mathbb{Z}_2$ and the induced morphisms are already injective. In cases (c) and (d) it is clear that there is an action of $\mathbb{Z}_2$ and that the induced morphisms are injective only after passing to the quotient with respect to that action. The only claim that is not a priori obvious is the existence of the local trivialization compatible with the action of $\mathbb{Z}_2$. We will prove it for the case (d) and assuming for simplicity that the base space $\hat{U}_{a,a;i,i}$ described in the proof of the previous proposition coincides with $\hat{G}_1 \times \hat{G}_2 \times \hat{G}_3$ (on that space we will actually only be interested in the part outside the diagonal $\hat{\Delta}_{23}$, since that case will have to be considered in the next proposition). In particular, we are assuming that there is only a significant sequence $(a,b;i,j) = (a,a;i,i)$. When the base space is smaller, we will have more indices to consider (and we will have to restrict the base spaces and the top spaces according to that), but the idea will be exactly the same.

Here even if $\hat{G}_2 = \hat{G}_3$, we will use both notations since the 2 schemes will play different roles in the product $\hat{G}_1 \times \hat{G}_2 \times \hat{G}_3$. If we write by $\hat{X}^2 := \hat{R}^2_{a;i}$ and $\hat{X}^3 := \hat{R}^3_{a;i}$ (these schemes are equal, but the previous remark applies), then we can write diagram (10.11) as follows:

$$
\begin{tikzcd}
\hat{R}_{a,a;i,i} \arrow[r, "\hat{\theta}^3"] \arrow[d, "\hat{\theta}^2"'] \arrow[dr, phantom, "\square(d)"] & \hat{X}^2 \times \hat{G}_3 \arrow[r] \arrow[d, "\hat{\psi}^2"] \arrow[dr, phantom, "\square(b)"] & \hat{X}^2 \arrow[d, "\hat{\varphi}^2"] \\
\hat{X}^3 \times \hat{G}_2 \arrow[r, "\hat{\psi}^3"] \arrow[d] \arrow[dr, phantom, "\square(c)"] & \hat{G}_1 \times \hat{G}_2 \times \hat{G}_3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square(a)"] & \hat{G}_1 \times \hat{G}_2 \arrow[d] \\
\hat{X}^3 \arrow[r, "\hat{\varphi}^3"'] & \hat{G}_1 \times \hat{G}_3 \arrow[r] & \hat{G}_1.
\end{tikzcd}
\tag{10.17}
$$

Here $\hat{\varphi}^2$ is a locally trivial fibration with fibers isomorphic to $\mathbb{P}^{a-1}$, so there exists an open covering $\{\hat{U}^2_\alpha\}_{\alpha \in A}$ of $\hat{G}_1 \times \hat{G}_2$ and trivializations

$$\hat{U}^2_\alpha \times \mathbb{P}^{a-1} \xrightarrow{\sim} (\hat{\varphi}_2)^{-1}(\hat{U}^2_\alpha). \tag{10.18}$$

Now by identifying $\hat{G}_2$ with $\hat{G}_3$ and $\hat{\varphi}^2$ with $\hat{\varphi}^3$, we can use the same covering in order to trivialize $\hat{\varphi}^3$. We denote by $\{\hat{U}^3_\beta\}_{\beta \in A}$ that covering (here β varies over the same set of the $\hat{U}^2_\alpha$'s, so we use A to denote also that set of indices). So we get trivializations:

$$\hat{U}^3_\beta \times \mathbb{P}^{a-1} \xrightarrow{\sim} (\hat{\varphi}^3)^{-1}(\hat{U}^3_\beta). \tag{10.19}$$

Then for every $\alpha, \beta \in A$ we consider

$$\hat{U}_{\alpha,\beta} := (\hat{U}^2_\alpha \times \hat{G}_3) \cap (\hat{U}^3_\beta \times \hat{G}_2) \subset \hat{G}_1 \times \hat{G}_2 \times \hat{G}_3.$$

This gives an open covering of $\hat{G}_1 \times \hat{G}_2 \times \hat{G}_3$; since $\hat{\psi}^2 = \hat{\varphi}^2 \times \mathrm{id}_{\hat{G}_3}$ and analogously for $\hat{\psi}^3$, then (10.18) and (10.19) induce trivializations of $\hat{\psi}^2$ and $\hat{\psi}^3$:

$$\lambda^2_\alpha : \hat{U}^2_\alpha \times \hat{G}_3 \times \mathbb{P}^{a-1} \xrightarrow{\sim} (\hat{\psi}^2)^{-1}(\hat{U}^2_\alpha \times \hat{G}_3) \subset \hat{X}^2 \times \hat{G}_3$$

$$\lambda^3_\beta : \hat{U}^3_\beta \times \hat{G}_2 \times \mathbb{P}^{a-1} \xrightarrow{\sim} (\hat{\psi}^3)^{-1}(\hat{U}_\beta \times \hat{G}_2) \subset \hat{X}^3 \times \hat{G}_2.$$

Then we consider the following diagram:

$$
\begin{tikzcd}
(\hat{U}^2_\alpha \times \hat{G}_3 \times \mathbb{P}^{a-1})_{pr_1} \times_{pr_2} (\hat{U}^3_\beta \times \hat{G}_2 \times \mathbb{P}^{e-1}) \arrow[rrr] \arrow[dddd] \arrow[dr, dashed, "\gamma_{\alpha,\beta}"] & & \arrow[r, phantom, "\curvearrowright"] & \hat{U}^2_\alpha \times \hat{G}_3 \times \mathbb{P}^{a-1} \arrow[d, "\wr\ \lambda^2_\alpha"] \arrow[ddd, bend left, "pr^2"] \\
& (\hat{\psi}^2)^{-1}(\hat{U}^2_\alpha \times \hat{G}_3) \times_{\hat{G}_1 \times \hat{G}_2 \times \hat{G}_3} (\hat{\psi}^3)^{-1}(\hat{U}^3_\beta \times \hat{G}_2) \arrow[rr] \arrow[ddd] \arrow[ddr, dashed, "\phi_{\alpha,\beta}"] \arrow[ddl, phantom, "\curvearrowright"] \arrow[dr, phantom, "\curvearrowright"] & & (\hat{\psi}^2)^{-1}(\hat{U}^2_\alpha \times \hat{G}_3) \arrow[d, hook, "i^2"] \arrow[r, phantom, "\curvearrowright"] & {} \\
& & & \\
& \arrow[r, phantom, "\curvearrowright"] & \hat{H} \arrow[r, "\hat{\theta}^3"] \arrow[d, "\hat{\theta}^2"] \arrow[dr, phantom, "\square"] & \hat{X}^2 \times \hat{G}_3 \arrow[d, "\hat{\psi}^2"] \\
\hat{U}^3_\beta \times \hat{G}_2 \times \mathbb{P}^{a-1} \arrow[r, "\lambda^3_\beta", "\sim"'] \arrow[rrr, bend right, "pr^3"', "\curvearrowright"] & (\hat{\psi}^3)^{-1}(\hat{U}^3_\beta \times \hat{G}_2) \arrow[r, hook, "i^3"] & \hat{X}^3 \times \hat{G}_2 \arrow[r, "\hat{\psi}^3"] & \hat{G}_1 \times \hat{G}_2 \times \hat{G}_3.
\end{tikzcd}
$$

Here we have that $\hat{\psi}^2 \circ i^2 \circ \lambda^2_\alpha = pr^2$, where pr^2 is the composition

$$pr^2 : \hat{U}^2_\alpha \times \hat{G}_3 \times \mathbb{P}^{a-1} \to \hat{U}^2_\alpha \times \hat{G}_3 \hookrightarrow \hat{G}_1 \times \hat{G}_2 \times \hat{G}_3,$$

because λ^2_α is a trivialization of $\hat{\psi}^2$, and analogously for pr^3. Both $\phi_{\alpha,\beta}$ and $\gamma_{\alpha,\beta}$ are induced by the universal properties of fiber products. Since both λ^2_α and λ^3_β are isomorphisms, using the universal properties of fiber products we get that also $\gamma_{\alpha,\beta}$ is an isomorphism. Moreover, since the sets

$$\{(\hat{\psi}^2)^{-1}(\hat{U}^2_\alpha \times \hat{G}_3)\}_{\alpha \in A} \quad \text{and} \quad \{(\hat{\psi}^3)^{-1}(\hat{U}^3_\beta \times \hat{G}_2)\}_{\beta \in A}$$

are open coverings of $\hat{X}^2 \times \hat{G}_3$ and $\hat{X}^3 \times \hat{G}_2$ respectively, then the sets

$$\widetilde{U}_{\alpha,\beta} := (\hat{\psi}^2)^{-1}(\hat{U}^2_\alpha \times \hat{G}_3) \times_{\hat{G}_1 \times \hat{G}_2 \times \hat{G}_3} (\hat{\psi}^3)^{-1}(\hat{U}^3_\beta \times \hat{G}_2)$$

for $\alpha, \beta \in A$ form an open covering of $\hat{R}$ and the morphisms $\phi_{\alpha,\beta}$ are open embeddings. Moreover, by construction of pr^2 and pr^3, we get isomorphisms

$$\widetilde{U}_{\alpha,\beta} \xrightarrow{(\gamma_{\alpha,\beta})^{-1}} (\hat{U}^2_\alpha \times \hat{G}_3 \times \mathbb{P}^{a-1})_{pr^2} \times_{pr^3} (\hat{U}^3_\beta \times \hat{G}_2 \times \mathbb{P}^{a-1}) \simeq \hat{U}_{\alpha,\beta} \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}.$$

Moreover, using the commutativity of the previous diagram, we get that such isomorphisms are compatible with the fibration $\hat{\psi}^2 \circ \hat{\theta}^3$, therefore the previous isomorphisms are local trivializations of such fibrations (a priori we only knew that we had trivializations, here we have an explicit description of a possible choice of trivializations, that will be useful immediately).

Now let us consider $\hat{G}_1 \times \hat{\Delta}_{23}$: this is a locally closed subscheme of $\hat{G}_1 \times \hat{G}_2 \times \hat{G}_3$ and we can define

$$\hat{M} := (\hat{\psi}^2 \circ \hat{\theta}^3)^{-1}(\hat{G}_1 \times (\hat{G}_2 \times \hat{G}_3 \smallsetminus \hat{\Delta}_{23})) = \hat{R}|_{\hat{G}_1 \times (\hat{G}_2 \times \hat{G}_3 \smallsetminus \Delta_{23})};$$

since $\hat{\psi}^2 \circ \hat{\theta}^3$ is a locally trivial fibration, we get that $\hat{M}$ is a locally closed subscheme of $\hat{R}$ and it has an open covering by subschemes of the form:

$$\widetilde{V}_{\alpha,\beta} := \widetilde{U}_{\alpha,\beta} \cap \hat{M}.$$

Moreover, the previous isomorphisms restrict to isomorphisms

$$\widetilde{V}_{\alpha,\beta} \xrightarrow{\sim} \hat{V}_{\alpha,\beta} \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$$

where $\hat{V}_{\alpha,\beta} := \hat{U}_{\alpha,\beta} \smallsetminus (\hat{G}_1 \times \Delta_{23})$ (these sets are locally closed in $\hat{G}_1 \times \hat{G}_2 \times \hat{G}_3$).

Now let us consider the action of $\mathbb{Z}_2$ on $\hat{G}_1 \times \hat{G}_2 \times \hat{G}_3$ given by $(x, y, y') \mapsto (x, y', y)$. If we denote by ε the non-trivial element of $\mathbb{Z}_2$, then $\varepsilon(\hat{U}_{\alpha,\beta}) = \hat{U}_{\beta,\alpha}$ and $\varepsilon(\hat{V}_{\alpha,\beta}) = \hat{V}_{\beta,\alpha}$, so we get that:

- $\varepsilon(\hat{V}_{\alpha,\alpha}) = \hat{V}_{\alpha,\alpha}$;
- if $\alpha \neq \beta$ and we set $\hat{W}_{\alpha,\beta} := \hat{V}_{\alpha,\beta} \cap \hat{V}_{\beta,\alpha}$, then $\varepsilon(\hat{W}_{\alpha,\beta}) = \hat{W}_{\alpha,\beta}$;
- if $\alpha \neq \beta$ and we set $\hat{Z}_{\alpha,\beta} := (\hat{V}_{\alpha,\beta} \cup \hat{V}_{\beta,\alpha}) \smallsetminus \hat{W}_{\alpha,\beta}$, then $\varepsilon(\hat{Z}_{\alpha,\beta}) = \hat{Z}_{\alpha,\beta}$.

Then the set:

$$\{\hat{T}_l\}_{l \in L} := \Big\{ \{\hat{V}_{\alpha,\alpha}\}_{\alpha \in A}, \quad \{\hat{W}_{\alpha,\beta}\}_{\alpha < \beta}, \quad \{\hat{Z}_{\alpha,\beta}\}_{\alpha < \beta} \Big\}$$

is a finite locally closed disjoint covering of $\hat{G}_1 \times (\hat{G}_2 \times \hat{G}_3 \smallsetminus \Delta_{23})$ and each of such subschemes is invariant under the action of $\mathbb{Z}_2$. Moreover, by restricting to any such subscheme we have a trivialization of the fibration $\hat{\psi}^2 \circ \hat{\theta}^3$ and that trivialization is compatible with the action of $\mathbb{Z}_2$ on $\hat{T}_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$. Finally, we consider the action of $PGL(N_1) \times PGL(N_2) \times PGL(N_3)$ on all these schemes and we conclude. □

Proof of proposition 7.1.3. Let us denote by $\hat{p}_{12}$ and $\hat{q}_{12}$ the projections from $\hat{G}_1 \times \hat{G}_2$ to its factors. Conditions (10.3), respectively (10.4), prove that $(k_1, n_1) \neq (k_2, n_2) = (k_3, n_3)$. Therefore, for all $t \in \hat{G}_1 \times \hat{G}_2$ we have that

$$\mathrm{Hom}((\hat{q}'_{12}, \hat{q}_{12})^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2)_t, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = 0.$$

Then we can apply proposition 4.6.3 and corollary 4.5.6 for $t = 2$ in order to get the following objects:

- for all $a \in \mathbb{N}$ such that $U_a \neq \varnothing$, a finite locally closed covering $\{\hat{U}_{a;i}\}_i$ of

$$\hat{U}_a := \{t \in \hat{G}_1 \times \hat{G}_2 \text{ s.t. } \dim \operatorname{Ext}^1((\hat{q}'_{12}, \hat{q}_{12})^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2)_t, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = a\}.$$

- for all i a locally free sheaf

$$\hat{\mathcal{H}}_{a;i} := \mathcal{E}xt^1_{\pi_{\hat{U}_{a;i}}}(((\hat{q}'_{12}, \hat{q}_{12})^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2)|_{\hat{U}_{a;i}}, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)|_{\hat{U}_{a;i}})^{\vee}$$

  over $\hat{U}_{a;i}$ and a grassmannian bundle

$$\hat{\theta}_{2;a;i} : \hat{R}_{2;a;i} := Grass(2, \hat{\mathcal{H}}_{a;i}) \longrightarrow \hat{U}_{a;i}$$

- a locally free sheaf $\overline{\mathcal{M}}_{2;a;i}$ of rank 2 over $\hat{R}_{2;a;i}$ and a universal family of non-degenerate extensions on the right of rank 2:

$$0 \to (\hat{\theta}'_{2;a;i}, \hat{\theta}_{2;a;i})^*(\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \to (\hat{\mathcal{E}}_{\hat{R}_{2;a;i}}, \hat{\mathcal{V}}_{\hat{R}_{2;a;i}}) \to$$
$$\to (\hat{\theta}'_{2;a;i}, \hat{\theta}_{2;a;i})^*(\hat{q}'_{12}, \hat{q}_{12})^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2) \otimes_{\hat{R}_{2;a;i}} \overline{\mathcal{M}}_{2;a;i} \to 0.$$

Now if we assume conditions (10.3), respectively (10.4), then lemma 10.1.2 proves that for each point r in $\hat{R}_{2;a;i}$ the coherent system

$$(E, V) := (\hat{\mathcal{E}}_{\hat{R}_{2;a;i}}, \hat{\mathcal{V}}_{\hat{R}_{2;a;i}})_r$$

is α_c^+-stable, respectively α_c^--stable. Therefore for each pair a, i we get an induced morphism from $\hat{R}_{2;a;i}$ to $G(\alpha_c^+; n, d, k)$, respectively to $G(\alpha_c^-; n, d, k)$. The rest of the proof follows the usual pattern. □

10.2 Canonical filtration of type (2,1)

We want to parametrize all those (E, V)'s that belong to $G^+(\alpha_c; n, d, k)$ or to $G^-(\alpha_c; n, d, k)$ and that have α_c-canonical filtration of type (2,1). Let us fix 3 α_c-stable coherent systems $(Q_i, W_i)_{i=1,2,3}$ with the same α_c-slope μ; then every (E, V) with graded $\oplus_{i=1}^3 (Q_i, W_I)$ and with α_c-canonical filtration of type (2,1) sits in an exact sequence of the form:

$$0 \to (Q_1, W_1) \oplus (Q_2, W_2) \xrightarrow{\alpha} (E, V) \xrightarrow{\beta} (Q_3, W_3) \to 0. \tag{10.20}$$

If (E, V) has canonical filtration of type $(2,1)$, then it has always the following proper α_c-semistable subobjects with α_c-slope μ:

- (Q_i, W_i) for $i = 1, 2$;
- $(Q_1, W_1) \oplus (Q_2, W_2)$.

This in general is not a complete list, see lemma 10.2.1. If we look only at the subobjects (Q_i, W_i) for $i = 1, 2$, then we have that for every (E, V) with α_c-canonical filtration of type (2,1), the following numerical conditions are necessary, but in general not sufficient, in order to have that (E, V) belongs to $G^+(\alpha_c; n, d, k)$:

$$\frac{k_i}{n_i} < \frac{k}{n} \quad \forall\, i = 1, 2. \tag{10.21}$$

A direct check proves that this implies $\frac{k_1+k_2}{n_1+n_2} < \frac{k}{n}$, so the subobject $(Q_1, W_1) \oplus (Q_2, W_2)$ does not destabilize (E, V) for α_c^+. Analogously, the following numerical conditions are necessary, but in general not sufficient, in order to have that (E, V) belongs to $G^-(\alpha_c; n, d, k)$:

$$\frac{k_i}{n_i} > \frac{k}{n} \quad \forall\, i = 1, 2. \tag{10.22}$$

Lemma 10.2.1. *Given any sequence (10.20) with conditions (10.21), respectively (10.22), then (E, V) belongs to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, if and only if there are no quotients $\zeta_i : (E, V) \twoheadrightarrow (Q_i, W_i)$ for $i = 1, 2$. Moreover, if this happens, then (E, V) has α_c-canonical filtration of type $(2, 1)$.*

Proof. Let us suppose that we use conditions (10.21), the other case is completely analogous. If there is any quotient ζ_i as in the claim, then the kernel (E', V') of ζ_i is an α_c-semistable subsystem of (E, V) with $k' = k - k_i$ and $n' = n - n_i$. Since $\mu_{\alpha_c}(E, V) = \mu_{\alpha_c}(E', V')$, using (10.21) we get that $\frac{k'}{n'} > \frac{k}{n}$, so (E, V) cannot be α_c^+-stable.

Conversely, if (E, V) is not α_c^+-stable, then there exists a subsystem (E', V') that destabilizes it for α_c^+. Using (10.21), the graded of (E', V') cannot contain only some (possibly all) objects of the form (Q_i, W_i) for $i \in \{1, 2\}$, so it contains (Q_3, W_3). Therefore the quotient $(E'', V'') := (E, V)/(E', V')$ contains only some (possibly all) objects of the form (Q_i, W_i) for $i \in \{1, 2\}$. If we consider any α_c-Jordan-Hölder filtration $(E''_l, V''_l)_{l=1,\cdots,t}$ of (E'', V''), we get that $(E'', V''/(E''_{t-1}, V''_{t-1})$ is isomorphic to some (Q_i, W_i) for $i \in \{1, 2\}$, so we get a quotient $(E, V) \twoheadrightarrow (E, V)/(E', V') = (E'', V'') \twoheadrightarrow (Q_i, W_i)$. So this proves the first part of the claim.

Now let us assume that there are no quotients $\zeta_i : (E, V) \twoheadrightarrow (Q_i, W_i)$ for $i = 1, 2$; we want to prove that the α_c-canonical filtration of (E, V) is of type $(2, 1)$. So let us consider the filtration of (E, V) given as follows:

$$0 = (E_0, V_0) \subset (E_1, V_1) := (Q_1, W_1) \oplus (Q_2, W_2) \subset (E_2, V_2) = (E, V). \tag{10.23}$$

Here $(E_2, V_2)/(E_1, V_1) = (Q_3, W_3)$ is α_c-stable and $(E_1, V_1)/(E_0, V_0) = (E_1, V_1)$ is α_c-polystable. Then by proposition 2.1.3 we get that (10.23) is the α_c-canonical filtration of (E, V) (and so (E, V) has α_c-canonical filtration of type $(2, 1)$) if and only if condition (c) of that proposition is satisfied. In our case the index t is equal to 2, so (10.23) is the α_c-canonical filtration of (E, V) if and only if for all $i = 1, 2, 3$ and for all non-zero morphisms $\gamma_i : (Q_i, W_i) \to (E, V)$ we have $\beta \circ \gamma_i = 0$. Now conditions (10.21) imply that $(Q_3, W_3) \not\simeq (Q_i, W_i)$ for all $i = 1, 2$. Since all the (Q_i, W_i)'s for $i = 1, 2, 3$ are α_c-stable of the same slope, then for all $i=1,2$ and for all $\gamma_i:(Q_i,W_i)\to(E,V)$ we have that $\beta\circ\gamma_i=0$. Then we conclude that for every (E,V) as in (10.20) the following conditions are equivalent:

(a) (10.23) is the α_c-canonical filtration of (E,V);

(b) for all non-zero morphisms $\gamma_3:(Q_3,W_3)\to(E,V)$ we have $\beta\circ\gamma_3=0$.

Now (b) is equivalent to saying that the sequence (10.20) is non-split, so in order to conclude the proof we need to verify that (10.20) is non-split if we assume the conditions of first part of the claim of lemma 10.2.1. If (10.20) is split, then for every $i=1,2$ we can write morphisms of the form

$$\zeta_i:\,(E,V)\simeq\oplus_{l=1}^3(Q_l,W_l)\twoheadrightarrow(Q_i,W_i).$$

Such a morphism cannot exist in the case under consideration, so we conclude. □

Now if we denote by μ any extension like (10.20), we get that we can identify μ with a pair

$$(\mu_1,\mu_2)\in\bigoplus_{i=1}^2\mathrm{Ext}^1\Big((Q_3,W_3),(Q_i,W_i)\Big).$$

This identification gives a diagram of the following form for $i=1,2$:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{l=1}^2(Q_l,W_l) \arrow[r, "\alpha"] \arrow[d, "pr_i"'] \arrow[dr, phantom, "\curvearrowright"] & (E,V) \arrow[r, "\beta"] \arrow[d, "\eta_i"'] \arrow[dr, phantom, "\curvearrowright"] & (Q_3,W_3) \arrow[r] \arrow[d, equal] & 0 & \mu \arrow[d, "\overline{pr_i}"] \\
0 \arrow[r] & (Q_i,W_i) \arrow[r, "\alpha_i"] & (E_{3i},V_{3i}) \arrow[r, "\beta_i"] & (Q_3,W_3) \arrow[r] & 0 & \mu_i,
\end{tikzcd} \tag{10.24}$$

where pr_i is the quotient $(Q_1,W_1)\oplus(Q_2,W_2)\twoheadrightarrow(Q_i,W_i)$. Then we have the following results.

Lemma 10.2.2. *Let us fix any triple $(Q_i,W_i)_{i=1,2,3}\in\prod_{i=1}^3 G_i$ with numerical conditions (10.21), respectively (10.22) (this automatically implies that $(n_3,k_3)\neq(n_i,k_i)$ for $i=1,2$) and such that $(Q_1,W_1)\not\simeq(Q_2,W_2)$. Then the (E,V)'s that belong to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, that have graded $\oplus_{i=1}^3(Q_i,W_i)$ and α_c-canonical filtration of type $(2,1)$ are parametrized by $\mathbb{P}(H_1)\times\mathbb{P}(H_2)$, where $H_i:=Ext^1\left((Q_3,W_3),(Q_i,W_i)\right)$ for $i=1,2$.*

Proof. Let us fix any extension $\mu=(\mu_1,\mu_2)$ represented by a sequence of the form (10.20). Using lemma 10.2.1, the following facts are equivalent

(a) (E,V) belongs to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, and it has α_c-canonical filtration of type $(2,1)$;

(b) there are no quotients $\zeta_i:\,(E,V)\twoheadrightarrow(Q_i,W_i)$ for $i=1,2$.

Since $(Q_3, W_3) \not\simeq (Q_i, W_i)$ for $i = 1, 2$, then this is also equivalent to

(c) for $i = 1, 2$ there are no quotients $\zeta_i : (E, V) \twoheadrightarrow (Q_i, W_i)$ such that $\zeta_i \circ \alpha \neq 0$.

Now since $(Q_1, W_1) \not\simeq (Q_2, W_2)$, then by lemma 3.3.1 we get that this is also equivalent to

(d) $\mu_1 \neq 0 \neq \mu_2$.

Now if we look at the sequence (10.20), we get that $\mathrm{Aut}((Q_1, W_1) \oplus (Q_2, W_2)) = \mathbb{C}^* \times \mathbb{C}^*$ (because $(Q_1, W_1) \not\simeq (Q_2, W_2)$) and $\mathrm{Aut}(Q_3, W_3) = \mathbb{C}^*$, so this proves the claim. □

Lemma 10.2.3. *Let us fix any triple $(Q_i, W_i)_{i=1,2,3} \in \prod_{i=1}^3 G_i$ with numerical conditions (10.21), respectively (10.22) (this automatically implies that $(n_3, k_3) \neq (n_i, k_i)$ for $i = 1, 2$), and let us suppose that $(Q_1, W_1) \simeq (Q_2, W_2)$. Then the (E, V)'s that belong to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, that have graded $\oplus_{i=1}^3 (Q_i, W_i)$ and α_c-canonical filtration of type $(2, 1)$ are parametrized by the grassmannian $Grass(2, H_1)$, where $H_1 := Ext^1((Q_3, W_3), (Q_1, W_1))$.*

Proof. As in the previous proof, we get that for every (E, V) that sits in a sequence (10.20) with numerical conditions (10.21), respectively (10.22), the following facts are equivalent:

(a) (E, V) belongs to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$ and it has α_c-canonical filtration of type (2,1);

(b) there are no quotients $\zeta_1 : (E, V) \twoheadrightarrow (Q_1, W_1)$ such that $\zeta_1 \circ \alpha \neq 0$.

In this case $(Q_1, W_1) \simeq (Q_2, W_2)$, so we can ignore in (b) the case $i = 2$. Since $(Q_1, W_1) \simeq (Q_2, W_2)$, then by lemma 3.3.2 we get that this is equivalent to

(c) μ_1, μ_2 linearly independent in H_1.

Now if we consider the exact sequence (10.20) with $(Q_1, W_1) \simeq (Q_2, W_2)$, we get that $\mathrm{Aut}(Q_3, W_3) = \mathbb{C}^*$, while $\mathrm{Aut}((Q_1, W_1) \oplus (Q_1, W_1)) = \mathrm{GL}(2, \mathbb{C})$, so we conclude. □

As in the previous section, we need to globalize the constructions of lemma 10.2.2 and of lemma 10.2.3 and we we have to distinguish between 3 cases, that are taken into account by propositions 7.2.1, 7.2.2 and 7.2.3 respectively. Once we use lemmas 10.2.2 and 10.2.3 instead of lemmas 10.1.1 and of lemma 10.1.2, the proofs are on the same line of the proofs of propositions 7.1.1, 7.1.2 and 7.1.3 respectively, so we omit the details.

Chapter 11

Parametrization of objects with canonical filtration of type $(1,3)$ and $(3,1)$

Having fixed any triple (n, d, k) and a critical value α_c for it, in this chapter we want to describe how to parametrize those (E, V)'s that have α_c-canonical filtration of type (1,3) or (3,1) and that belong to $G^+(\alpha_c; n, d, k)$ or to $G^-(\alpha_c; n, d, k)$.

11.1 Canonical filtration of type (1,3)

Let us fix any object $\oplus_{i=1}^4 (Q_i, W_i)$, with all the (Q_i, W_i)'s α_c-stable coherent systems with the same α_c-slope μ; let us suppose that (E, V) has such a graded at α_c and that it has α_c-canonical filtration of type (1,3). Then every (E, V) that we want to parametrize sits in an exact sequence of the form:

$$0 \to (Q_1, W_1) \xrightarrow{\alpha} (E, V) \xrightarrow{\beta} (Q_2, W_2) \oplus (Q_3, W_3) \oplus (Q_4, W_4) \to 0. \tag{11.1}$$

Any (E, V) as in this extension has always the following proper α_c-semistable subobjects with α_c-slope μ:

- the only α_c-stable one is (Q_1, W_1) (if any other (Q_i, W_i) is a subobject, then the α_c-canonical filtration is no more of type (1,3));
- for all $i \in \{2, 3, 4\}$, any extension (E_{i1}, V_{i1}) of (Q_i, W_i) by (Q_1, W_1);
- for all $i \neq j \in \{2, 3, 4\}$, any extension (E_{ji1}, V_{ji1}) of (Q_j, W_j) by (E_{i1}, V_{i1}).

Actually, if (E, V) has α_c-canonical filtration of type (1,3), then the previous list is complete. Therefore, for any such (E, V) we have that (E, V) belongs to $G^+(\alpha_c; n, d, k)$ if and only if the following numerical conditions hold:

$$\frac{k_1}{n_1} < \frac{k}{n}, \quad \frac{k_1 + k_j}{n_1 + n_j} < \frac{k}{n} \quad \forall\, j \in \{2, 3, 4\},$$

$$\frac{k_1+k_j+k_l}{n_1+n_j+n_l}<\frac{k}{n}\quad \forall\, j\neq l\in\{2,3,4\}. \tag{11.2}$$

For every pair j,l as before, let us denote by i the index in $\{2,3,4\}$ different from j and l. Then if we use the fact that $\mu_{\alpha_c}(E_{jl1},V_{jl1})=\mu_{\alpha_c}(E,V)$, we get that the last line is equivalent to:

$$\frac{k_i}{n_i}>\frac{k}{n}\quad \forall\, i\in\{2,3,4\}. \tag{11.3}$$

Actually, if we assume this condition, then we get

$$\frac{k_i+k_l}{n_i+n_l}>\frac{k}{n}\quad \forall\, i\neq l\in\{2,3,4\},\qquad \frac{k_2+k_3+k_4}{n_2+n_3+n_4}>\frac{k}{n}.$$

The first inequality implies the second condition of (11.2), while the second inequality implies the first inequality of (11.2). Therefore, we conclude that given any (E,V) with α_c-canonical filtration of type (1,3), then (E,V) belongs to $G^{+}(\alpha_c;n,d,k)$ if and only if (11.3) holds. Analogously, given any (E,V) with α_c-canonical filtration of type (1,3), we have that (E,V) belongs to $G^{-}(\alpha_c;n,d,k)$ if and only if:

$$\frac{k_i}{n_i}<\frac{k}{n}\quad \forall\, i\in\{2,3,4\}. \tag{11.4}$$

Now if we denote by μ any extension like (11.1), we get that we can identify μ with a triple

$$(\mu_2,\mu_3,\mu_4)\in\bigoplus_{i=2}^{4}\mathrm{Ext}^1\Big((Q_i,W_i),(Q_1,W_1)\Big).$$

For every $i=2,3,4$, this identification gives a diagram of the form:

$$\begin{tikzcd}
0 \arrow[r] & (Q_1,W_1) \arrow[r, "\alpha"] \arrow[d, equal] \arrow[dr, phantom, "\curvearrowright"] & (E,V) \arrow[r, "\beta"] \arrow[dr, phantom, "\curvearrowright"] & \bigoplus_{l=2}^{4}(Q_l,W_l) \arrow[r] & 0 & \mu \arrow[d, "\overline{\varepsilon_i}"] \\
0 \arrow[r] & (Q_1,W_1) \arrow[r, "\alpha_i"] & (E_{i1},V_{i1}) \arrow[r, "\beta_i"] \arrow[u, hook, "\delta_i"'] & (Q_i,W_i) \arrow[r] \arrow[u, hook, "\varepsilon_i"'] & 0 & \mu_i,
\end{tikzcd} \tag{11.5}$$

where ε_i is the embedding of (Q_i,W_i) in $\oplus_{l=2}^{4}(Q_l,W_l)$ for $i=2,3,4$. Then we have the following results.

Lemma 11.1.1. *Let us fix any quadruple $(Q_i,W_i)_{i=1,\cdots,4}\in\prod_{i=1}^{4}G_i$ such that conditions (11.3), respectively (11.4), are satisfied (this automatically implies that $(n_1,k_1)\neq(n_i,k_i)$ for $i=2,3,4$). Moreover, let us suppose that*

$$(Q_i,W_i)\not\simeq(Q_j,W_j)\ \text{for all}\ i\neq j\in\{2,3,4\}$$

(this condition can be omitted if $(n_i,k_i)\neq(n_j,k_j)$ for every $i\neq j\in\{2,3,4\}$). Then the (E,V)'s that belong to $G^{+}(\alpha_c;n,d,k)$, respectively to $G^{-}(\alpha_c;n,d,kk)$, that have α_c-canonical

filtration of type $(1,3)$ *and graded* $\oplus_{i=1}^{4}(Q_i, W_i)$ *are parametrized by* $\prod_{i=2}^{4}\mathbb{P}(H_i)$*, where* $H_i = Ext^1((Q_i, W_i), (Q_1, W_1))$.

Proof. Let us fix any extensions μ with representative (11.1). Then we have that (E, V) has a filtration of the form

$$0 = (E_0, V_0) \subset (E_1, V_1) := (Q_1, W_1) \subset (E_2, V_2) = (E, V). \tag{11.6}$$

Here $(E_1, V_1)/(E_0, V_0) = (Q_1, W_1)$ is α_c-stable and $(E_2, V_2)/(E_1, V_1) = (Q_2, W_2) \oplus (Q_3, W_3) \oplus (Q_4, W_4)$ is α_c-polystable. Then by proposition 2.1.3 we get that (11.6) is the α_c-canonical filtration of (E, V) (and so (E, V) has α_c-canonical filtration of type $(1,3)$), if and only if condition (c) of that proposition is satisfied. In our case the index t is equal to 2, so (E, V) has α_c-canonical filtration of type $(1,3)$ if and only if for all (Q_i, W_i) for $i = 1, \cdots, 4$ and for all non-zero morphisms $\gamma_i : (Q_i, W_i) \to (E, V)$ we have $\beta \circ \gamma_i = 0$. Now by hypothesis $(Q_1, W_1) \not\simeq (Q_i, W_i)$ for all $i = 2, 3, 4$. Since all the (Q_i, W_i)'s for $i = 1, \cdots, 4$ are α_c-stable of the same slope, then for all $\gamma_1 : (Q_1, W_1) \to (E, V)$ we have that $\beta \circ \gamma_1 = 0$. So for every (E, V) as in (11.1) the following conditions are equivalent:

(a) (E, V) has α_c-canonical filtration of type $(1,3)$;

(b) there are no morphisms $\gamma_i : (Q_i, W_i) \to (E, V)$ for $i = 2, 3, 4$ such that $\beta \circ \gamma_i = 0$.

By hypothesis we have that $(Q_i, W_i) \not\simeq (Q_j, W_j)$ for all $j \neq j$ in $\{2, 3, 4\}$, so by lemma 3.3.2, (b) is equivalent to

(c) $\mu_i \neq 0$ for all $i = 2, 3, 4$.

Now if we look at the sequence (11.1), we get that $\mathrm{Aut}(Q_1, W_1) = \mathbb{C}^*$ and $\mathrm{Aut}((Q_2, W_2) \oplus (Q_3, W_3) \oplus (Q_4, W_4)) = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ (because $(Q_i, W_i) \not\simeq (Q_j, W_j)$ for all $i \neq j \in \{2, 3, 4\}$), so we conclude. □

Lemma 11.1.2. *Let us any fix any quadruple* $(Q_i, W_i)_{i=1,\cdots,4} \in \prod_{i=1}^{4} G_i$ *such that conditions (11.3), respectively (11.4), are satisfied (this automatically implies that* $(n_1, k_1) \neq (n_i, k_i)$ *for* $i = 2, 3, 4$*). Moreover, let us suppose that* $(n_2, k_2) = (n_3, k_3)$ *and*

$$(Q_2, W_2) \simeq (Q_3, W_3) \not\simeq (Q_4, W_4).$$

Then the (E, V)*'s that belong to* $G^+(\alpha_c; n, d, k)$*, respectively to* $G^-(\alpha_c; n, d, k)$*, that have* α_c*-canonical filtration of type* $(1,3)$ *and graded* $\oplus_{i=1}^{4}(Q_i, W_i)$ *are parametrized by*

$$Grass(2, Ext^1((Q_2, W_2), (Q_1, W_1))) \times \mathbb{P}(Ext^1((Q_4, W_4), (Q_1, W_1))).$$

Proof. As in the previous proof, we get that for any (E, V) that sits in a sequence (11.1) the following facts are equivalent

(a) (E, V) has α_c-canonical filtration of type (1,3);

(b) there are no morphisms $\gamma_i : (Q_i, W_i) \to (E, V)$ for $i = 2, 4$ such that $\beta \circ \gamma_i \neq 0$.

In this case $(Q_2, W_2) \simeq (Q_3, W_3)$, so we can ignore in (b) the case $i = 3$. Then by lemma 3.3.2 we get that (b) is equivalent to

(c) $\mu_4 \neq 0$ and μ_2, μ_3 linearly independent in $\mathrm{Ext}^1((Q_2, W_2), (Q_1, W_1))$.

Now if we consider the exact sequence (11.1) with $(Q_2, W_2) \simeq (Q_3, W_3) \not\simeq (Q_4, W_4)$, then we get that $\mathrm{Aut}(Q_1, W_1) = \mathbb{C}^*$, while $\mathrm{Aut}((Q_2, W_2) \oplus (Q_2, W_2) \oplus (Q_4, W_4)) = \mathrm{GL}(2, \mathbb{C}) \times \mathbb{C}^*$, so we conclude. $\square$

Lemma 11.1.3. *Let us fix any quadruple $(Q_i, W_i)_{i=1,\cdots,4} \in \prod_{i=1}^{4} G_i$ such that conditions (11.3), respectively (11.4), are satisfied (this automatically implies that $(n_1, k_1) \neq (n_i, k_i)$ for $i = 2, 3, 4$). Moreover, let us suppose that $(n_2, k_2) = (n_3, k_3) = (n_4, k_4)$ and that*

$$(Q_2, W_2) \simeq (Q_3, W_3) \simeq (Q_4, W_4).$$

Then the (E, V)'s that belong to $G^+(\alpha_c; n, d, k)$, respectively $G^-(\alpha_c; n, d, k)$, that have α_c-canonical filtration of type $(1,3)$ and graded $\oplus_{i=1}^{4}(Q_i, W_i)$ are parametrized by

$$Grass(3, Ext^1((Q_2, W_2), (Q_1, W_1))).$$

Proof. As in the previous proof, we get that for any (E, V) that sits in a sequence (11.1), the following facts are equivalent.

(a) (E, V) has α_c-canonical filtration of type $(1,3)$;

(b) there are no morphisms $\gamma_2 : (Q_2, W_2) \to (E, V)$ such that $\beta \circ \gamma_2 \neq 0$.

In this case, $(Q_2, W_2) \simeq (Q_3, W_3) \simeq (Q_4, W_4)$, so we can ignore the cases $i = 3, 4$ in (b). Then by lemma 3.3.2 we get that (b) is equivalent to

(c) μ_2, μ_3 and μ_4 are linearly independent in $\mathrm{Ext}^1((Q_2, W_2), (Q_1, W_1))$.

Now if we consider the exact sequence (11.1) with $(Q_2, W_2) \simeq (Q_3, W_3) \simeq (Q_4, W_4)$, we get that $\mathrm{Aut}(Q_1, W_1) = \mathbb{C}^*$, while $\mathrm{Aut}((Q_2, W_2) \oplus (Q_2, W_2) \oplus (Q_2, W_2)) = \mathrm{GL}(3, \mathbb{C})$, so we conclude. $\square$

Now we give a global parametrization of the objects described before, i.e. we describe families of schemes that parametrize various types of (E, V)'s when the graded $\oplus_{i=1}^{4}(Q_i, W_i)$ varies over $\prod_{i=1}^{4} G_i$ and the α_c-canonical filtration is of type (1,3). Since the order of the objects (Q_i, W_i) for $i = 2, 3, 4$ is not important, we can assume that we have fixed any order that satisfies the following properties:

- if $(n_i, k_i) \neq (n_j, k_j)$ for $i \neq j \in \{2, 3, 4\}$, then we use the lexicographic order on the set $\{(n_i, k_i)\}_{i=2,3,4}$;

- if exactly 2 (n_i, k_i)'s are equal for $i \in \{2,3,4\}$, then we assume that they are (n_2, k_2) and (n_3, k_3); by using the fact that $\mu_{\alpha_c}(n_i, d_i, k_i)$ is the same for all $i = 1, \cdots, 4$, we get that $d_2 = d_3$;

- if all the (n_i, k_i)'s are equal for $i \in \{2,3,4\}$ (this implies that $d_2 = d_3 = d_4$) and if exactly 2 among the corresponding (Q_i, W_i)'s are isomorphic, we order them so that $(Q_2, W_2) \simeq (Q_3, W_3) \not\simeq (Q_4, W_4)$.

Let us write

$$H_i := \mathrm{Ext}^1\left((Q_i, W_i), (Q_1, W_1)\right) \quad \forall\, i = 2,3,4.$$

Then we need to distinguish the following subcases

(1) If $(n_2, k_2) \neq (n_3, k_3) \neq (n_4, k_4)$ (this implies that $(n_2, k_2) \neq (n_4, k_4)$ since we are using the lexicographic order), then having fixed the graded, the corresponding (E, V)'s are in bijection with the points of $\mathbb{P}(H_2) \times \mathbb{P}(H_3) \times \mathbb{P}(H_4)$.

(2) If $(n_2, k_2) = (n_3, k_3) \neq (n_4, k_4)$ and $(Q_2, W_2) \not\simeq (Q_3, W_3)$, then the corresponding (E, V)'s are parametrized by $H_{23} \times \mathbb{P}(H_4)$, where $H_{23} := (\mathbb{P}(H_2) \times \mathbb{P}(H_3))/\mathbb{Z}_2$.

(3) If $(n_2, k_2) = (n_3, k_3) \neq (n_4, k_4)$ and $(Q_2, W_2) \simeq (Q_3, W_3)$, then $H_2 = H_3$ and the corresponding (E, V)'s are parametrized by $Grass(2, H_2) \times \mathbb{P}(H_4)$.

(4) Let us assume that $(n_2, k_2) = (n_3, k_3) = (n_4, k_4)$ and $(Q_2, W_2) \not\simeq (Q_3, W_3)$ (using the hypothesis on the ordering, this implies that (Q_4, W_4) is not isomorphic to (Q_i, W_i) for $i = 2, 3$). Then the corresponding (E, V)'s are parametrized by $(\mathbb{P}(H_2) \times \mathbb{P}(H_3) \times \mathbb{P}(H_4))/S_3$.

(5) Let us assume that $(n_2, k_2) = (n_3, k_3) = (n_4, k_4)$ and $(Q_2, W_2) \simeq (Q_3, W_3) \not\simeq (Q_4, W_4)$. Then $H_2 = H_3$ and the corresponding (E, V)'s are parametrized by $Grass(2, H_2) \times \mathbb{P}(H_4)$.

(6) If $(n_2, k_2) = (n_3, k_3) = (n_4, k_4)$ and $(Q_2, W_2) \simeq (Q_3, W_3) \simeq (Q_4, W_4)$, then $H_2 = H_3 = H_4$ and the corresponding (E, V)'s are parametrized by $Grass(3, H_2)$.

Note that in this way the cases (3) and (5) coincide if we fix the graded. However, we will have to give different global descriptions for them because the base spaces we will work on will be different.

The previous 6 cases are taken into account by propositions 7.3.1, 7.3.2, 7.3.3, 7.3.4, 7.3.5 and 7.3.6 respectively. We give below the proof of those 6 results.

Proof of proposition 7.3.1. Let us fix any triple $(a, b, c) \in \mathbb{N}^3$; as in the proof of proposition 7.1.1 we define sets of data $\mathscr{D}^2_a$ and $\mathscr{D}^3_b$; moreover, we consider also a third set of data of the form $\mathscr{D}^4_c$, i.e. the data consisting of a tree with 2 leaves with associated invariants (n_1, k_1), (n_4, k_4) and c. For each set of that type we apply proposition 5.0.5 and we get projective fibrations and universal families as usual.

If we consider only $\mathscr{D}_a^2$ and $\mathscr{D}_b^3$, then we can proceed as in the proof of proposition 7.1.1 in order to get a diagram as (10.11). In particular, we will need to use the following morphisms obtained by that diagram:

$$\hat{R}_{a,b;i,j} \xrightarrow{\hat{\theta}^3_{b;j}} \hat{A}_{a,b;i,j} \xrightarrow{\hat{\psi}^2_{a;i}} \hat{U}_{a,b;i,j} \xrightarrow{\hat{r}^3_{b;j}} \hat{U}^2_{a;i} \xrightarrow{\hat{p}^2_{a;i}} \hat{G}_1.$$

Using again the same computations of that proposition, we get an exact sequence of families of coherent systems parametrized by $\hat{R}_{a,b;i,j}$

$$0 \to (\overline{\mathcal{Q}}_1, \overline{\mathcal{W}}_1) \to (\hat{\mathcal{E}}_{a,b;i,j}, \hat{\mathcal{V}}_{a,b;i,j}) \to (\overline{\mathcal{Q}}_2, \overline{\mathcal{W}}_2) \oplus (\overline{\mathcal{Q}}_3, \overline{\mathcal{W}}_3) \to 0 \tag{11.7}$$

as we got in (10.15). Now let us apply proposition 5.0.5 to the set of data U^4_c. So we get a finite locally closed disjoint covering $\{\hat{U}^4_{c;k}\}_k$ of $\hat{U}^4_c$ and a family of projective fibrations $\{\hat{R}^4_{c;k} \xrightarrow{\hat{\varphi}^4_{c;k}} \hat{\mathcal{U}}^4_{c;k}\}_k$. Moreover, for each k we get a family of non-splitting extensions parametrized by $\hat{R}^4_{c;k}$, obtained as in (10.8):

$$\begin{aligned} 0 &\to (\hat{\varphi}^{4'}_{c;k}, \hat{\varphi}^4_{c;k})^*(\hat{p}^{4'}_{c;k}, \hat{p}^4_{c;k})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \to (\hat{\mathcal{E}}^4_{c;k}, \hat{\mathcal{V}}^4_{c;k}) \otimes_{\hat{R}^4_{c;k}} \mathcal{O}^4_{c;k}(-1) \to \\ &\to (\hat{\varphi}^{4'}_{c;k}, \hat{\varphi}^4_{c;k})^*(\hat{q}^{4'}_{c;k}, \hat{q}^4_{c;k})^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4) \otimes_{\hat{R}^4_{c;k}} \mathcal{O}^4_{c;k}(-1) \to 0. \end{aligned} \tag{11.8}$$

Now for every triple (i,j,k) we consider the following cartesian diagram, constructed in several steps starting from (a):

$$\begin{tikzcd}
\hat{R}_{a,b,c;i,j,k} \arrow[r, "\hat{\phi}_1"] \arrow[d] \arrow[dr, phantom, "\square(f)"] & \hat{C}_{a,b,c;i,j,k} \arrow[r] \arrow[d, "\hat{\phi}_2"] \arrow[dr, phantom, "\square(e)"] & \hat{R}_{a,b;i,j} \arrow[d, "\hat{\theta}^3_{b;j}"] \\
\hat{R}^{4''}_{c;k} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square(d)"] & \hat{B}_{a,b,c;i,j,k} \arrow[r] \arrow[d, "\hat{\phi}_3"] \arrow[dr, phantom, "\square(c)"] & \hat{A}_{a,b;i,j} \arrow[d, "\hat{\psi}^2_{a;i}"] \\
\hat{R}^{4'}_{c;k} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square(b)"] & \hat{U}_{a,b,c;i,j,k} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square(a)"] & \hat{U}_{a,b;i,j} \arrow[d, "\hat{p}^2_{a;i} \circ \hat{r}^3_{b;j}"] \\
\hat{R}^4_{c;k} \arrow[r, "\hat{\varphi}^4_{c;k}"] & \hat{U}^4_{c;k} \arrow[r, "\hat{p}^4_{c;k}"] & \hat{G}_1.
\end{tikzcd} \tag{11.9}$$

Then we proceed as in the proof of proposition 7.1.1: given the sequences (11.7) and (11.8), we pullback both of them to $\hat{R}_{a,b,c;i,j,k}$ and we sum them in order to obtain an extension parametrized by that space, as follows:

$$0 \to (\widetilde{\mathcal{Q}}_1, \widetilde{\mathcal{W}}_1) \to (\hat{\mathcal{E}}_{a,b,c;i,j,k}, \hat{\mathcal{V}}_{a,b,c;i,j,k}) \to (\widetilde{\mathcal{Q}}_2, \widetilde{\mathcal{W}}_2) \oplus (\widetilde{\mathcal{Q}}_3, \widetilde{\mathcal{W}}_3) \oplus (\widetilde{\mathcal{Q}}_4, \widetilde{\mathcal{W}}_4) \to 0. \tag{11.10}$$

Here the various $(\widetilde{\mathcal{Q}}_i, \widetilde{\mathcal{W}}_i)$'s are suitable pullbacks from the corresponding $\hat{G}_i$'s to $\hat{R}_{a,b,c;i,j,k}$ of the local universal families $(\hat{\mathcal{Q}}_i, \hat{\mathcal{W}}_i)$'s. Then we conclude the proof in the same way of the proof of proposition 7.1.1. We only need to consider the additional action of $PGL(N_4)$ on the various spaces. The morphisms ϕ_1, ϕ_2 and ϕ_3 satisfy the claim of the proposition since $\hat{\phi}_1$, $\hat{\phi}_2$ and $\hat{\phi}_3$ are obtained in diagram (11.9) as pullbacks of projective fibrations with fibers isomorphic to $\mathbb{P}^{c-1}, \mathbb{P}^{b-1}$ and $\mathbb{P}^{a-1}$ respectively. $\square$

Proof of proposition 7.3.2. Let us consider the family of schemes of the form $R_{a,b,c;i,j,k}$ obtained in the proof of proposition 7.3.1. In cases (a) and (b) there is nothing to prove since there is no action of $\mathbb{Z}_2$. In cases (c) and (d) it is clear that there is an action of $\mathbb{Z}_2$ and that the induced morphisms to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, are injective only after passing to the quotient with respect to that action. The only claim that a priori is not obvious is the existence of the local trivializations that are compatible with the action of $\mathbb{Z}_2$.

We give all the details only for the case (d) and assuming for simplicity that the every index a, b, c, i, j, k can assume only one value. In particular, this means that we have the following identities:

$$\hat{U}_{a,a;i,j} = \hat{G}_1 \times \hat{G}_2 \times \hat{G}_3, \quad \hat{U}^4_{c;k} = \hat{G}_1 \times \hat{G}_4, \quad \hat{U}_{a,a,c;i,j,k} = \hat{G}_1 \times \hat{G}_2 \times \hat{G}_3 \times \hat{G}_4.$$

So we can rewrite diagram (11.9) as follows:

$$\begin{tikzcd}
\hat{R} \arrow[r, "\hat{\phi}_1"] \arrow[d] \arrow[dr, phantom, "\square"] & \hat{C} \arrow[r] \arrow[d, "\hat{\phi}_2"] \arrow[dr, phantom, "\square"] & \hat{R}^0 \arrow[d, "\hat{\theta}^3"] \\
\hat{R}^{4''} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \hat{B} \arrow[r] \arrow[d, "\hat{\phi}_3"] \arrow[dr, phantom, "\square"] & \hat{A} \arrow[d, "\hat{\psi}^2"] \\
\hat{R}^{4'} \arrow[r, "\hat{\varphi}^{4'}"] \arrow[d] \arrow[dr, phantom, "\square"] & \hat{G}_1 \times \hat{G}_2 \times \hat{G}_3 \times \hat{G}_4 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \hat{G}_1 \times \hat{G}_2 \times \hat{G}_3 \arrow[d, "\hat{p}^2 \circ \hat{r}^3"] \\
\hat{R}^4 \arrow[r, "\hat{\varphi}^4"] & \hat{G}_1 \times \hat{G}_4 \arrow[r, "\hat{p}^4"] & \hat{G}_1.
\end{tikzcd} \tag{11.11}$$

As in the proof of proposition 7.1.2, we can describe an open covering $\{\hat{V}_{\alpha,\beta}\}_{\alpha,\beta\in A}$ of $\hat{G}_1 \times (\hat{G}_2 \times \hat{G}_3 \smallsetminus \Delta_{23})$ such that if we denote by $\widetilde{V}_{\alpha,\beta}$ the subscheme of $\hat{R}$ over $\hat{V}_{\alpha,\beta}$, then we have trivializations $\lambda_{\alpha,\beta}$ of $\hat{\psi}^2 \circ \hat{\theta}^3$ as follows

$$\begin{tikzcd}
\widetilde{V}_{\alpha,\beta} \arrow[rr, "\lambda_{\alpha,\beta}", "\sim"'] \arrow[dr, "\hat{\psi}^2\circ\hat{\theta}^3|_{\widetilde{V}_{\alpha,\beta}}"'] & \arrow[d, phantom, "\curvearrowright"] & \hat{V}_{\alpha,\beta} \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}. \arrow[dl, "pr"] \\
& \hat{V}_{\alpha,\beta} &
\end{tikzcd}$$

As in the proof of proposition 7.1.2, if ε denotes the non-trivial element of $\mathbb{Z}_2$, then we have that ε acts on $\hat{G}_1 \times \hat{G}_2 \times \hat{G}_3 = \hat{G}_1 \times \hat{G}_2 \times \hat{G}_2$ so that $\varepsilon(\hat{V}_{\alpha,\beta}) = \hat{V}_{\beta,\alpha}$. Then we write

$$\hat{V}'_{\alpha,\beta} := \hat{V}_{\alpha,\beta} \times \hat{G}_4 \subset \hat{G}_1 \times (\hat{G}_2 \times \hat{G}_3 \smallsetminus \Delta_{23}) \times \hat{G}_4.$$

In the case when $\hat{U}^4_{c;k}$ does not coincide with the whole $\hat{G}_1 \times \hat{G}_4$, this should be defined as:

$$\hat{V}'_{\alpha,\beta} := \hat{V}_{\alpha,\beta} \times_{\hat{G}_1} \hat{U}^4_{c;k}.$$

In the case under consideration we can simply use the previous definition. Now the previous action of ε on $\hat{G}_1 \times \hat{G}_2 \times \hat{G}_3$ extends to an action on $\hat{G}_1 \times \hat{G}_2 \times \hat{G}_3 \times \hat{G}_4$, so that $\varepsilon(\hat{V}'_{\alpha,\beta}) = V'_{\beta,\alpha}$. Since the fibration $\hat{\theta}^3 \circ \hat{\psi}^2$ is trivial on $\hat{V}_{\alpha,\beta}$, then we get that the fibration $\hat{\phi}_3 \circ \hat{\phi}_2$ is trivial over each $V'_{\alpha,\beta}$. Moreover, this trivialization is compatible with the action of $\mathbb{Z}_2$ on the base and on the top space.

Since $\hat{\varphi}^4$ is a locally trivial fibration, then there is a covering $\{\hat{U}^4_\gamma\}_\gamma$ of $\hat{U}^4 = \hat{G}_1 \times \hat{G}_4$ and trivializations

$$\lambda_\gamma : \hat{R}^4|_{\hat{U}^4_\gamma} \xrightarrow{\sim} \hat{U}^4_\gamma \times \mathbb{P}^{c-1}.$$

Then we define

$$\hat{V}'_{\alpha,\beta,\gamma} := \hat{V}'_{\alpha,\beta} \cap (\hat{U}^4_\gamma \times \hat{G}_2 \times \hat{G}_3) = \hat{V}'_{\alpha,\beta} \times_{\hat{G}_1} \hat{U}^4_\gamma \subset \hat{G}_1 \times (\hat{G}_2 \times \hat{G}_3 \smallsetminus \Delta) \times \hat{G}_4.$$

By pullback we get induced trivializations of $\hat{\varphi}^{4'}$:

$$\lambda'_\gamma : \hat{R}^{4'}|_{V'_{\alpha,\beta,\gamma}} \xrightarrow{\sim} \hat{V}'_{\alpha,\beta,\gamma} \times \mathbb{P}^{c-1}.$$

For every triple (α,β,γ) let us define

$$\widetilde{V}'_{\alpha,\beta,\gamma} := \hat{R}|_{\hat{V}'_{\alpha,\beta\,\gamma}}$$

By construction, we have

$$\widetilde{V}'_{\alpha,\beta,\gamma} := \left(\hat{C}|_{\hat{V}'_{\alpha,\beta,\gamma}}\right) \times_{\hat{V}'_{\alpha,\beta,\gamma}} \left(\hat{R}^{4'}|_{\hat{V}'_{\alpha,\beta,\gamma}}\right).$$

Since both both $\hat{\phi}_3 \circ \hat{\phi}_2$ and $\hat{\varphi}^{4'}$ are trivial fibrations if restricted to $\hat{V}'_{\alpha,\beta,\gamma}$, then we get that

$$\widetilde{V}'_{\alpha,\beta,\gamma} \simeq \hat{V}'_{\alpha,\beta,\gamma} \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{c-1}.$$

The action of $\mathbb{Z}_2$ extends to an action also on $\hat{G}_1 \times \hat{G}_2 \times \hat{G}_3 \times \hat{G}_4$ and on $\hat{R}$. We have that $\varepsilon(\hat{V}'_{\alpha,\beta,\gamma}) = \hat{V}'_{\beta,\alpha,\gamma}$ and analogously for $\widetilde{V}'_{\alpha,\beta,\gamma}$. In particular, we have:

- $\varepsilon(\hat{V}'_{\alpha,\alpha,\gamma}) = \hat{V}'_{\alpha,\alpha,\gamma}$;
- if $\alpha \neq \beta$ and we set $\hat{W}_{\alpha,\beta,\gamma} := \hat{V}'_{\alpha,\beta,\gamma} \cap \hat{V}'_{\beta,\alpha,\gamma}$, then $\varepsilon(\hat{W}_{\alpha,\beta,\gamma}) = \hat{W}_{\alpha,\beta,\gamma}$;
- if $\alpha \neq \beta$ and we set $\hat{Z}_{\alpha,\beta,\gamma} := (\hat{V}'_{\alpha,\beta,\gamma} \cup \hat{V}'_{\beta,\alpha,\gamma}) \smallsetminus \hat{W}_{\alpha,\beta,\gamma}$, then $\varepsilon(\hat{Z}_{\alpha,\beta,\gamma}) = \hat{Z}_{\alpha,\beta,\gamma}$.

Then the set:

$$\{\hat{T}_l\}_{l\in L} := \left\{\{\hat{V}_{\alpha,\alpha,\gamma}\}_{\alpha\in A}, \quad \{\hat{W}_{\alpha,\beta,\gamma}\}_{\alpha<\beta}, \quad \{\hat{Z}_{\alpha,\beta,\gamma}\}_{\alpha<\beta}\right\}_{\gamma}$$

is a locally closed covering of $\hat{G}_1 \times (\hat{G}_2 \times \hat{G}_3 \smallsetminus \Delta_{23}) \times \hat{G}_4$ and each of such subschemes is invariant under the action of $\mathbb{Z}_2$. Moreover, by restricting to any subscheme $\hat{T}_l$ we have a trivialization of the fibration $\hat{\phi}_3 \circ \hat{\phi}_2 \circ \hat{\phi}_1$ and that trivialization is compatible with the action of $\mathbb{Z}_2$ on $\hat{T}_i \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{c-1}$. Finally, we consider the free action of $PGL(N_1) \times PGL(N_2) \times PGL(N_3) \times PGL(N_4)$ on all these schemes and we conclude. □

Proof of proposition 7.3.3. Let us fix any pair of invariants $(a,b) \in \mathbb{N}^2$. Let us denote by $\hat{p}_{12}$ and $\hat{q}_{12}$ the projections from $\hat{G}_1 \times \hat{G}_2$ to its factors. If the subscheme

$$\hat{U}^2_a := \{t \in \hat{G}_1 \times \hat{G}_2 \text{ s.t. } \dim \operatorname{Ext}^1((\hat{q}'_{12}, \hat{q}_{12})^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2)_t, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = a\}$$

is non-empty, then it has a finite disjoint locally closed covering $\{\hat{U}^2_{a;i}\}_i$ such that for every i the sheaf

$$\hat{E}_{a;i} := \mathcal{E}xt^1_{\pi_{\hat{U}_{a;i}}}((\hat{q}'_{12}, \hat{q}_{12})^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2), (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1))^{\vee}$$

is locally free of rank a and commutes with base change. By conditions (11.3), respectively (11.4), and lemma 1.0.4, we get that

$$\operatorname{Hom}((\hat{q}'_{12}, \hat{q}_{12})^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2)_t, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = 0$$

for all $t \in \hat{G}_1 \times \hat{G}_2$. Therefore, we can apply corollary 4.5.6 for $t = 2$, so we get that there exists a grassmannian bundle

$$\hat{\theta}_{2;a;i} : \hat{Q}_{2;a;i} := Grass(2, \hat{E}_{a;i}) \longrightarrow \hat{U}^2_{a;i}$$

associated to the sheaf $\hat{E}_{2;i}$. Moreover, there exists a universal non-degenerate extension of rank 2 on the right:

$$0 \to (\hat{\theta}'_{2;a;i}, \hat{\theta}_{2;a;i})^*(\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \to (\mathcal{E}^{123}_{a;i}, \mathcal{V}^{123}_{a;i}) \to$$

$$\to (\hat{\theta}'_{2;a;i}, \hat{\theta}_{2;a;i})^*(\hat{q}'_{12}, \hat{q}_{12})^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2) \otimes_{\hat{Q}_{2;a;i}} \overline{\mathcal{M}}_{2;a;i} \to 0, \tag{11.12}$$

where $\overline{\mathcal{M}}_{2;a;i}$ is a locally free sheaf on $\hat{Q}_{2;a;i}$ of rank 2. In particular, we have that for every $t \in Grass(2, \hat{E}_{a;i})$ the restriction of the previous extension to t gives an extension of the form

$$0 \to (Q_1, W_1) \to (E^{123}, V^{123}) \to (Q_2, W_2)^{\oplus 2} \to 0$$

that is a representative for an object $\mu = (\mu_2, \mu_3)$ such that μ_2 and μ_3 are linearly independent vectors of $\mathrm{Ext}^1((Q_2, W_2), (Q_1, W_1))$.

Now let us consider a set of data $\mathscr{D}^4_b$ given by

- $r = 2$, i.e. we are considering a tree with only 2 leaves and an internal node;
- the invariants (n_1, k_1) and (n_4, k_4) associated to the first leaf, respectively to the second leaf;
- any non-negative integer b such that the subscheme

$$\hat{U}^4_b := \{t \in \hat{G}_1 \times \hat{G}_4 \text{ s.t. } \dim \mathrm{Ext}^1((\hat{q}'_{14}, \hat{q}_{14})^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)_t, (\hat{p}'_{14}, \hat{p}_{14})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = b\}$$

  is non-empty. Here $\hat{p}_{14}$ and $\hat{q}_{14}$ are the projections from $\hat{G}_1 \times \hat{G}_4$ to its 2 factors.

Then by proposition 5.0.5 we get a family of induced locally trivial fibrations of rank b:

$$\{\hat{\varphi}^4_{b;j} : \hat{R}^4_{b;j} \longrightarrow \hat{U}^4_{b;j}\}_j,$$

where $\{\hat{U}^4_{b;j}\}_j$ is a finite disjoint locally closed covering of $\hat{U}^4_b$. Moreover, for every j we get a universal family of non-splitting extensions parametrized by $\hat{R}^4_{b;j}$

$$0 \to (\hat{\varphi}^{4'}_{b;j}, \hat{\varphi}^4_{b;j})^*(\hat{p}'_{14}, \hat{p}_{14})^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \to (\hat{\mathcal{E}}^4_{b;j}, \hat{\mathcal{V}}^4_{b;j}) \otimes_{\hat{R}^4_{b;j}} \mathcal{O}_{b;j}(-1) \to$$

$$\to (\hat{\varphi}^{4'}_{b;j}, \hat{\varphi}^4_{b;j})^*(\hat{q}'_{14}, \hat{q}_{14})^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4) \otimes_{\hat{R}^4_{b;j}} \mathcal{O}_{b;j}(-1) \to 0. \tag{11.13}$$

Then for every pair (i, j) we consider the fiber product

$$\begin{tikzcd}
\hat{R}_{a,b;i,j} \arrow[r, "\hat{p}_2"] \arrow[d, "\hat{p}_4"] \arrow[dr, phantom, "\square"] & \hat{A}_{a,b;i,j} \arrow[r, "\hat{q}_2"] \arrow[d] \arrow[dr, phantom, "\square"] & \hat{Q}_{2;a;i} \arrow[d, "\hat{\theta}_{2;a;i}"] \\
\hat{R}^{4'}_{b;j} \arrow[r] \arrow[d, "\hat{q}_4"] \arrow[dr, phantom, "\square"] & \hat{U}_{a,b;i,j} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \hat{U}^2_{a;i} \arrow[d, "\hat{p}_{12}"] \\
\hat{R}^4_{b;j} \arrow[r, "\hat{\varphi}^4_{b;j}"] & \hat{U}^4_{b;j} \arrow[r, "\hat{p}_{14}"] & \hat{G}_1.
\end{tikzcd}$$

Note that actually $\hat{A}_{a,b;i,j} = Grass(2, \hat{F}_{a;i})$, where $\hat{F}_{a;i}$ is the pullback of $\hat{E}_{a;i}$ from $\hat{U}^2_{a;i}$ to $\hat{U}_{a,b;i,j}$.

Then we consider the pullback of (11.12) from $\hat{Q}_{2;a;i}$ to $\hat{R}_{a,b;i,j}$ and the pullback of (11.13) from $\hat{R}^4_{b;j}$ to $\hat{R}_{a,b;i,j}$ and we sum the resulting extensions. Then we get an extension parametrized by $\hat{R}_{a,b;i,j}$ of the form

$$0 \to (\widetilde{\mathcal{Q}}_1, \widetilde{\mathcal{W}}_1) \to (\mathcal{E}_{a,b;i,j}, \mathcal{V}_{a,b;i,j}) \to (\widetilde{\mathcal{Q}}_2, \widetilde{\mathcal{W}}_2) \otimes_{\hat{R}_{a,b;i,j}} \widetilde{\mathcal{M}}_{2;a;i} \oplus (\widetilde{\mathcal{Q}}_4, \widetilde{\mathcal{W}}_4) \to 0.$$

By using lemma 11.1.2 we get that for every $r \in \hat{R}_{a,b;i,j}$ the central term of the previous extension restricts to a coherent system (E, V) that belongs to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$. Therefore, by the universal properties of such schemes, there exists an induced morphism $\hat{\omega}_{a,b;i,j}$ from $\hat{R}_{a,b;i,j}$ to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$. Then we consider the free action of $PGL(N_1) \times PGL(N_2) \times PGL(N_4)$ on $\hat{R}_{a,b;i,j}$ and we denote by $R_{a,b;i,j}$ the quotient. The morphism $\hat{\omega}_{a,b;i,j}$ is invariant under such an action, so it induces a morphism $\omega_{a,b;i,j}$ from $R_{a,b,c;i,j}$ to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$. Such a morphism is injective by lemma 11.1.2, so we conclude. $\square$

Proof of proposition 7.3.4. Let us consider the family of schemes of the form $\hat{R}_{a,b,c;i,j,k}$ obtained in the proof of proposition 7.3.1. For the schemes of the form (a),(b) and (e) there is nothing to prove, since there are no actions, so we conclude directly. For the schemes of the form (c),(d) and (f)-(i) we can proceed as in the proof of proposition 7.3.3. Then we have only to describe the action of S_3 on the schemes of the form (j). For simplicity, let us assume that the indices a and i assume only one value. Otherwise, the proof involves more indices, but the idea is exactly the same. In that case, in the proof of proposition 7.3.1 for every $l = 2, 3, 4$ we have that

$$\hat{U}^l_{a;i} = \hat{G}_1 \times \hat{G}_2$$

and the base $\hat{U} = \hat{U}_{a,a,a;i,i,i}$ is constructed as the fiber product

$$\hat{U} = \hat{U}^2 \times_{\hat{G}_1} \hat{U}^3_{a;i} \times_{\hat{G}_1} \hat{U}^4_{a;i} = \hat{G}_1 \times \hat{G}_2 \times \hat{G}_2 \times \hat{G}_2.$$

For $l = 2, 3, 4$, let us write:

$$\hat{X}^l := \hat{R}^2_{a;i} \xrightarrow{\hat{\varphi}^l = \hat{\varphi}^2_{a;i}} \hat{U}^2_{a;i} = \hat{G}_1 \times \hat{G}_2.$$

This is the same fibration, we give 3 different names to that fibration in order to distinguish the various components of

$$\hat{R}_{a,a,a;i,i,i} = \hat{X}^2 \times_{\hat{G}_1} \hat{X}^3 \times_{\hat{G}_1} \hat{X}^4.$$

Since $\hat{\varphi}^2$ is a locally trivial fibration with fibers isomorphic to $\mathbb{P}^{a-1}$, then there exists an open covering $\{\hat{U}^2_\alpha\}_{\alpha \in A}$ of $\hat{G}_1 \times \hat{G}_2$ and trivializations

$$\hat{U}^2_\alpha \times \mathbb{P}^{a-1} \xrightarrow{\sim} \hat{X}^2|_{\hat{U}^2_\alpha}.$$

If we identify $\hat{X}^2 = \hat{X}^3 = \hat{X}^4$, we get that the same covering trivializes also $\hat{\varphi}^3$ and $\hat{\varphi}^4$. We denote those coverings of $\hat{X}^3$ and $\hat{X}^4$ by $\{\hat{U}^3_\beta\}_{\beta \in A}$ and $\{\hat{U}^4_\gamma\}_{\gamma \in A}$ respectively. Now for every triple $(\alpha, \beta, \gamma) \in A^3$ we consider

$$\begin{gathered} \hat{U}_{\alpha,\beta,\gamma} := (\hat{U}^2_\alpha \times \hat{G}_3 \times \hat{G}_4) \cap (\hat{U}^3_\beta \times \hat{G}_2 \times \hat{G}_4) \cap (\hat{U}^4_\gamma \times \hat{G}_2 \times \hat{G}_3) \subset \\ \subset \hat{G}_1 \times \hat{G}_2 \times \hat{G}_3 \times \hat{G}_4 = \hat{G}_1 \times \hat{G}_2 \times \hat{G}_2 \times \hat{G}_2. \end{gathered}$$

In the case when $\hat{U}^2_{a;i}$ does not coincide with the whole $\hat{G}_1 \times \hat{G}_2$, this should be defined as:

$$\hat{U}_{\alpha,\beta,\gamma} := \hat{U}^2_\alpha \times_{\hat{G}_1} \hat{U}^3_\beta \times_{\hat{G}_1} \hat{U}^4_\gamma.$$

The set $\{\hat{U}_{\alpha,\beta,\gamma}\}_{\alpha,\beta,\gamma \in A}$ is an open covering of $\hat{U}$. Let us consider:

$$\widetilde{U}_{\alpha,\beta,\gamma} := \hat{R}|_{\hat{U}_{\alpha,\beta,\gamma}} = \hat{X}^2|_{\hat{U}^2_\alpha} \times_{\hat{G}_1} \hat{X}^3|_{\hat{U}^3_\beta} \times_{\hat{G}_1} \hat{X}^4|_{\hat{U}^2_\gamma}.$$

Since we have trivializations as before, we get that

$$\widetilde{U}_{\alpha,\beta,\gamma} \simeq \hat{U}_{\alpha,\beta,\gamma} \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$$

and this isomorphism is compatible with the fibration $\hat{\phi}_1 \circ \hat{\phi}_2 \circ \hat{\phi}_3$. We denote by $\hat{\Delta}$ the big diagonal of $\hat{G}_2 \times \hat{G}_2 \times \hat{G}_2$, i.e. the set of all triples of objects such that at least 2 of them are isomorphic. Then we define the scheme

$$\hat{M} := \hat{R}|_{\hat{G}_1 \times (\hat{G}_2 \times \hat{G}_2 \times \hat{G}_2 \smallsetminus \hat{\Delta})}$$

For every triple (α, β, γ) we write:

$$\hat{V}_{\alpha,\beta,\gamma} := \hat{U}_{\alpha,\beta,\gamma} \cap \hat{G}_1 \times (\hat{G}_2 \times \hat{G}_2 \times \hat{G}_2 \smallsetminus \hat{\Delta}).$$

Then we have that $\hat{M}$ is covered by open subschemes $\widetilde{V}_{\alpha,\beta,\gamma}$ defined as $\widetilde{U}_{\alpha,\beta,\gamma} \cap \hat{M}$ and we have trivializations:

$$\widetilde{V}_{\alpha,\beta,\gamma} \simeq \hat{V}_{\alpha,\beta,\gamma} \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$$

compatible with the fibration $\hat{\phi}_1 \circ \hat{\phi}_2 \circ \hat{\phi}_3$. Moreover, we have a natural action of S_3 both on $\hat{G}_1 \times (\hat{G}_2 \times \hat{G}_2 \times \hat{G}_2 \smallsetminus \hat{\Delta})$ and on $\hat{M}$, and such an action is compatible with this trivializations. Now for every $\sigma \in S_3$ we have that $\sigma(\hat{V}_{\alpha,\beta,\gamma}) = \hat{V}_{\sigma(\alpha),\sigma(\beta),\sigma(\gamma)}$ where σ acts by permutations on the ordered set $\{\alpha, \beta, \gamma\}$. Then we set the following notation:

- If $\alpha \neq \gamma$, we set $\hat{W}^0_{\alpha,\gamma} := (\hat{V}_{\alpha,\alpha,\gamma} \cap \hat{V}_{\alpha,\gamma,\alpha}) \cup (\hat{V}_{\alpha,\alpha,\gamma} \cap \hat{V}_{\gamma,\alpha,\alpha}) \cup (\hat{V}_{\alpha,\gamma,\alpha} \cap \hat{V}_{\gamma,\alpha,\alpha})$. Then we define:

$$\hat{W}^3_{\alpha,\gamma} := \hat{V}_{\alpha,\alpha,\gamma} \cap \hat{V}_{\alpha,\gamma,\alpha} \cap \hat{V}_{\gamma,\alpha,\alpha},$$

$$\hat{W}^1_{\alpha,\gamma} := (\hat{V}_{\alpha,\alpha,\gamma} \cup \hat{V}_{\alpha,\gamma,\alpha} \cup \hat{V}_{\gamma,\alpha,\alpha}) \smallsetminus \hat{W}^0_{\alpha,\gamma},$$

$$\hat{W}^2_{\alpha,\gamma} := W^0_{\alpha,\gamma} \smallsetminus \hat{W}^3_{\alpha,\gamma}.$$

- If α, β, γ are 3 distinct indices, then for every $i = 1, \cdots, 6$ we define $\hat{Z}^i_{\alpha,\beta,\gamma}$ as the set of all the $t \in \hat{V}_{\sigma(\alpha),\sigma(\beta),\sigma(\gamma)}$ (for some $\sigma \in S_3$) that belong to exactly i sets of the form $\hat{V}_{\eta(\alpha),\eta(\beta),\eta(\gamma)}$ for $\eta \in S_3$. For example,

$$\hat{Z}^1_{\alpha,\beta,\gamma} := (\cup_{\sigma \in S_3} \hat{V}_{\sigma(\alpha),\sigma(\beta),\sigma(\gamma)}) \smallsetminus (\cup_{\sigma \neq \eta \in S_3} \hat{V}_{\sigma(\alpha),\sigma(\beta),\sigma(\gamma)} \cap \hat{V}_{\eta(\alpha),\eta(\beta),\eta(\gamma)})$$

and

$$\hat{Z}^6_{\alpha,\beta,\gamma} := \cap_{\sigma \in S_3} \hat{V}_{\sigma(\alpha),\sigma(\beta),\sigma(\gamma)}.$$

Each of these sets is invariant under the action of S_3 and we have that the set

$$\{\hat{T}_l\}_{l \in L} := \left\{ \{\hat{V}_{\alpha,\alpha,\alpha}\}_{\alpha \in A}, \quad \{\hat{W}^l_{\alpha,\gamma}\}_{\substack{l=1,2,3, \\ \alpha<\gamma}}, \quad \{\hat{Z}^l_{\alpha,\beta,\gamma}\}_{\substack{l=1,\cdots,6 \\ \alpha<\beta<\gamma}} \right\}$$

is a disjoint locally closed covering of $\hat{G}_1 \times (\hat{G}_2 \times \hat{G}_2 \times \hat{G}_2 \smallsetminus \Delta)$ and each of such subschemes is invariant under the action of S_3. By restricting to any subscheme $\hat{T}_l$ we have a trivialization of the fibration $\hat{\phi}_3 \circ \hat{\phi}_2 \circ \hat{\phi}_1$ and that trivialization is compatible with the action of S_3 on $T_l \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1} \times \mathbb{P}^{a-1}$. Finally, we consider the free action of $PGL(N_1) \times PGL(N_2) \times PGL(N_2) \times PGL(N_2)$ on all these schemes and we conclude. $\square$

Proof of proposition 7.3.5. The proof is on the same line of the proof of proposition 7.3.3. The only significant difference is that we need to substitute the scheme $\hat{U}_{a,b;i,j}$ in that proposition by the scheme

$$\hat{V}_{a,b;i,j} := \hat{U}_{a,b;i,j} \cap (\hat{G}_1 \times (\hat{G}_2 \times \hat{G}_2 \smallsetminus \Delta)).$$

Moreover, we have to replace the grassmannian fibration $\hat{A}_{a,b;i,j}$ over $\hat{U}_{a,b;i,j}$ by its restriction $\hat{B}_{a,b;i,j}$ over $\hat{V}_{a,b;i,j}$. Then the rest of the proof is analogous, so we omit it. $\square$

The proof of proposition 7.3.6 is on the same line of the proof of proposition 7.1.3, so we omit the details.

11.2 Canonical filtration of type (3,1)

Let us fix any object $\oplus_{i=1}^4(Q_i,W_i)$, with all the (Q_i,W_i)'s α_c-stable coherent systems with the same α_c-slope μ; let us suppose that (E,V) has such a graded at α_c and that it has α_c-canonical filtration of type (3,1). Then every (E,V) that we want to parametrize sits in an exact sequence of the form:

$$0 \to (Q_1,W_1)\oplus(Q_2,W_2)\oplus(Q_3,W_3) \xrightarrow{\alpha} (E,V) \xrightarrow{\beta} (Q_4,W_4) \to 0. \tag{11.14}$$

If (E,V) has canonical filtration of type (3,1), then it has always the following proper α_c-semistable subobjects with α_c-slope μ:

(a) (Q_i,W_i) for $i=1,2,3$;

(b) for all $i\neq j\in\{1,2,3\}$, $(Q_i,W_i)\oplus(Q_j,W_j)$;

(c) $(Q_1,W_1)\oplus(Q_2,W_2)\oplus(Q_3,W_3)$.

This is not a complete list, see lemma 11.2.1. Given any (E,V) as in (11.14) with α_c-filtration of type (3,1), the following numerical conditions are necessary in order to have that (E,V) is not destabilized for α_c^+ by subobjects of type (b) and (c):

$$\frac{k_i+k_j}{n_i+n_j}<\frac{k}{n}\quad\forall\, i\neq j\in\{1,2,3\},\qquad \frac{k_1+k_2+k_3}{n_1+n_2+n_3}<\frac{k}{n}. \tag{11.15}$$

Actually, both conditions are implied by

$$\frac{k_i}{n_i}<\frac{k}{n}\quad\forall\, i\in\{1,2,3\} \tag{11.16}$$

and these conditions ensure also that (E,V) is not destabilized also by subobjects of type (a). Therefore we get that conditions (11.16) are necessary (but in general not sufficient) in order to have that (E,V) belongs to $G^+(\alpha_c;n,d,k)$: Analogously, the following numerical conditions are necessary (but in general not sufficient) in order to have that (E,V) belongs to $G^-(\alpha_c;n,d,k)$:

$$\frac{k_i}{n_i}>\frac{k}{n}\quad\forall\, i\in\{1,2,3\}. \tag{11.17}$$

Lemma 11.2.1. *Given any (E,V) as in (11.14) with conditions (11.16), respectively (11.17), then (E,V) belongs to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, if and only if there are no quotients $\zeta_i:(E,V)\twoheadrightarrow(Q_i,W_i)$ for $i=1,2,3$. Moreover, if this happens, then (E,V) has α_c-canonical filtration of type $(3,1)$.*

Proof. Let us suppose that we use conditions (11.16), the other case is completely analogous. If there is any quotient ζ_i as in the claim, then the kernel (E', V') of ζ_i is an α_c-semistable subsystem of (E, V) with $k' = k - k_i$ and $n' = n - n_i$. Since $\mu_{\alpha_c}(E, V) = \mu_{\alpha_c}(E', V')$, using (11.16) we get that $\frac{k'}{n'} > \frac{k}{n}$, so (E, V) cannot be α_c^+-stable.

Conversely, if (E, V) is not α_c^+-stable, then there exists a subsystem (E', V') that destabilizes it for α_c^+. Using (11.16) and (11.15), the graded of (E', V') cannot contain only some (possibly all) objects of the form (Q_i, W_i) for $i \in \{1, 2, 3\}$, so it contains (Q_4, W_4). Therefore the quotient $(E'', V'') := (E, V)/(E', V')$ contains only some (possibly all) objects of the form (Q_i, W_i) for $i \in \{1, 2, 3\}$. If we consider a Jordan-Hölder filtration $(E''_l, V''_l)_{l=1,\cdots,t}$ of (E'', V''), we get that $(E'', V'')/(E''_{t-1}, V''_{t-1})$ is isomorphic to some (Q_i, W_i) for $i \in \{1, 2, 3\}$, so we get a quotient $(E, V) \twoheadrightarrow (E, V)/(E', V') = (E'', V'') \twoheadrightarrow (Q_i, W_i)$.

Now let us assume that there are no quotients $\zeta_i : (E, V) \twoheadrightarrow (Q_i, W_i)$ for $i = 1, 2, 3$; we want to prove that the α_c-canonical filtration of (E, V) is of type $(3, 1)$. So let us consider the filtration of (E, V) given as follows:

$$0 = (E_0, V_0) \subset (E_1, V_1) := (Q_1, W_1) \oplus (Q_2, W_2) \oplus (Q_3, W_3) \subset (E_2, V_2) = (E, V). \tag{11.18}$$

Here $(E_2, V_2)/(E_1, V_1) = (Q_4, W_4)$ is α_c-stable and $(E_1, V_1)/(E_0, V_0) = (E_1, V_1)$ is α_c-polystable. Then by proposition 2.1.3 we get that (11.18) is the α_c-canonical filtration of (E, V) (and so (E, V) has α_c-canonical filtration of type $(3, 1)$) if and only if condition (c) of that proposition is satisfied. In our case the index t is equal to 2, so (11.18) is the α_c-canonical filtration of (E, V) if and only if for all $i = 1, \cdots, 4$ and for all non-zero morphisms $\gamma_i : (Q_i, W_i) \to (E, V)$ we have $\beta \circ \gamma_i = 0$. Now by hypothesis $(Q_4, W_4) \not\simeq (Q_i, W_i)$ for all $i = 1, 2, 3$. Since all the (Q_i, W_i)'s for $i = 1, \cdots, 4$ are α_c-stable of the same slope, then for all $i = 1, 2, 3$ and for all $\gamma_i : (Q_i, W_i) \to (E, V)$ we have that $\beta \circ \gamma_i = 0$. Then we conclude that for every (E, V) as in (11.14) the following conditions are equivalent:

(a) (11.18) is the α_c-canonical filtration of (E, V);

(b) for all morphisms $\gamma_4 : (Q_4, W_4) \to (E, V)$ we have $\beta \circ \gamma_4 = 0$.

Now proving (b) is equivalent to proving that the sequence (11.14) is non-split. By contradiction, let us suppose that it is split. Then for every $i = 1, 2, 3$ we can write morphisms of the form

$$\zeta_i : (E, V) \simeq \oplus_{l=1}^4 (Q_l, W_l) \twoheadrightarrow (Q_i, W_i).$$

But this is impossible in our hypothesis, so we conclude. □

Now if we denote by μ any extension like (11.14), we get that we can identify μ with a triple

$$(\mu_1,\mu_2,\mu_3)\in\bigoplus_{i=1}^{3}\mathrm{Ext}^1\Big((Q_4,W_4),(Q_i,W_i)\Big).$$

For every $i=1,2,3$, this identification gives a diagram of the form:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{l=1}^{3}(Q_l,W_l) \arrow[r, "\alpha"] \arrow[d, "pr_i"'] \arrow[dr, phantom, "\curvearrowright"] & (E,V) \arrow[r, "\beta"] \arrow[d, "\eta_i"'] \arrow[dr, phantom, "\curvearrowright"] & (Q_4,W_4) \arrow[r] \arrow[d, equal] & 0 & \mu \arrow[d, "\overline{pr_i}"] \\
0 \arrow[r] & (Q_i,W_i) \arrow[r, "\alpha_i"] & (E_{4i},V_{4i}) \arrow[r, "\beta_i"] & (Q_4,W_4) \arrow[r] & 0 & \mu_i,
\end{tikzcd} \tag{11.19}$$

where pr_i is the quotient $\oplus_{l=1}^{3}(Q_l,W_l)\twoheadrightarrow(Q_i,W_i)$ for every $i=1,2,3$. Then we have the following results.

Lemma 11.2.2. *Let us fix any quadruple $(Q_i,W_i)_{i=1,\cdots,4}\in\prod_{i=1}^{4}G_i$ with numerical conditions (11.16), respectively (11.17), and let us suppose that $(Q_i,W_i)\not\simeq(Q_j,W_j)$ for all $i\neq j\in\{1,2,3\}$. Then the (E,V)'s that belong to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, that have graded $\oplus_{i=1}^{4}(Q_i,W_i)$ and canonical filtration of type $(3,1)$ are parametrized by $\mathbb{P}(H_1)\times\mathbb{P}(H_2)\times\mathbb{P}(H_3)$, where $H_i:=Ext^1((Q_4,W_4),(Q_i,W_i))$ for $i=1,2,3$.*

Proof. To any (E,V) that we want to parametrize we can associate a triple (μ_1,μ_2,μ_3). Using the previous lemma, the following facts are equivalent

(a) (E,V) belongs to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, and it has α_c-canonical filtration of type $(3,1)$;

(b) there are no quotients $\zeta_i:(E,V)\twoheadrightarrow(Q_i,W_i)$ for $i=1,2,3$.

Since $(Q_4,W_4)\not\simeq(Q_i,W_i)$ for $i=1,2,3$, then (b) is equivalent to

(d) for all quotients $\zeta_i:(E,V)\twoheadrightarrow(Q_i,W_i)$ for $i=1,2,3$ we have that $\alpha\circ\zeta_i=0$.

Since $(Q_i,W_i)\not\simeq(Q_j,W_j)$ for all $i\neq j$ in $\{1,2,3\}$, then by lemma 3.3.2 we get that (c) is equivalent to

(d) $\mu_i\neq 0$ for all $i=1,2,3$.

Now if we look at the sequence (11.14), we get that $\mathrm{Aut}(Q_4,W_4)=\mathbb{C}^*$ and $\mathrm{Aut}((Q_1,W_1)\oplus(Q_2,W_2)\oplus(Q_3,W_3))=\mathbb{C}^*\times\mathbb{C}^*\times\mathbb{C}^*$ (because $(Q_i,W_i)\not\simeq(Q_j,W_j)$ for all $i\neq j\in\{1,2,3\}$), so we conclude. □

Lemma 11.2.3. *Let us fix any quadruple $(Q_i,W_i)_{i=1,\cdots,4}\in\prod_{i=1}^{4}G_i$ with numerical conditions (11.16), respectively (11.17), and let us suppose that $(Q_1,W_1)\simeq(Q_2,W_2)\not\simeq(Q_3,W_3)$. Then the (E,V)'s that belong to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, that have graded $\oplus_{i=1}^{4}(Q_i,W_i)$ and α_c-canonical filtration of type $(3,1)$ are parametrized by $Grass(2,H_1)\times\mathbb{P}(H_3)$, where $H_i=Ext^1\left((Q_4,W_4),(Q_i,W_i)\right)$ for $i=1,3$*

Proof. As in the previous proof, we get that for any (E, V) that sits in a sequence (11.14) with conditions (11.16), respectively (11.17), the following facts are equivalent:

(a) (E, V) belongs to $G^{+}(\alpha_c; n, d, k)$, respectively to $G^{-}(\alpha_c; n, d, k)$, and it has α_c-canonical filtration of type (3,1);

(b) for all quotients $\zeta_i : (E, V) \twoheadrightarrow (Q_i, W_i)$ for $i = 1, 3$ we have $\alpha \circ \zeta = 0$.

In this case $(Q_1, W_1) \simeq (Q_2, W_2)$, so we can ignore the case $i = 2$ in (b). Using lemma 3.3.2, we get that (b) is equivalent to

(c) $\mu_3 \neq 0$ and μ_1, μ_2 linearly independent in H_1.

Now if we consider the exact sequence (11.14) with $(Q_1, W_1) \simeq (Q_2, W_2) \not\simeq (Q_3, W_3)$, we get that $\mathrm{Aut}(Q_4, W_4) = \mathbb{C}^*$, while $\mathrm{Aut}((Q_1, W_1) \oplus (Q_1, W_1) \oplus (Q_3, W_3)) = \mathrm{GL}(2, \mathbb{C}) \times \mathbb{C}^*$, so we conclude. □

Lemma 11.2.4. *Let us fix any quadruple $(Q_i, W_i)_{i=1,\cdots,4} \in \prod_{i=1}^{4} G_i$ with numerical conditions (11.16), respectively (11.17), and let us suppose that $(Q_1, W_1) \simeq (Q_2, W_2) \simeq (Q_3, W_3)$. Then the (E, V)'s that belong to $G^{+}(\alpha_c; n, d, k)$, respectively to $G^{-}(\alpha_c; n, d, k)$, that have graded $\oplus_{i=1}^{4}(Q_i, W_i)$ and α_c-canonical filtration of type $(3, 1)$ are parametrized by Grass$(3, H_1)$, where we set $H_1 = \mathit{Ext}^1 ((Q_4, W_4), (Q_1, W_1))$.*

Proof. As in the previous proof, we get that for any (E, V) that sits in a sequence (11.14) with conditions (11.16), respectively (11.17), the following facts are equivalent.

(a) (E, V) belongs to $G^{+}(\alpha_c; n, d, k)$, respectively to $G^{-}(\alpha_c; n, d, k)$ and it has α_c-canonical filtration of type $(3, 1)$;

(b) for all quotients $\zeta_1 : (E, V) \twoheadrightarrow (Q_1, W_1)$ we have $\zeta_1 \circ \alpha = 0$.

In this case, $(Q_1, W_1) \simeq (Q_2, W_2) \simeq (Q_3, W_3)$, so we can ignore the cases $i = 2, 3$ in (b). Since $(Q_1, W_1) \simeq (Q_2, W_2) \simeq (Q_3, W_3)$, by lemma 3.3.2 we have that (b) is equivalent to

(c) μ_1, μ_2 and μ_3 are linearly independent in H_1.

Now if we consider the exact sequence (11.14) with $(Q_1, W_1) \simeq (Q_2, W_2) \simeq (Q_3, W_3)$, we get that $\mathrm{Aut}(Q_4, W_4) = \mathbb{C}^*$, while $\mathrm{Aut}((Q_1, W_1) \oplus (Q_1, W_1) \oplus (Q_1, W_1)) = \mathrm{GL}(3, \mathbb{C})$, so we conclude. □

Now we give a global parametrization of the objects described before, i.e. we describe families of schemes that parametrize various types of (E, V)'s when the graded $\oplus_{i=1}^{4}(Q_i, W_i)$ varies over $\prod_{i=1}^{4} G_i$ and the α_c-canonical filtration is of type (3,1). Since the order of the objects (Q_i, W_i) for $i = 1, 2, 3$ is not important, we can assume that we have fixed any order that satisfies the following properties:

- if $(n_i, k_i) \neq (n_j, k_j)$ for $i \neq j \in \{1, 2, 3\}$, then we use the lexicographic order on the set $\{(n_i, k_i)\}_{i=1,2,3}$;

- if exactly 2 (n_i, k_i)'s are equal for $i \in \{1,2,3\}$, then we assume that they are (n_1, k_1) and (n_2, k_2); in this case we have automatically that $d_1 = d_2$;
- if all the (n_i, k_i)'s are equal for $i \in \{1,2,3\}$ (and so also $d_1 = d_2 = d_3$) and if exactly 2 among the corresponding (Q_i, W_i)'s are isomorphic, we order them so that $(Q_1, W_1) \simeq (Q_2, W_2)$.

Let us write

$$H_i := \operatorname{Ext}^1\left((Q_4, W_4), (Q_i, W_i)\right) \quad \forall\, i = 1,2,3.$$

Then we need to distinguish the following subcases:

(1) If $(n_1, k_1) \neq (n_2, k_2) \neq (n_3, k_3)$ (this implies that $(n_1, k_1) \neq (n_3, k_3)$ since we are using the lexicographic order), then having fixed the graded, the corresponding (E, V)'s are in bijection with the points of $\mathbb{P}(H_1) \times \mathbb{P}(H_2) \times \mathbb{P}(H_3)$.

(2) If $(n_1, k_1) = (n_2, k_2) \neq (n_3, k_3)$ and $(Q_1, W_1) \not\simeq (Q_2, W_2)$, then the corresponding (E, V)'s are parametrized by $H_{12} \times \mathbb{P}(H_3)$, where $H_{12} := (\mathbb{P}(H_1) \times \mathbb{P}(H_2))/\mathbb{Z}_2$.

(3) If $(n_1, k_1) = (n_2, k_2) \neq (n_3, k_3)$ and $(Q_1, W_1) \simeq (Q_2, W_2)$, then $H_1 = H_2$ and the corresponding (E, V)'s are parametrized by $Grass(2, H_1) \times \mathbb{P}(H_3)$.

(4) Let us assume that $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$ and $(Q_1, W_1) \not\simeq (Q_2, W_2)$ (using the hypothesis on the ordering, this implies that (Q_3, W_3) is not isomorphic to (Q_i, W_i) for $i = 1, 2$). Then the corresponding (E, V)'s are parametrized by $(\mathbb{P}(H_1) \times \mathbb{P}(H_2) \times \mathbb{P}(H_3))/S_3$.

(5) Let us assume that $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$ and $(Q_1, W_1) \simeq (Q_2, W_2) \not\simeq (Q_3, W_3)$. Then $H_1 = H_2$ and the corresponding (E, V)'s are parametrized by $Grass(2, H_1) \times \mathbb{P}(H_3)$.

(6) If $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$ and $(Q_1, W_1) \simeq (Q_2, W_2) \simeq (Q_3, W_3)$, then $H_1 = H_2 = H_3$ and the corresponding (E, V)'s are parametrized by $Grass(3, H_1)$.

Note that in this way cases (3) and (5) coincide if we fix the graded. However, we will have to give different global descriptions for them because the base spaces we will work on will be different.

The previous 6 cases are taken into account by propositions 7.4.1, 7.4.2, 7.4.3, 7.4.4, 7.4.5 and 7.4.6 respectively. The proofs of those results are omitted since they are analogous to those of the previous section. The only significant difference is that we use lemmas 11.2.2, 11.2.3 and 11.2.4 instead of lemmas 11.1.1, 11.1.2 and 11.1.3 respectively.

Chapter 12

Parametrization of objects with canonical filtration of type (2,1,1), (1,2,1) and (1,1,2)

Having fixed any triple (n, d, k) and a critical value α_c for it, in this chapter we want to describe how to parametrize those (E, V)'s that have α_c-canonical filtration of type (2,1,1), (1,2,1) and (1,1,2) and that belong to $G^+(\alpha_c; n, d, k)$ or to $G^-(\alpha_c; n, d, k)$.

Remark 12.0.1. We have a complete pointwise description for all the cases (2,1,1), (1,2,1) and (1,1,2), as described below. Regrettably, we are able to get complete global results only for the cases (2,1,1) and (1,1,2); at the moment it is possible to get explicit results only for 4 of the 8 subcases involved in the case (1,2,1) (see below for the details).

Remark 12.0.2. We will give the pointwise descriptions for every quadruple $(n_i, k_i)_{i=1,\cdots,4}$ (with the additional numerical conditions such that the corresponding (E, V)'s belong to $G^+(\alpha_c; n, d, k)$ or to $G^-(\alpha_c; n, d, k)$). Anyway, we will need the results of this chapter only for the case when $n = 4$ and $k = 1$; in this case the subschemes $G^+(\alpha_c; n, d, k)$ will be associated to a quadruple $(n_i, k_i)_{i=1,\cdots,4}$ such that

$$(1, 0) = (n_1, k_1) = (n_2, k_2) = (n_3, k_3) \neq (n_4, k_4) = (1, 1);$$

analogously, $G^-(\alpha_c; n, d, k)$ will be associated to a quadruple $(n_i, k_i)_{i=1,\cdots,4}$ such that

$$(1, 1) = (n_1, k_1) \neq (n_2, k_2) = (n_3, k_3) = (n_4, k_4) = (1, 0).$$

So we will give explicitly the global results by restricting to the case when $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$ in case of α_c-canonical filtration of type (2,1,1) and in the first subcase associated to α_c-canonical filtrations of type (1,2,1). We will restrict to the condition that $(n_2, k_2) = (n_3, k_3) = (n_4, k_4)$ in case of α_c-canonical filtration of type (1,1,2) and in the remaining subcases associated to α_c-canonical filtrations of type (1,2,1).

12.1 Canonical filtration of type (2,1,1)

If (E,V) has α_c-canonical filtration of type (2,1,1), then its α_c-canonical filtration is given by:

$$0 \subset (E_1,V_1) \subset (E_2,V_2) \subset (E_3,V_3) = (E,V)$$

where $(E_1,V_1) \simeq (Q_1,W_1) \oplus (Q_2,W_2)$; we write $(Q_3,W_3) := (E_2,V_2)/(E_1,V_1)$ and $(Q_4,W_4) := (E,V)/(E_2,V_2)$. All the (Q_i,W_i)'s for $i = 1,\cdots,4$ are α_c-stable coherent systems with the same α_c-slope μ. Then we can associate to every (E,V) that we want to parametrize a pair of exact sequences of the form:

$$0 \to (Q_3,W_3) \xrightarrow{\alpha_2} (E'',V'') \xrightarrow{\beta_2} (Q_4,W_4) \to 0; \tag{12.1}$$

$$0 \to (Q_1,W_1) \oplus (Q_2,W_2) \xrightarrow{\alpha_1} (E,V) \xrightarrow{\beta_1} (E'',V'') \to 0. \tag{12.2}$$

We denote by μ and ν the classes of those 2 exact sequences. If (E,V) has α_c-canonical filtration of type (2,1,1), then it has always the following proper α_c-semistable subobjects with α_c-slope μ:

(a) (Q_i,W_i) for $i = 1,2$;

(b) $(Q_1,W_1) \oplus (Q_2,W_2)$;

(c) an extension of (Q_3,W_3) by $(Q_1,W_1) \oplus (Q_2,W_2)$.

This is not a complete list, see lemma 12.1.2 and remark 12.1.1 for a complete list. If we consider only the subobjects (a) and (c), we get that the following conditions are necessary (but in general not sufficient) in order to have that (E,V) as in (12.2) belongs to $G^+(\alpha_c;n,d,k)$:

$$\frac{k_i}{n_i} < \frac{k}{n} \quad \forall\, i \in \{1,2\}, \qquad \frac{k_1+k_2+k_3}{n_1+n_2+n_3} < \frac{k}{n}. \tag{12.3}$$

We remark that the first condition implies that

$$\frac{k_1+k_2}{n_1+n_2} < \frac{k}{n}, \tag{12.4}$$

so also the subobject of type (b) does not destabilize (E,V) for α_c^+. Analogously, we get that the following conditions are necessary (but in general not sufficient) in order to have that (E,V) belongs to $G^-(\alpha_c;n,d,k)$:

$$\frac{k_i}{n_i} > \frac{k}{n} \quad \forall\, i \in \{1,2\}, \qquad \frac{k_1+k_2+k_3}{n_1+n_2+n_3} > \frac{k}{n}. \tag{12.5}$$

Now let us consider the following long exact sequence obtained by applying the functor $\mathrm{Hom}(-,(Q_1,W_1)\oplus(Q_2,W_2))$ to (12.1):

$$\cdots \to \mathrm{Ext}^1((Q_4,W_4),(Q_1,W_1)\oplus(Q_2,W_2)) \xrightarrow{\overline{\beta_2}} \mathrm{Ext}^1((E'',V''),(Q_1,W_1)\oplus(Q_2,W_2)) \xrightarrow{\overline{\alpha_2}}$$

$$\xrightarrow{\overline{\alpha_2}} \mathrm{Ext}^1((Q_3, W_3), (Q_1, W_1) \oplus (Q_2, W_2)) \to \cdots \tag{12.6}$$

If we apply $\overline{\alpha_2}$ to ν we get a diagram of this form:

$$\begin{tikzcd}
0 \arrow[r] & \oplus_{i=1}^2 (Q_i, W_i) \arrow[r, "\alpha_1"] \arrow[d, equal] \arrow[dr, phantom, "\curvearrowright"] & (E,V) \arrow[r, "\beta_1"] \arrow[dr, phantom, "\curvearrowright"] & (E'', V'') \arrow[r] & 0 & \nu \arrow[d, "\overline{\alpha_2}"] \\
0 \arrow[r] & \oplus_{i=1}^2 (Q_i, W_i) \arrow[r, "\alpha_1'"] & (E_2, V_2) \arrow[r, "\beta_1'"] \arrow[u, hook, "\delta"'] & (Q_3, W_3) \arrow[r] \arrow[u, hook, "\alpha_2"'] & 0 & \overline{\alpha_2}(\nu).
\end{tikzcd} \tag{12.7}$$

By the snake lemma and (12.1), we have an induced short exact sequence

$$0 \to (E_2, V_2) \xrightarrow{\delta} (E, V) \xrightarrow{\eta} (Q_4, W_4) \to 0. \tag{12.8}$$

We can identify ν with a pair

$$(\nu_1, \nu_2) \in \bigoplus_{i=1}^{2} \mathrm{Ext}^1\Big((E'', V''), (Q_i, W_i)\Big).$$

For every $i = 1, 2$, this identification gives a diagram of the form:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{l=1}^2 (Q_l, W_l) \arrow[r, "\alpha_1"] \arrow[d, two heads, "pr_i"'] \arrow[dr, phantom, "\curvearrowright"] & (E,V) \arrow[r, "\beta_1"] \arrow[d, two heads, "\eta_i"'] \arrow[dr, phantom, "\curvearrowright"] & (E'', V'') \arrow[r] \arrow[d, equal] & 0 & \nu \arrow[d, "\overline{pr_i}"] \\
0 \arrow[r] & (Q_i, W_i) \arrow[r, "\alpha_{43i}"] & (E_{43i}, V_{43i}) \arrow[r, "\beta_{43i}"] & (E'', V'') \arrow[r] & 0 & \nu_i,
\end{tikzcd} \tag{12.9}$$

where pr_i is the quotient $\oplus_{l=1}^2 (Q_l, W_l) \twoheadrightarrow (Q_i, W_i)$ for $i = 1, 2$. If we denote by j the index in $\{1, 2\}$ different from i, then the snake lemma proves that we have a short exact sequence

$$0 \to (Q_j, W_j) \xrightarrow{\delta_i} (E, V) \xrightarrow{\eta_i} (E_{43i}, V_{43i}) \to 0. \tag{12.10}$$

Having fixed all those notations, let us state and prove the following results.

Lemma 12.1.1. *Let us fix any pair of exact sequences as (12.1) and (12.2), let us denote by μ and ν their classes and let us suppose that $(Q_4, W_4) \not\simeq (Q_i, W_i) \quad \forall\, i \in \{1, 2, 3\}$. Then the following facts are equivalent.*

(a) (E, V) *has α_c-canonical filtration of type* $(2, 1, 1)$*;*

(b) $\mu \neq 0$ *and* $\overline{\alpha_2}(\nu) \neq 0$.

Proof. Let us assume (b) and let us prove that (a) holds. By definition of $\overline{\alpha_2}(\nu)$, we have a diagram as (12.7). We claim that the α_c-canonical filtration of (E,V) is given by

$$0 \subset (Q_1,W_1) \oplus (Q_2,W_2) =: (E_1,V_1) \subset (E_2,V_2) \subset (E_3,V_3) = (E,V). \tag{12.11}$$

The second line of (12.7) proves that $(E_2,V_2)/(E_1,V_1) = (Q_3,W_3)$; moreover, by (12.8) we get that $(E,V)/(E_2,V_2) \simeq (Q_4,W_4)$. So for all $i = 1,2,3$ the objects of the form $(E_i,V_i)/(E_{i-1},V_{i-1})$ are α_c-stable or polystable. Then by proposition 2.1.3 we get that (12.11) is the α_c-canonical filtration of (E,V) if and only if condition (c) of that proposition is satisfied. In our case the index t is equal to 3, $gr(E,V) = \oplus_{i=1}^4 (Q_i,W_i)$ and

$$gr((E,V)/(E_1,V_1)) = gr(E'',V'') = (Q_3,W_3) \oplus (Q_4,W_4).$$

So (12.11) is the α_c-canonical filtration of (E,V) if and only if the following two conditions hold:

(i) for all $i = 1,\cdots,4$ and for all morphisms $\gamma_i : (Q_i,W_i) \to (E,V)$ we have $\beta_1 \circ \gamma_i = 0$;

(ii) for all $i = 3,4$ and for all morphisms $\widetilde{\gamma}_i : (Q_i,W_i) \to (E'',V'')$ we have $\beta_2 \circ \widetilde{\gamma}_i = 0$.

Let us consider (ii): here we can ignore $\widetilde{\gamma}_3$ because $(Q_3,W_3) \not\simeq (Q_4,W_4)$ by hypothesis. Moreover, there is a morphism $\widetilde{\gamma}_4$ such that $\beta_2 \circ \widetilde{\gamma}_4 \neq 0$ if and only if (12.1) is split, i.e. if and only if $\mu = 0$. Now let us consider (i): let us suppose that there is any non-zero morphism $\gamma_4 : (Q_4,W_4) \to (E,V)$ and let us write $\widetilde{\gamma}_4 := \beta_1 \circ \gamma_4$. Since $(Q_4,W_4) \not\simeq (Q_i,W_i)$ for $i = 1,2,3$, then necessarily we have that $\beta_2 \circ \widetilde{\gamma}_4 \neq 0$, so we can apply what we said in the previous lines.

Moreover, if in (i) there exists any morphism γ_i for $i = 1$ or 2, such that $\beta_1 \circ \gamma_i \neq 0$, then this implies that we have a non-zero morphism from (Q_i,W_i) to (E'',V''). Since the graded of (E'',V'') is $(Q_3,W_3) \oplus (Q_4,W_4)$ and since (Q_4,W_4) is not isomorphic to (Q_i,W_i) for $i = 1,2$, then we have that necessarily $(Q_i,W_i) \simeq (Q_3,W_3)$; so the morphism γ_i that we are considering is a morphism of the form γ_3.

So (12.11) is the α_c-canonical filtration of (E,V) if and only if the following two conditions hold:

- for all morphisms $\gamma_3 : (Q_3,W_3) \to (E,V)$ we have $\beta_1 \circ \gamma_3 = 0$;
- $\mu \neq 0$.

Now let us suppose that we have a morphism $\gamma_3 : (Q_3,W_3) \to (E,V)$ such that $\beta_1 \circ \gamma_3 \neq 0$ and let us consider (12.8). We have that necessarily $\eta \circ \gamma_3 = 0$ because (Q_3,W_3) and (Q_4,W_4) are α_c-stable coherent systems and they are not isomorphic. Then by exactness of (12.8) we get an induced non-zero morphism $\gamma_3' : (Q_3,W_3) \to (E_2,V_2)$ such that $\gamma_3 = \delta \circ \gamma_3'$. Now by definition of γ_3' and by commutativity of (12.7) we have:

$$\alpha_2 \circ \beta_1' \circ \gamma_3' = \beta_1 \circ \delta \circ \gamma_3' = \beta_1 \circ \gamma_3 \neq 0.$$

So we get that in particular $\beta_1' \circ \gamma_3' \neq 0$, so it belongs to $\mathrm{Aut}(Q_3, W_3) = \mathbb{C}^*$. Therefore γ_3' gives a splitting of the second line of (12.7), so $\overline{\alpha_2}(\nu) = 0$.

So we conclude that if (12.11) is the α_c-canonical filtration of (E, V), then $\mu \neq 0$ and $\overline{\alpha_2}(\nu) \neq 0$, so (b) is verified. Conversely, if $\mu = 0$, then the α_c-canonical filtration of (E, V) is of type (2,2), (3,1) or (4), but not of type (2,1,1). If $\overline{\alpha_2}(\nu) = 0$, then the second line of (12.7) is split, so

$$(E, V) \supset (E_2, V_2) \simeq (Q_1, W_1) \oplus (Q_2, W_2) \oplus (Q_3, W_3).$$

So the α_c-canonical filtration of (E, V) is of type (3,1) or (4), but not of type (2,1,1). So we have proved that (a) and (b) are equivalent. $\square$

Lemma 12.1.2. *Let us fix any pair of exact sequences as (12.1) and (12.2) with conditions (12.3), respectively (12.5). Moreover, let us suppose that $(Q_4, W_4) \not\simeq (Q_i, W_i)\, \forall\, i \in \{1, 2, 3\}$ and that*

$$\frac{k_i + k_3}{n_i + n_3} < \frac{k}{n} \quad \forall\, i \in \{1, 2\}, \quad \textit{resp.} \quad \frac{k_i + k_3}{n_i + n_3} > \frac{k}{n} \quad \forall\, i \in \{1, 2\}. \tag{12.12}$$

Let us also suppose that (E, V) has α_c-canonical filtration of type $(2, 1, 1)$. Then the following facts are equivalent.

(a) (E, V) belongs to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$;

(b) there are no quotients $\zeta_i : (E, V) \twoheadrightarrow (Q_i, W_i)$ for $i = 1, 2$.

Proof. Let us assume conditions (12.3) and the first part of (12.12); the other case is completely analogous.

If there is any quotient $\zeta_i : (E, V) \twoheadrightarrow (Q_i, W_i)$ for some $i \in \{1, 2\}$, then the kernel (E', V') of ζ_i is an α_c-semistable subsystem of (E, V) with the same α_c-slope as (E, V) and with $k' = k - k_i$, $n' = n - n_i$. Since $\mu_{\alpha_c}(E, V) = \mu_{\alpha_c}(E', V')$, using (12.3) we get that $\frac{k'}{n'} > \frac{k}{n}$, so (E, V) cannot be α_c^+-stable.

Conversely, if (E, V) is not α_c^+-stable, then there exists a proper subsystem (E', V') of (E, V) that destabilizes it for α_c^+. Such an object is necessarily α_c-semistable with the same α_c-slope as (E, V); the length of a Jordan-Hölder filtration of (E', V') can be equal to 1, 2 or 3, so we have to handle all these cases. By definition of α_c-canonical filtration of type (2,1,1), we have that (Q_1, W_1) and (Q_2, W_2) are the only α_c-stable subobjects of (E, V); therefore we have necessarily that $(Q_i, W_i) \subset (E', V')$ for some $i = 1, 2$.

Length of any α_c-Jordan-Hölder filtration of (E', V') equal to 1. In this case (E', V') coincides with (Q_1, W_1) or (Q_2, W_2), that don't destabilize (E, V) because we are using (12.3).

Length of any α_c-Jordan-Hölder filtration of (E',V') equal to 2. If (E',V') is an extension of (Q_2,W_2) by (Q_1,W_1) or conversely, then it does not destabilize (E,V) because of (12.4). If (E',V') is an extension of (Q_3,W_3) by (Q_i,W_i) for $i=1$ or 2, then (E',V') does not destabilize (E,V) for α_c^+ since we are using the first part of (12.12).

Lastly, we have to consider the case when (E',V') sits in a non-split exact sequence of the form:

$$0 \to (Q_i,W_i) \xrightarrow{\alpha} (E',V') \xrightarrow{\beta} (Q_4,W_4) \to 0 \tag{12.13}$$

for some $i=1,2$; we denote by γ the inclusion of (E',V') in (E,V). Let us consider the exact sequences (12.1) and (12.2): if $\beta_1 \circ \gamma = 0$, this induces an injective morphism $\gamma' : (E',V') \to (Q_1,W_1) \oplus (Q_2,W_2)$ such that $\gamma = \alpha_1 \circ \gamma'$, but this is impossible since the graded of (E',V') contains (Q_4,W_4) and $(Q_i,W_i) \not\simeq (Q_4,W_4)$ for $i=1,2$. Therefore, we have that $\beta_1 \circ \gamma : (E',V') \to (E'',V'')$ is non-zero. Then we have to consider 2 subcases as follows

- If $\beta_1 \circ \gamma \circ \alpha = 0$, then by exactness of (12.13), we get an induced morphism $\gamma'' : (Q_4,W_4) \to (E'',V'')$ such that $\gamma'' \circ \beta = \beta_1 \circ \gamma \neq 0$, so in particular $\gamma'' \neq 0$. Since $(Q_3,W_3) \not\simeq (Q_4,W_4)$, then γ'' gives a splitting of (12.1), so $\mu = 0$, but this is impossible because of lemma 12.1.1. Therefore, this case cannot happen.

- If $\beta_1 \circ \gamma \circ \alpha \neq 0$, then such a morphism is injective and we get that (Q_i,W_i) is contained in $gr_{\alpha_c}(E'',V'') = (Q_3,W_3) \oplus (Q_4,W_4)$. Since $(Q_4,W_4) \not\simeq (Q_i,W_i)$, we conclude that in this case $(Q_i,W_i) \simeq (Q_3,W_3)$. Then we have

  $$\frac{k_i+k_4}{n_i+n_4} = \frac{k_3+k_4}{n_3+n_4} > \frac{k}{n}$$

  because of (12.4). Therefore, the object (E',V') destabilizes (E,V) for α_c^+. In this case, the morphism γ induces an exact sequence

  $$0 \to (E',V') \xrightarrow{\gamma} (E,V) \xrightarrow{\zeta} (\widetilde{E},\widetilde{V}) \to 0.$$

  By looking at the graded objects associated to the first 2 coherent systems we get that $gr_{\alpha_c}(\widetilde{E},\widetilde{V}) = (Q_j,W_j) \oplus (Q_3,W_3)$ where j is the index in $\{1,2\}$ different from i. In this case we have already prove that $(Q_i,W_i) \simeq (Q_3,W_3)$, so

  $$gr_{\alpha_c}(\widetilde{E},\widetilde{V}) \simeq (Q_i,W_i) \oplus (Q_j,W_j) = (Q_1,W_1) \oplus (Q_2,W_2).$$

  In particular, we have a quotient $\zeta' : (\widetilde{E},\widetilde{V}) \twoheadrightarrow (Q_i,W_i)$ for some $i=1,2$. By composing such a quotient with ζ we conclude that we have a quotient $\zeta_i : (E,V) \twoheadrightarrow (Q_i,W_i)$ for some $i=1,2$.

So we conclude that if there exists a suboject (E', V') of (E, V) with length of any α_c-Jordan-Hölder filtration equal to 2, that destabilizes (E, V) for α_c^+, then there exists a quotient $\zeta_i : (E, V) \twoheadrightarrow (Q_i, W_i)$ for some $i = 1, 2$.

Length of any α_c-Jordan-Hölder filtration of (E', V') **equal to** 3. In this case we denote by $(\widetilde{E}, \widetilde{V})$ the quotient $(E, V)/(E', V')$. This is a coherent system that is α_c-stable. If it equal to (Q_4, W_4), then (E', V') does not destabilize (E, V) for α_c^+ because of (12.3). If it is equal to (Q_i, W_i) for some $i = 1, 2$, then (E', V') do destabilize (E, V) for α_c^+.

Lastly, if $(\widetilde{E}, \widetilde{V})$ is equal to (Q_3, W_3), then we denote by $\zeta_3 : (E, V) \twoheadrightarrow (Q_3, W_3)$ the quotient and we distinguish 2 cases as follows.

- If $(Q_3, W_3) \not\simeq (Q_i, W_i)$ for some $i = 1, 2$, we consider the exact sequence (12.2) and we get that necessarily there is an induced quotient $\zeta_3' : (E'', V'') \twoheadrightarrow (Q_3, W_3)$ such that $\zeta_3 = \zeta_3' \circ \beta_1$. Then we consider (12.1): since $(Q_4, W_4) \not\simeq (Q_3, W_3)$, then we get that $\zeta_3' \circ \alpha_2 \neq 0$, so it belongs to $\mathrm{Aut}(Q_3, W_3) = \mathbb{C}^*$. Therefore, ζ_3' gives a splitting of (12.2), so $\mu = 0$, but this is impossible by lemma 12.1.1, so this case cannot happen in our hypothesis.
- If $(Q_3, W_3) \simeq (Q_i, W_i)$ for some $i = 1, 2$, then (E', V') destabilize (E, V) for α_c^+.

By putting everything together, we get that if (E, V) is not α_c^+-stable, then there exists a quotient $\zeta_i : (E, V) \twoheadrightarrow (Q_i, W_i)$ for some $i = 1, 2$. Together with the first part of the proof, this is sufficient to prove that (a) and (b) are equivalent. □

Remark 12.1.1. Let us fix any ordering (i, j) of $\{1, 2\}$. In the previous proof we considered also the case of a suboject (E', V') of (E, V) that is an extension of (Q_3, W_3) by (Q_i, W_i) for some $i = 1, 2$. Using the extra hypothesis given in the first part of (12.12) we got that if such an object exists, then it does not destabilize (E, V) for α_c^+. If such an extra condition is not verified, than we have to impose that (E', V') does not exist, otherwise it destabilizes (E, V) for α_c^+ (same statement for the second part of (12.12) and α_c^-). One can prove that (E, V) has a subobject of this form if and only if $\overline{p_j} \circ \overline{\alpha_2}(\nu) = 0$, where $\overline{p_j}$ is the morphism

$$\overline{p_j} : \mathrm{Ext}^1((Q_3, W_3), (Q_1, W_1) \oplus (Q_2, W_2)) \to \mathrm{Ext}^1((Q_3, W_3), (Q_j, W_j))$$

induced by any quotient $p_j : (Q_1, W_1) \oplus (Q_2, W_2) \to (Q_j, W_j)$. If $(Q_1, W_1) \not\simeq (Q_2, W_2)$, p_j is the projection to the j-th component (up to multiplication by non-zero scalars); if $(Q_1, W_1) \simeq (Q_2, W_2)$, p_j is any morphism of the form (a, b) for $(a, b) \in \mathbb{C}^2 \smallsetminus \{0\}$.

So by proceeding as in the previous lemma one can prove the following result. We will not need to use it, we just state this result for completeness.

Lemma 12.1.3. *Let us fix any pair of exact sequences as (12.1) and (12.2) with conditions (12.3), respectively (12.5). Moreover, let us suppose that* $(Q_4, W_4) \not\simeq (Q_i, W_i)$ *for all* $i = 1, 2, 3$

and that (E, V) has α_c-canonical filtration of type (2,1,1). Let us also fix an ordering (i, j) of the set $\{1, 2\}$ and let us suppose that

$$\frac{k_i + k_3}{n_i + n_3} \geq \frac{k}{n} \quad \textit{and} \quad \frac{k_j + k_3}{n_j + n_3} < \frac{k}{n}, \tag{12.14}$$

respectively

$$\frac{k_i + k_3}{n_i + n_3} \leq \frac{k}{n} \quad \textit{and} \quad \frac{k_j + k_3}{n_j + n_3} > \frac{k}{n}. \tag{12.15}$$

Then the following facts are equivalent:

(a) (E, V) belongs to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$;

(b) there are no quotients $\zeta_i : (E, V) \twoheadrightarrow (Q_i, W_i)$ for $i = 1, 2$ and $\overline{p_j} \circ \overline{\alpha_2}(\nu) \neq 0$ for all quotients $p_j : (Q_1, W_1) \oplus (Q_2, W_2) \twoheadrightarrow (Q_j, W_j)$.

If we replace conditions (12.14), respectively (12.15), by

$$\frac{k_i + k_3}{n_i + n_3} \geq \frac{k}{n} \quad \forall\, i = 1, 2, \tag{12.16}$$

respectively

$$\frac{k_i + k_3}{n_i + n_3} \leq \frac{k}{n} \quad \forall\, i = 1, 2, \tag{12.17}$$

then (a) is equivalent to

(c) there are no quotients $\zeta_i : (E, V) \twoheadrightarrow (Q_i, W_i)$ for $i = 1, 2$ and $\overline{p_j} \circ \overline{\alpha_2}(\nu) \neq 0$ for all quotients $p_j : (Q_1, W_1) \oplus (Q_2, W_2) \twoheadrightarrow (Q_j, W_j)$ and for all $j = 1, 2$.

Now the second condition of (12.3) is implied by the first condition of (12.3) together with the first part of (12.12) (the same for (12.5) and the second part of (12.12)). Therefore, as a corollary of lemmas 12.1.1 and 12.1.2 we get:

Corollary 12.1.4. *Let us fix any quadruple $(Q_i, W_i)_{i=1,\cdot,4} \in \prod_{i=1}^{4} G_i$ such that $(Q_4, W_4) \not\simeq (Q_3, W_3)$ and let us suppose that*

$$\frac{k_i}{n_i} < \frac{k}{n} \quad \forall\, i \in \{1, 2\}, \quad \frac{k_i + k_3}{n_i + n_3} < \frac{k}{n} \quad \forall\, i \in \{1, 2\}, \tag{12.18}$$

respectively that

$$\frac{k_i}{n_i} > \frac{k}{n} \quad \forall\, i \in \{1, 2\}, \quad \frac{k_i + k_3}{n_i + n_3} > \frac{k}{n} \quad \forall\, i \in \{1, 2\} \tag{12.19}$$

(automatically, we have that $(Q_4, W_4) \not\simeq (Q_i, W_i)$ for $i = 1, 2$). Then the (E, V)'s that belong to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, that have α_c-canonical filtration of type $(2, 1, 1)$ and graded $\oplus_{i=1}^{4}(Q_i, W_i)$ are those induced by pairs of exact sequences as (12.1) and (12.2), such that:

- *$\mu \neq 0$;*
- *$\overline{\alpha_2}(\nu) \neq 0$;*
- *there are no quotients $\zeta_i : (E,V) \twoheadrightarrow (Q_i, W_i)$ for $i = 1,2$.*

Now we will have to state 2 lemmas according to the relation between (Q_1, W_1) and (Q_2, W_2). Note that the description of next lemma can be simplified by considering together the various sets $M_i([\mu])$ described below for $i = 1,2,3$. We prefer to use this description because we will need it when we will globalize it below. Indeed, if the invariants (n_1, k_1) and (n_2, k_2) coincide, then we have a natural action of $\mathbb{Z}_2$ to take into account. It will turn out that such an action fixes the bundle obtained by globalizing the set $M_1([\mu])$, while it interchanges the bundles obtained by globalizing $M_2([\mu])$ and $M_3([\mu])$.

Lemma 12.1.5. *Let us fix any quadruple $(Q_i, W_i)_{i=1,\cdots,4} \in \prod_{i=1}^4 G_i$ with numerical conditions (12.18), respectively (12.19), and such that:*

$$(Q_1, W_1) \not\simeq (Q_2, W_2), \quad (Q_4, W_4) \not\simeq (Q_3, W_3)$$

(automatically, we have that $(Q_4, W_4) \not\simeq (Q_i, W_i)$ for $i = 1,2$). Let us denote by μ any class of a non-split extension of the form

$$0 \to (Q_3, W_3) \xrightarrow{\alpha_2} (E'', V'') \xrightarrow{\beta_2} (Q_4, W_4) \to 0. \tag{12.20}$$

Having fixed $[\mu] \in \mathbb{P}(Ext^1((Q_4, W_4), (Q_3, W_3)))$, let us consider the morphisms

$$Ext^1((Q_4, W_4), (Q_i, W_i)) \xrightarrow{\overline{\beta_2^i}} Ext^1((E'', V''), (Q_i, W_i)) \quad \text{for } i = 1,2$$

induced by the morphism β_2, so that the morphism $\overline{\beta_2}$ in (12.6) coincides with the pair $(\overline{\beta_2^1}, \overline{\beta_2^2})$. Moreover, let us write:

$$M_1([\mu]) := \left(Ext^1((E'', V''), (Q_1, W_1)) \smallsetminus Im(\overline{\beta_2^1})\right) \oplus \left(Ext^1((E'', V''), (Q_2, W_2)) \smallsetminus Im(\overline{\beta_2^2})\right),$$

$$M_2([\mu]) := \left(Ext^1((E'', V''), (Q_1, W_1)) \smallsetminus Im(\overline{\beta_2^1})\right) \oplus \left(Im(\overline{\beta_2^2}) \smallsetminus \{0\}\right),$$

$$M_3([\mu]) := \left(Im(\overline{\beta_2^1}) \smallsetminus \{0\}\right) \oplus \left(Ext^1((E'', V''), (Q_2, W_2)) \smallsetminus Im(\overline{\beta_2^2})\right).$$

Each of these sets has a natural action of $\mathbb{C}^* \times \mathbb{C}^*$ on it (given by multiplication by scalars on the 2 components). Then we have that the set of all the (E,V)'s that belong to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, that have α_c-canonical filtration of type $(2,1,1)$ and graded $\oplus_{i=1}^4 (Q_i, W_i)$ is given by a fibration over $\mathbb{P}(Ext^1((Q_4, W_4), (Q_3, W_3)))$. The fiber over any point $[\mu]$ in that space with μ represented by (12.20) is given by

$$\overline{M}([\mu]) = M_1([\mu])/(\mathbb{C}^* \times \mathbb{C}^*) \amalg$$
$$\amalg\, M_2([\mu])/(\mathbb{C}^* \times \mathbb{C}^*) \amalg M_3([\mu])/(\mathbb{C}^* \times \mathbb{C}^*). \tag{12.21}$$

In addition, if we write:

$$b := \dim \, Ext^1((E'', V''), (Q_1, W_1)), \quad c := \dim \, Ext^1((Q_4, W_4), (Q_1, W_1)),$$
$$d := \dim \, Ext^1((E'', V''), (Q_2, W_2)), \quad e := \dim \, Ext^1((Q_4, W_4), (Q_2, W_2)),$$

then for every $[\mu]$ we have the following description.

(a) If no (Q_i, W_i)'s are isomorphic for $i = 1, 2, 3$, then:

$$\overline{M}([\mu]) \simeq (\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}) \times (\mathbb{P}^{d-1} \smallsetminus \mathbb{P}^{e-1}) \amalg (\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}) \times \mathbb{P}^{e-1} \amalg \mathbb{P}^{c-1} \times (\mathbb{P}^{d-1} \smallsetminus \mathbb{P}^{e-1}).$$

(b) If $(Q_1, W_1) \simeq (Q_3, W_3) \not\simeq (Q_2, W_2)$, then

$$\overline{M}([\mu]) \simeq (\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-2}) \times (\mathbb{P}^{d-1} \smallsetminus \mathbb{P}^{e-1}) \amalg (\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-2}) \times \mathbb{P}^{e-1} \amalg \mathbb{P}^{c-2} \times (\mathbb{P}^{d-1} \smallsetminus \mathbb{P}^{e-1}).$$

(c) If $(Q_1, W_1) \not\simeq (Q_2, W_2) \simeq (Q_3, W_3)$, then

$$\overline{M}([\mu]) \simeq (\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}) \times (\mathbb{P}^{d-1} \smallsetminus \mathbb{P}^{e-2}) \amalg (\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}) \times \mathbb{P}^{e-2} \amalg \mathbb{P}^{c-1} \times (\mathbb{P}^{d-1} \smallsetminus \mathbb{P}^{e-2}).$$

Proof. To any (E, V) that we want to parametrize, we can associate a triple $(\mu, \nu_1, \nu_2) = (\mu, \nu)$, where μ has a representative as (12.20) and $\nu = (\nu_1, \nu_2)$ is as in (12.2) and (12.9). Then by using corollary 12.1.4, the following facts are equivalent

(a) (E, V) belongs to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, and it has α_c-canonical filtration of type $(2, 1, 1)$;

(b) $\mu \neq 0$, $\overline{\alpha_2}(\nu) \neq 0$ and there are no quotients $\zeta_i : (E, V) \twoheadrightarrow (Q_i, W_i)$ for $i = 1, 2$.

Now let us suppose that $\mu \neq 0$ and that there is a quotient $\zeta_i : (E, V) \twoheadrightarrow (Q_i, W_i)$ for some $i = 1, 2$. If $\zeta_i \circ \alpha_1 = 0$, then this induces a quotient $(E'', V'') \twoheadrightarrow (Q_i, W_i)$. Since $(Q_i, W_i) \not\simeq (Q_4, W_4)$, we have that $(Q_i, W_i) \simeq (Q_3, W_3)$ and $\mu = 0$, but this is impossible in our case. Therefore, if $\mu \neq 0$, then $\zeta_i \circ \alpha_1 \neq 0$. Therefore, we can rewrite (b) as

(a) $\mu \neq 0$, $\overline{\alpha_2}(\nu) \neq 0$ and there are no quotients $\zeta_i : (E, V) \twoheadrightarrow (Q_i, W_i)$ for $i = 1, 2$ such that $\zeta_i \circ \alpha_1 \neq 0$.

Since $(Q_1, W_1) \not\simeq (Q_2, W_2)$, by lemma 3.3.1 we get that the following facts are equivalent:

(i) there are no quotients $\zeta_i : (E, V) \twoheadrightarrow (Q_i, W_i)$ for $i = 1, 2$ such that $\zeta_i \circ \alpha_1 \neq 0$;

(ii) $\nu_i \neq 0$ for $i = 1, 2$.

By substituting in (c) we get that (a) is equivalent to:

(d) $\mu \neq 0$, $\overline{\alpha_2}(\nu) \neq 0$ and $\nu_i \neq 0$ for $i = 1, 2$.

Now the condition that $\overline{\alpha_2}(\nu)$ is not zero can be rewritten by imposing that $\nu = (\nu_1, \nu_2)$ does not belong to the image of the morphism $\overline{\beta_2} = (\overline{\beta_2^1}, \overline{\beta_2^2})$:

$$\mathrm{Ext}^1((Q_4, W_4), (Q_1, W_1) \oplus (Q_2, W_2)) \xrightarrow{\overline{\beta_2}} \mathrm{Ext}^1((E'', V''), (Q_1, W_1) \oplus (Q_2, W_2))$$

or, equivalently, that the following 2 conditions are not verified at the same time:

- ν_1 belongs to the image of $\overline{\beta_2^1}$;
- ν_2 belongs to the image of $\overline{\beta_2^2}$.

Now let us look at the sequence (12.2). The set of all possible (E'', V'')'s in that sequence is given by all possible $\mu \neq 0$, modulo the action of $\mathbb{C}^*$ because $(Q_3, W_3) \not\simeq (Q_4, W_4)$; so the (E'', V'')'s are parametrized by

$$\mathbb{P}(\mathrm{Ext}^1((Q_4, W_4), (Q_3, W_3))).$$

Moreover, $\mathrm{Aut}((Q_1, W_1) \oplus (Q_2, W_2)) = \mathbb{C}^* \times \mathbb{C}^*$ because $(Q_1, W_1) \not\simeq (Q_2, W_2)$ and $\mathrm{Aut}(E'', V'') = \mathbb{C}^*$. So having fixed

$$[\mu] \in \mathbb{P}(\mathrm{Ext}^1((Q_4, W_4), (Q_3, W_3))),$$

we have a natural action of $\mathbb{C}^* \times \mathbb{C}^*$ on the set of all possible $\nu = (\nu_1, \nu_2)$'s. Now such an action restricts to an action on the set $M([\mu])$ of all pairs (ν_1, ν_2) such that $\nu_i \neq 0$ for $i = 1, 2$ and such that (ν_1, ν_2) is not in the image of $(\overline{\beta_2^1}, \overline{\beta_2^2})$. So having fixed any point $[\mu] \in \mathbb{P}(\mathrm{Ext}^1((Q_4, W_4), (Q_3, W_3)))$ with representative (12.20) for μ, we have that the set of all possible (E, V)'s that we want to parametrize is given by

$$\overline{M}([\mu]) = M([\mu])/(\mathbb{C}^* \times \mathbb{C}^*) = (M_1([\mu]) \amalg M_2([\mu]) \amalg M_3([\mu])) / (\mathbb{C}^* \times \mathbb{C}^*),$$

where the $M_i([\mu])$'s are described in the claim of the lemma. The action of $\mathbb{C}^* \times \mathbb{C}^*$ sends every $M_i([\mu])$ to itself, so this proves (12.21).

If we apply the functor $\mathrm{Hom}(-, (Q_1, W_1))$ to the sequence (12.20), we get the long exact sequence:

$$\cdots \to \mathrm{Hom}((E'', V''), (Q_1, W_1)) \to \mathrm{Hom}((Q_3, W_3), (Q_1, W_1)) \xrightarrow{\delta^1}$$

$$\xrightarrow{\delta^1} \mathrm{Ext}^1((Q_4, W_4), (Q_1, W_1)) \xrightarrow{\overline{\beta_2^1}} \mathrm{Ext}^1((E'', V'''), (Q_1, W_1)) \to \cdots$$

Now let us suppose that there is any non-zero morphism γ from (E'', V'') to (Q_1, W_1); since $(Q_4, W_4) \not\simeq (Q_1, W_1)$, this implies that $(Q_3, W_3) \simeq (Q_1, W_1)$ and that γ gives a splitting of (12.20), but this is impossible since $\mu \neq 0$. Therefore, the first space of the previous exact sequence is zero, so δ^1 is injective. Now if $(Q_3, W_3) \not\simeq (Q_1, W_1)$, then $\overline{\beta_2^1}$ is injective, so $\dim(\mathrm{Im}\ \overline{\beta_2^1}) = c$; in the opposite case $\overline{\beta_2^1}$ has a kernel of dimension 1, so $\dim(\mathrm{Im}\ \overline{\beta_2^1}) = c - 1$.

Analogously, if $(Q_3, W_3) \not\simeq (Q_2, W_2)$, then $\dim(\mathrm{Im}\ \overline{\beta_2^2}) = e$; in the opposite case $\dim(\mathrm{Im}\ \overline{\beta_2^1}) = e - 1$, so we conclude. □

Lemma 12.1.6. *Let us fix any quadruple $(Q_i, W_i)_{i=1,\cdots,4} \in \prod_{i=1}^4 G_i$ with numerical conditions (12.18), respectively (12.19), and such that:*

$$(Q_1, W_1) \simeq (Q_2, W_2), \quad (Q_4, W_4) \not\simeq (Q_3, W_3).$$

Let us denote by μ any class of a non-split extension of the form

$$0 \to (Q_3, W_3) \xrightarrow{\alpha_2} (E'', V'') \xrightarrow{\beta_2} (Q_4, W_4) \to 0. \tag{12.22}$$

Having fixed $[\mu] \in \mathbb{P}(Ext^1((Q_4, W_4), (Q_3, W_3)))$, let us consider the morphisms $\overline{\beta_2^1}$ and $\overline{\beta_2^2}$ induced by β_2 as in the previous lemma; since $(Q_1, W_1) \simeq (Q_2, W_2)$, we can identify those 2 morphisms. Then we have that the set of all the (E, V)'s that belong to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, that have α_c-canonical filtration of type (2,1,1) and graded $\oplus_{i=1}^4 (Q_i, W_i)$ is given by a fibration over $\mathbb{P}(Ext^1((Q_4, W_4), (Q_3, W_3)))$. The fiber over any point $[\mu]$ in that space with μ represented by (12.22) is given by

$$\overline{M}([\mu]) := Grass(2, H([\mu])) \smallsetminus Grass(2, H'([\mu])),$$

where

$$H([\mu]) := Ext^1((E'', V''), (Q_1, W_1))$$

and $H'([\mu])$ is the subvector space of $H([\mu])$ defined as the image of $\overline{\beta_2^1}$. If we write:

$$b := dim\ Ext^1((E'', V''), (Q_1, W_1)), \quad c := dim\ Ext^1((Q_4, W_4), (Q_1, W_1)),$$

then we have that:

- *if $(Q_1, W_1) \simeq (Q_2, W_2) \not\simeq (Q_3, W_3)$, then*

$$\overline{M}([\mu]) \simeq Grass(2, b) \smallsetminus Grass(2, c);$$

- *if $(Q_1, W_1) \simeq (Q_2, W_2) \simeq (Q_3, W_3)$, then*

$$\overline{M}([\mu]) \simeq Grass(2, b) \smallsetminus Grass(2, c - 1).$$

Proof. To any (E, V) that we want to parametrize we can associate a triple $(\mu, \nu_1, \nu_2) = (\mu, \nu)$, where μ and ν have representatives of the form (12.1), respectively (12.2), and ν_2, ν_3 are as in diagram (12.9). Then by corollary 12.1.4, the following facts are equivalent

(a) (E, V) belongs to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, and it has α_c-canonical filtration of type $(2, 1, 1)$;

(b) $\mu \neq 0$, $\overline{\alpha_2}(\nu) \neq 0$ and there are no quotients $\zeta_1 : (E, V) \twoheadrightarrow (Q_1, W_1)$.

In (b) we have omitted the case of (Q_2, W_2) since by hypothesis such an object is isomorphic to (Q_1, W_1). As in the previous lemma, we get that (b) is equivalent to

(c) $\mu \neq 0$, $\overline{\alpha_2}(\nu) \neq 0$ and there are no quotients $\zeta_1 : (E, V) \twoheadrightarrow (Q_1, W_1)$ such that $\zeta_1 \circ \alpha_1 \neq 0$.

Since $(Q_1, W_1) \simeq (Q_2, W_2)$, by lemma 3.3.1 we get that the following facts are equivalent:

(i) there are no quotients $\zeta_1 : (E, V) \twoheadrightarrow (Q_1, W_1)$ such that $\zeta_1 \circ \alpha_1 \neq 0$;

(ii) ν_1 and ν_2 are linearly independent in

$$H([\mu]) = \mathrm{Ext}^1((E'', V''), (Q_1, W_1)) = \mathrm{Ext}^1((E'', V''), (Q_2, W_2)).$$

Then we can substitute this into (c) and we get that (a) is equivalent to

(d) $\mu \neq 0$, $\overline{\alpha_2}(\nu) \neq 0$ and ν_1, ν_2 linearly independent in $H([\mu])$.

Now as in the previous lemma, $\overline{\alpha_2}(\nu)$ is non-zero if and only if the following 2 conditions are not verified at the same time:

- ν_1 belongs to the image of $\overline{\beta_2^1}$;
- ν_2 belongs to the image of $\overline{\beta_2^2}$.

Since $(Q_1, W_1) \simeq (Q_2, W_2)$, we can identify $\overline{\beta_2^1}$ and $\overline{\beta_2^2}$. Therefore, having fixed $[\mu] \in \mathbb{P}(\mathrm{Ext}^1((Q_4, W_4), (Q_3, W_3)))$, we have to remove from the set of all the (ν_1, ν_2)'s that are linearly independent in $H([\mu])$ the subset of all (ν_1, ν_2)'s that are linearly independent in $H'([\mu]) := \mathrm{Im}\ \overline{\beta_2^1} \subset H([\mu])$. Now if we look at the exact sequence (12.2) with $(Q_1, W_1) \simeq (Q_2, W_2)$, we get that as in the previous lemma $\mathrm{Aut}(E'', V'') = \mathbb{C}^*$, while $\mathrm{Aut}((Q_1, W_1) \oplus (Q_1, W_1)) = \mathrm{GL}(2, \mathbb{C})$. So having fixed $[\mu]$, the corresponding (E, V)'s that we want to parametrize are in bijection with the set

$$\overline{M}([\mu]) := Grass(2, H([\mu])) \smallsetminus Grass(2, H'([\mu])).$$

The same dimension counting of the previous lemma proves that the dimension of $H'([\mu])$ is c if $(Q_3, W_3) \not\simeq (Q_1, W_1)$ and it is $c - 1$ if $(Q_3, W_3) \simeq (Q_1, W_1)$. □

Now we give a global parametrization of the objects described before, i.e. we describe families of schemes that parametrize various types of (E, V)'s when the graded $\oplus_{i=1}^4 (Q_i, W_i)$ varies over $\prod_{i=1}^4 G_i$ and the α_c-canonical filtration is of type (2,1,1). <Since the α_c-canonical filtration is of type (2,1,1), then the order of (Q_1, W_1) and of (Q_2, W_2) is not important. As we said in remark 12.0.2, we will state only the global results for the case when $(n_1, k_1) =$

$(n_2, k_2) = (n_3, k_3)$; the cases when this condition does not hold are actually simpler to manage and they are not needed for computing the Hodge-Deligne polynomials of $G(\alpha; 4, d, 1)$.

Let us denote by $\oplus_{i=1}^4 (Q_i, W_i)$ a fixed graded with conditions (12.18), respectively (12.19) and such that $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$; in this case the first condition of (12.18), respectively of (12.19), implies the second one, so if $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$ imposing (12.18), respectively (12.19) is equivalent to imposing that:

$$\frac{k_1}{n_1} < \frac{k}{n}, \tag{12.23}$$

respectively that

$$\frac{k_1}{n_1} > \frac{k}{n}. \tag{12.24}$$

If $(Q_1, W_1) \not\simeq (Q_2, W_2)$, then by lemma 12.1.5 the corresponding (E, V)'s are parametrized by triples $([\mu], [\nu_1], [\nu_2])$ with $[\mu] \in \mathbb{P}(\mathrm{Ext}^1((Q_4, W_4), (Q_3, W_3)))$ and representative (12.20) for μ and

$$([\nu_1], [\nu_2]) \in \overline{M}([\mu]) \subset \prod_{i=1}^{2} \mathbb{P}(\mathrm{Ext}^1((E'', V''), (Q_i, W_i))).$$

We are considering the case when the (Q_i, W_i)'s are all of the same type for $i = 1, 2, 3$. Therefore we need to take into account the possible isomorphisms between them. So, having fixed $[\mu]$, we need to consider separately the following cases.

(1) If $(Q_1, W_1) \not\simeq (Q_2, W_2) \simeq (Q_3, W_3)$, then the roles of (Q_1, W_1) and of (Q_2, W_2) are not interchangeable, so we need to consider *ordered* pairs $([\nu_1], [\nu_2])$.

(2) If $(Q_i, W_i) \not\simeq (Q_j, W_j)$ for all $i \neq j \in \{1, 2, 3\}$, then the roles of (Q_1, W_1) and of (Q_2, W_2) are interchangeable, so we need to consider *unordered* pairs $([\nu_1], [\nu_2])$.

Note that since the order of (Q_1, W_1) and (Q_2, W_2) is not important, we don't need to consider also the case $(Q_1, W_1) \simeq (Q_3, W_3) \not\simeq (Q_2, W_2)$.

If $(Q_1, W_1) \simeq (Q_2, W_2)$, then by lemma 12.1.6 the corresponding (E, V)'s are parametrized by pairs $([\mu], < \nu_1, \nu_2 >)$ with $[\mu] \in \mathbb{P}(\mathrm{Ext}^1((Q_4, W_4), (Q_3, W_3)))$ and representative (12.22) for μ and

$$< \nu_1, \nu_2 > \in \overline{M}([\mu]) \subset Grass(2, \mathrm{Ext}^1((E'', V''), (Q_1, W_1))).$$

Having fixed $[\mu]$, we need to consider separately the following cases.

(3) If $(Q_1, W_1) \simeq (Q_2, W_2) \not\simeq (Q_3, W_3)$, then the corresponding (E, V)'s are parametrized by a difference of grassmannians (see lemma 12.1.6);

(4) If $(Q_1, W_1) \simeq (Q_2, W_2) \simeq (Q_3, W_3)$, then also in this case the corresponding (E, V)'s are parametrized by a difference of grassmannians, but with different dimensions than the previous one.

The 4 cases are taken into account by propositions 7.5.1, 7.5.2, 7.5.3 and 7.5.4 respectively; we give below the proofs of those 4 results.

Proof of proposition 7.5.1. Let us fix any sequence $(a, b, c, d) \in \mathbb{N}^4$ and let us consider the locally closed subscheme of $\hat{G}_3 \times \hat{G}_4$ defined as:

$$\hat{U}_a := \{t \in \hat{G}_3 \times \hat{G}_4 \text{ s.t. } \dim \operatorname{Ext}^1((\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)_t, (\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)_t) = a\},$$

where $\hat{p}_3$ and $\hat{p}_4$ are the projections from $\hat{G}_3 \times \hat{G}_4$ to its factors. The numerical condition (12.23), respectively (12.24), prove that for every quadruple $(Q_i, W_i)_i \in \prod_{i=1}^4 G_i$ we have that $(Q_4, W_4) \not\simeq (Q_i, W_i)\, \forall\, i = 1, 2, 3$. Therefore, we have in particular that

$$\operatorname{Hom}((\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)_t, (\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)_t) = 0$$

for all t in $\hat{G}_3 \times \hat{G}_4$. So we can apply proposition 5.0.5 for $r = 2$ and we get that there is a finite disjoint covering $\{\hat{U}_{a;i}\}_i$ of $\hat{U}_a$ by locally closed subschemes; for every i there is a locally free sheaf on $\hat{U}_{a;i}$:

$$\hat{\mathcal{H}}_{a;i} := \mathcal{E}xt^1_{\pi_{\hat{U}_{a;i}}}\Big((\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4), (\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)\Big)^\vee$$

and a projective bundle

$$\hat{\varphi}_{a;i} : \hat{R}_{a;i} := \mathbb{P}(\hat{\mathcal{H}}_{a;i}) \longrightarrow \hat{U}_{a;i} \subset \hat{G}_3 \times \hat{G}_4$$

with fibers isomorphic to $\mathbb{P}^{a-1}$. By abuse of notation, we denote by $\hat{\varphi}_{a;i}$ also the composition $\hat{R}_{a;i} \to \hat{G}_3 \times \hat{G}_4$. Moreover, there exists a family of classes of non-split extensions parametrized by $\hat{R}_{a;i}$:

$$\begin{aligned} 0 \to (\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3) \otimes_{\hat{R}_{a;i}} \mathcal{O}_{\hat{R}_{a;i}}(1) \stackrel{\alpha_{2;a;i}}{\longrightarrow} \\ \stackrel{\alpha_{2;a;i}}{\longrightarrow} (\hat{\mathcal{E}}''_{a;i}, \hat{\mathcal{V}}''_{a;i}) \stackrel{\beta_{2;a;i}}{\longrightarrow} (\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4) \to 0. \end{aligned} \tag{12.25}$$

Such an extension is universal in the sense of corollary 4.4.4. Now let us fix any index i, let us consider the projections

$$\hat{p}_1 : \hat{G}_1 \times \hat{R}_{a;i} \longrightarrow \hat{G}_1, \quad \hat{p}_{34} : \hat{G}_1 \times \hat{R}_{a;i} \longrightarrow \hat{R}_{a;i}$$

and let us define the following scheme:

$$\hat{U}_{a,b,c,d;i} := \{t \in \hat{G}_1 \times \hat{R}_{a;i} \text{ s.t. } \dim \operatorname{Ext}^1((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i}, \hat{\mathcal{V}}''_{a;i})_t, (\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = b,$$

$$\dim\ \mathrm{Ext}^1((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_4,\hat{p}_4)^*(\hat{\mathcal{Q}}_4,\hat{\mathcal{W}}_4)_t,(\hat{p}'_1,\hat{p}_1)^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)_t)=c,$$
$$\dim\ \mathrm{Ext}^1((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i},\hat{\mathcal{V}}''_{a;i})_t,(\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)_t)=d,$$
$$\mathrm{Hom}((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3.\hat{\mathcal{W}}_3)_t,(\hat{p}'_1,\hat{p}_1)^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)_t)=0\}.$$

By proposition 1.0.5, this is a locally closed subscheme of $\hat{G}_1\times\hat{R}_{a;i}$. Moreover, by applying several times lemma 4.6.1, we get that it has a finite disjoint locally closed covering $\{\hat{U}_{a,b,c,d;i,j}\}_j$ such that all the following sheaves are locally free on $\hat{U}_{a,b,c,d;i,j}$ and commute with base change:

$$\hat{E}^1_{a,b,c,d;i,j}:=\mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c,d;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i},\hat{\mathcal{V}}''_{a;i}),(\hat{p}'_1,\hat{p}_1)^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)\Big),$$
$$\hat{F}^1_{a,b,c,d;i,j}:=\mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c,d;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_4,\hat{p}_4)^*(\hat{\mathcal{Q}}_4,\hat{\mathcal{W}}_4),(\hat{p}'_1,\hat{p}_1)^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)\Big),$$
$$\hat{E}^2_{a,b,c,d;i,j}:=\mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c,d;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i},\hat{\mathcal{V}}''_{a;i}),$$
$$(\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)\Big).$$

Moreover, by base change we have that also the following sheaf is locally free:

$$\hat{F}^2_{a,b,c,d;i,j}:=\mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c,d;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_4,\hat{p}_4)^*(\hat{\mathcal{Q}}_4,\hat{\mathcal{W}}_4),$$
$$(\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)\Big)=\hat{p}^*_{34}\hat{\varphi}^*_{a;i}\hat{\mathcal{H}}^\vee_{a;i}.$$

In addition, by base change, lemma 4.1.9 and definition of $\hat{U}_{a;i}$, also the following sheaf is locally free of rank 1:

$$\hat{G}^2_{a,b,c,d;i,j}:=\mathcal{H}om_{\pi_{\hat{U}_{a,b,c,d;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*((\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)\otimes_{\hat{R}_{a;i}}\mathcal{O}_{\hat{R}_{a;i}}(1)),$$
$$(\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)\Big) \tag{12.26}$$

By construction of $\hat{U}_{a,b,c,d;i,j}$ for every point t of $\hat{U}_{a,b,c,d;i,j}$ we have that:

$$\mathrm{Hom}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i},\hat{\mathcal{V}}''_{a;i})_t,(\hat{p}'_1,\hat{p}_1)^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)_t\Big)=0,$$
$$\mathrm{Hom}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i},\hat{\mathcal{V}}''_{a;i})_t,(\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)_t\Big)=0 \tag{12.27}$$

(see the end of the proof of lemma 12.1.5). Moreover, by construction we have already said that for $l=1,2$ the sheaf $\hat{E}^l_{a,b,c,d;i,j}$ commutes with base change. Therefore, by proposition 4.4.1 and corollary 4.4.4 we have that there exists a projective bundle

$$\hat{\varphi}^1_{a,b,c,d;i,j}:\hat{P}^1_{a,b,c,d;i,j}:=\mathbb{P}((\hat{E}^1_{a,b,c,d;i,j})^\vee)\longrightarrow\hat{U}_{a,b,c,d;i,j}$$

and a universal extension (in the sense of corollary 4.4.4):

$$0\to(\hat{\varphi}^{1'}_{a,b,c,d;i,j},\hat{\varphi}^1_{a,b,c,d;i,j})^*(\hat{p}'_1,\hat{p}_1)^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)\otimes_{\hat{P}^1_{a,b,c,d;i,j}}\mathcal{O}_{\hat{P}^1_{a,b,c,d;i,j}}(1)\to$$

$$\to (\hat{\mathcal{E}}^1_{a,b,c,d;i,j}, \hat{\mathcal{V}}^1_{a,b,c,d;i,j}) \to (\hat{\varphi}^{1'}_{a,b,c,d;i,j}, \hat{\varphi}^1_{a,b,c,d;i,j})^*(\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i}, \hat{\mathcal{V}}''_{a;i}) \to 0 \tag{12.28}$$

parametrized by $\hat{P}^1_{a,b,c,d;i,j}$. Moreover, there exists a projective bundle

$$\hat{\varphi}^2_{a,b,c,d;i,j} : \hat{P}^2_{a,b,c,d;i,j} := \mathbb{P}((\hat{E}^2_{a,b,c,d;i,j})^\vee) \longrightarrow \hat{U}_{a,b,c,d;i,j}$$

and a universal extension

$$0 \to (\hat{\varphi}^{2'}_{a,b,c,d;i,j}, \hat{\varphi}^2_{a,b,c,d;i,j})^*(\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3) \otimes_{\hat{P}^2_{a,b,c,d;i,j}} \mathcal{O}_{\hat{P}^2_{a,b,c,d;i,j}}(1) \to$$

$$\to (\hat{\mathcal{E}}^2_{a,b,c,d;i,j}, \hat{\mathcal{V}}^2_{a,b,c,d;i,j}) \to (\hat{\varphi}^{2'}_{a,b,c,d;i,j}, \hat{\varphi}^2_{a,b,c,d;i,j})^*(\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i}, \hat{\mathcal{V}}''_{a;i}) \to 0 \tag{12.29}$$

parametrized by $\hat{P}^2_{a,b,c,d;i,j}$. Now let us apply the functor

$$\mathcal{H}om_{\pi_{\hat{U}_{a,b,c,d;i,j}}}\Big(-, (\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)\Big)$$

to the pullback via $\hat{p}_{34}$ of the exact sequence (12.25). Then we get a long exact sequence as follows:

$$\longrightarrow \mathcal{H}om_{\pi_{\hat{U}_{a,b,c,d;i,j}}}\Big((\hat{p}'_{34}, \hat{p}_{34})^*((\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3) \otimes_{\hat{R}_{a;i}} \mathcal{O}_{\hat{R}_{a;i}}(1)),$$
$$(\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)\Big) \xrightarrow{\delta^l}$$

$$\xrightarrow{\delta^l} \mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c,d;i,j}}}\Big((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4), (\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)\Big) \xrightarrow{\overline{\beta^1_{2;a;i}}}$$

$$\xrightarrow{\overline{\beta^1_{2;a;i}}} \mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c,d;i,j}}}\Big((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i}, \hat{\mathcal{V}}''_{a;i}), (\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)\Big) \longrightarrow \cdots \tag{12.30}$$

By the last line of the definition of $\hat{U}_{a,b,c,d;i,j}$ and base change, the first sheaf of the previous sequence is zero. Now by the previous construction both the second and the third sheaf of the previous sequence are locally free, so we can rewrite that sequence as an injective morphism of vector bundles:

$$\overline{\beta^1_{2;a;i}} : \hat{F}^1_{a,b,c,d;i,j} \hookrightarrow \hat{E}^1_{a,b,c,d;i,j}.$$

So it makes sense to consider the projective bundle over $\hat{U}_{a,b,c,d;i,j}$:

$$\hat{Q}^1_{a,b,c,d;i,j} := \mathbb{P}((\hat{F}^1_{a,b,c,d;i,j})^\vee) \subseteq \hat{P}^1_{a,b,c,d;i,j}.$$

Let us also apply the functor

$$\mathcal{H}om_{\pi_{\hat{U}_{a,b,c,d;i,j}}}\Big(-, (\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)\Big)$$

to the pullback via $\hat{p}_{34}$ of the exact sequence (12.25). Then we get a long exact sequence as follows:

$$
\begin{aligned}
\cdots \longrightarrow \mathcal{H}om_{\pi_{\hat{U}_{a,b,c,d;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i},\hat{\mathcal{V}}''_{a;i}),(\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\varphi_{a;i})^*(\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)\Big) \longrightarrow \\
\longrightarrow \mathcal{H}om_{\pi_{\hat{U}_{a,b,c,d;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*((\hat{\varphi}'_{a;i},\varphi_{a;i})^*(\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)\otimes_{\hat{R}_{a;i}}\mathcal{O}_{\hat{R}_{a;i}}(1)), \\
(\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)\Big) \xrightarrow{\delta^2} \\
\xrightarrow{\delta^2} \mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c,d;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\varphi_{a;i})^*(\hat{p}'_4,\hat{p}_4)^*(\hat{\mathcal{Q}}_4,\hat{\mathcal{W}}_4), \\
(\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)\Big) \xrightarrow{\overline{\beta^2_{2;a;i}}} \\
\xrightarrow{\overline{\beta^2_{2;a;i}}} \mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c,d;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i},\hat{\mathcal{V}}''_{a;i}),(\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)\Big).
\end{aligned}
\tag{12.31}
$$

By (12.27) and base change the first sheaf is always zero, so δ^2 is injective. Moreover, the second sheaf coincides with $\hat{G}^2_{a,b,c,d;i,j}$, so it is locally free of rank 1. By the previous construction both the third and last sheaves of the previous sequence are locally free, so we can rewrite that sequence as an exact sequence of vector bundles:

$$
0 \to \hat{G}^2_{a,b,c,d;i,j} \xrightarrow{\delta^2} \hat{F}^2_{a,b,c,d;i,j} \xrightarrow{\overline{\beta^2_{2;a;i}}} \hat{E}^2_{a,b,c,d;i,j}.
$$

Then the rank of $\overline{\beta^2_{2;a;i}}$ is constant, so its image $\overline{F}^2_{a,b,c,d;i,j}$ is locally free of rank $a-1$. So it makes sense to consider the projective bundle over $\hat{U}_{a,b,c,d;i,j}$:

$$
\hat{Q}^2_{a,b,c,d;i,j} := \mathbb{P}((\overline{F}^2_{a,b,c,d;i,j})^\vee) \subseteq \hat{P}^2_{a,b,c,d;i,j}.
$$

Now let us consider the following fiber product:

$$
\begin{tikzcd}
\hat{R}^1_{a,b,c,d;i,j} \arrow[r, "pr_2"] \arrow[d, "pr_1"] \arrow[dr, phantom, "\square"] & \hat{P}^2_{a,b,c,d;i,j} \smallsetminus \hat{Q}^2_{a,b,c,d;i,j} \arrow[d, "\hat{\varphi}^2_{a,b,c,d;i,j}"] \\
\hat{P}^1_{a,b,c,d;i,j} \smallsetminus \hat{Q}^1_{a,b,c,d;i,j} \arrow[r, "\hat{\varphi}^1_{a,b,c,d;i,j}"] & \hat{U}_{a,b,c,d;i,j}.
\end{tikzcd}
$$

Then we consider the pullbacks of the sequences (12.28) and (12.29) via pr_1 and pr_2 respectively. We sum the 2 new extensions and we get an extension parametrized by $\hat{R}^1_{a,b,c,d;i,j}$ of the form:

$$
0 \to (\overline{\mathcal{Q}}_1,\overline{\mathcal{W}}_1) \oplus (\overline{\mathcal{Q}}_3,\overline{\mathcal{W}}_3) \to (\hat{\mathcal{E}}_{a,b,c,d;i,j},\hat{\mathcal{V}}_{a,b,c,d;i,j}) \to (\overline{\mathcal{E}}''_{a;i},\overline{\mathcal{V}}''_{a;i}) \to 0,
$$

where the objects on the left and on the right are suitable pullbacks of the families $(\hat{\mathcal{Q}}_l,\hat{\mathcal{W}}_l)$ for $l=1,3$ and of $(\hat{\mathcal{E}}''_{a;i},\hat{\mathcal{V}}''_{a;i})$. Given any point r in $\hat{R}^1_{a,b,c,d;i,j}$, let us denote by

$$
0 \to (Q_1,W_1) \oplus (Q_3,W_3) \to (E,V) \to (E'',V'') \to 0
$$

the restriction of the previous sequence to r. We denote by $\nu = (\nu_1, \nu_3)$ the class of this extension. By construction, (E'', V'') sits in a non-split exact sequences of the form

$$0 \to (Q_3, W_3) \xrightarrow{\alpha_2} (E'', V'') \xrightarrow{\beta_2} (Q_4, W_4) \to 0;$$

we denote by μ the class of this extension. Then by construction we get that $\nu = (\nu_1, \nu_3)$ belongs to the set $M_1([\mu])$ described in lemma 12.1.5. Now let us assume condition (12.23) (the proof for condition (12.24) is analogous). Then lemma 12.1.5 proves that (E, V) belongs to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$. So by the universal property of such a scheme we get an induced morphism

$$\hat{\omega}^1_{a,b,c,d;i,j} : \hat{R}^1_{a,b,c,d;i,j} \longrightarrow G^+(\alpha_c; n, d, k).$$

Again by lemma 12.1.5, such a morphism becomes injective once we quotient by the free action of $PGL(N_1) \times \cdots \times PGL(N_4)$; we denote the induced morphism by

$$\omega^1_{a,b,c,d;i,j} : R^1_{a,b,c,d;i,j} \longrightarrow G^+(\alpha_c; n, d, k).$$

This construction generalizes the pointwise construction of the set

$$\overline{M}_1([\mu])/(\mathbb{C}^* \times \mathbb{C}^*)$$

described in lemma 12.1.5. Analogously, we can construct the following fiber products

$$\begin{tikzcd}
\hat{R}^2_{a,b,c,d;i,j} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \hat{Q}^2_{a,b,c,d;i,j} \arrow[d, "\hat{\varphi}^2_{a,b,c,d;i,j}"] \\
\hat{P}^1_{a,b,c,d;i,j} \smallsetminus \hat{Q}^1_{a,b,c,d;i,j} \arrow[r, "\hat{\varphi}^1_{a,b,c,d;i,j}"] & \hat{U}_{a,b,c,d;i,j},
\end{tikzcd}$$

and

$$\begin{tikzcd}
\hat{R}^3_{a,b,c,d;i,j} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \hat{P}^2_{a,b,c,d;i,j} \smallsetminus \hat{Q}^2_{a,b,c,d;i,j} \arrow[d, "\hat{\varphi}^2_{a,b,c,d;i,j}"] \\
\hat{Q}^1_{a,b,c,d;i,j} \arrow[r, "\hat{\varphi}^1_{a,b,c,d;i,j}"] & \hat{U}_{a,b,c,d;i,j}
\end{tikzcd}$$

that generalize the pointwise constructions of $\overline{M}_l([\mu])/(\mathbb{C}^* \times \mathbb{C}^*)$ for $l = 2, 3$. The construction of the families parametrized by these schemes is analogous to the previous one, so we omit the details. This is enough to conclude. □

Proof of proposition 7.5.2. First of all, let us construct schemes of the form $\hat{R}^1_{a,b,c,d,e;i,j}$: these are defined similarly to the schemes of the form $\hat{R}^1_{a,b,c,d;i,j}$ of proposition 7.5.1. The only significant differences are the following.

- Having fixed any index i, we consider the projections

$$\hat{p}_l : \hat{G}_1 \times \hat{G}_2 \times \hat{R}_{a;i} \longrightarrow \hat{G}_l \text{ for } l = 1,2, \quad \hat{p}_{34} : \hat{G}_1 \times \hat{G}_2 \times \hat{R}_{a;i} \longrightarrow \hat{R}_{a;i}$$

and we define a scheme

$$\begin{gathered}
\hat{U}_{a,b,c,d,e;i} := \{t \in \hat{G}_1 \times \hat{G}_2 \times \hat{R}_{a;i} \text{ s.t.} \\
\dim \operatorname{Ext}^1((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i}, \hat{\mathcal{V}}''_{a;i})_t, (\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = b, \\
\dim \operatorname{Ext}^1((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)_t, (\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = c, \\
\dim \operatorname{Ext}^1((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i}, \hat{\mathcal{V}}''_{a;i})_t, (\hat{p}'_2, \hat{p}_2)^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2)_t) = d, \\
\dim \operatorname{Ext}^1((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)_t, (\hat{p}'_2, \hat{p}_2)^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2)_t) = e, \\
\operatorname{Hom}((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)_t, (\hat{p}'_l, \hat{p}_l)^*(\hat{\mathcal{Q}}_l, \hat{\mathcal{W}}_l)_t) = 0 \quad \forall\, l = 1,2, \\
\operatorname{Hom}((\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t, (\hat{p}'_2, \hat{p}_2)^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2)_t) = 0\}.
\end{gathered}$$

- The sheaves $\hat{E}^l_{a,b,c,d;i,j}$ and $\hat{F}^l_{a,b,c,d;i,j}$ for $l = 1,2$ are replaced by

$$\hat{E}^l_{a,b,c,d,e;i,j} := \mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c,d,e;i,j}}}\Big((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i}, \hat{\mathcal{V}}''_{a;i}), (\hat{p}'_l, \hat{p}_l)^*(\hat{\mathcal{Q}}_l, \hat{\mathcal{W}}_l)\Big),$$

$$\hat{F}^l_{a,b,c,d,e;i,j} := \mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c,d,e;i,j}}}\Big((\hat{p}'_{34}, \hat{p}_{34})^*(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_4, \hat{p}_4)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4), (\hat{p}'_l, \hat{p}_l)^*(\hat{\mathcal{Q}}_l, \hat{\mathcal{W}}_l)\Big)$$

and the line bundle $\hat{G}^2_{a,b,c,d;i,j}$ is replaced by the zero sheaf.

- the morphisms $\overline{\beta^1_{2;a;i}}$ and $\overline{\beta^2_{2;a;i}}$ are both injective. For $l = 1,2$ we define consequently the projective bundles

$$\hat{\varphi}^l_{a,b,c,d,e;i,j} : \hat{P}^l_{a,b,c,d,e;i,j} := \mathbb{P}((\hat{E}^l_{a,b,c,d,e;i,j})^\vee) \longrightarrow \hat{U}_{a,b,c,d,e;i,j}$$

and the subbundles

$$\hat{Q}^l_{a,b,c,d,e;i,j} := \mathbb{P}((\hat{F}^l_{a,b,c,d,e;i,j})^\vee) \subseteq \hat{P}^l_{a,b,c,d,e;i,j}.$$

Finally, we define the various schemes $\hat{R}^l_{a,b,c,d,e;i,j}$ for $l = 1,2$ in the same way of the corresponding schemes in the previous proof (we don't need to consider the schemes for $l = 3$, see below) and we get families of extensions over them analogously to the previous case.

Now let us assume condition (12.23) and let us consider the induced morphism

$$\omega^1_{a,b,c,d,e;i,j} : R^1_{a,b,c,d,e;i,j} = \hat{R}^1_{a,b,c,d,e;i,j}/(PGL(N_1)\times\cdots\times PGL(N_4)) \longrightarrow G^+(\alpha_c;n,d,k).$$

If we impose that $(b,c)\neq(d,e)$, then we get that such a morphism is injective and that the images of $\omega^1_{a,b,c,d,e;i,j}$ and of $\omega^1_{a,d,e,b,c;i,j}$ coincide, so we have to consider only those sequences (a,b,c,d,e) such that $(b,c)<(d,e)$ (with lexicographic order). If $(b,c)=(d,e)$, then $\omega^1_{a,b,c,b,c;i,j}$ is injective only up to quotienting by an action of $\mathbb{Z}_2$ described as in the claim of the proposition. The description of the $\mathbb{Z}_2$-invariant covering of $U_{a,b,c,d,e;i,j}$ follows the lines of the previous proofs for analogous cases.

For what concerns the schemes of the form $R^2_{a,b,c,d,e;i,j}$, the description is as follows: for every choice of indices $(a,b,c,d,e)\in\mathbb{N}^5$ their images via the induced morphisms $\omega^2_{a,b,c,d,e;i,j}$ are all disjoint in $G^+(\alpha_c;n,d,k)$. Differently from the previous proposition, we don't need any scheme of the form $R^3_{a,b,c,d,e;i,j}$ because the image of any such scheme would coincide with the image of a scheme of the form $R^2_{a,d,e,b,c;i,j}$. Moreover, also if $(b,c)=(d,e)$ in this case there is not any induced action of $\mathbb{Z}_2$ since the roles of (Q_1,W_1) and of (Q_2,W_2) are not interchangeable: indeed the objects $[\nu_1]$ and $[\nu_2]$ belong to complementary spaces, so we cannot interchange them. Therefore also the morphisms of the form $\omega^2_{a,b,c,b,c;i,j}$ are injective, so we conclude. $\square$

Proof of proposition 7.5.3. The construction of these spaces follows the lines of the proof of proposition 7.5.1 in order to get a family of scheme $\{\hat{R}_{a;i}\}_i$ and universal families of extensions as in (12.25). Now let us fix any index i, let us consider the projections

$$\hat{p}_1 : \hat{G}_1\times\hat{R}_{a;i}\longrightarrow\hat{G}_1,\quad \hat{p}_{34} : \hat{G}_1\times\hat{R}_{a;i}\longrightarrow\hat{R}_{a;i}$$

and let us define the following scheme

$$\begin{aligned}\hat{U}_{a,b,c;i} := \{t\in\hat{G}_1\times\hat{R}_{a;i}\text{ s.t. }&\dim\mathrm{Ext}^1((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i},\hat{\mathcal{V}}''_{a;i})_t,(\hat{p}'_1,\hat{p}_1)^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)_t)=b,\\ &\dim\mathrm{Ext}^1((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_4,\hat{p}_4)^*(\hat{\mathcal{Q}}_4,\hat{\mathcal{W}}_4)_t,(\hat{p}'_1,\hat{p}_1)^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)_t)=c,\\ &\mathrm{Hom}((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)_t,(\hat{p}'_l,\hat{p}_1)^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)_t)=0\}.\end{aligned}$$

By proposition 1.0.5, this is a locally closed subscheme of $\hat{G}_1\times\hat{R}_{a;i}$. Moreover, by applying several times lemma 4.6.1, we get that it has a finite disjoint locally closed covering $\{\hat{U}_{a,b,c;i,j}\}_j$ such that the following sheaves are both locally free on $\hat{U}_{a,b,c;i,j}$ and commute with base change:

$$\hat{E}_{a,b,c;i,j} := \mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i},\hat{\mathcal{V}}''_{a;i}),(\hat{p}'_1,\hat{p}_1)^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)\Big),$$

$$\hat{F}_{a,b,c;i,j} := \mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c;i,j}}}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_4,\hat{p}_4)^*(\hat{\mathcal{Q}}_4,\hat{\mathcal{W}}_4),(\hat{p}'_1,\hat{p}_1)^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)\Big).$$

By construction of $\hat{U}_{a,b,c;i,j}$ for every point t of such a scheme we have that:

$$\mathrm{Hom}\Big((\hat{p}'_{34},\hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i},\hat{\mathcal{V}}''_{a;i})_t,(\hat{p}'_1,\hat{p}_1)^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)_t\Big)=0$$

(see the end of the proof of lemma 12.1.5). Moreover, by construction we have already said that the sheaf $\hat{E}_{a,b,c;i,j}$ commutes with base change. Therefore, by proposition 4.5.1 and corollary 4.5.4 there exists a grassmannian fibration

$$\hat{\theta}_{2;a,b,c;i,j}:\hat{Q}_{a,b,c;i,j}:=Grass(2,\hat{E}^{\vee}_{a,b,c;i,j})\longrightarrow\hat{U}_{a,b,c;i,j}$$

and a universal extension (in the sense of corollary 4.5.4) parametrized by $\hat{Q}_{a,b,c;i,j}$

$$0\to(\hat{\theta}'_{2;a,b,c;i,j},\hat{\theta}_{2;a,b,c;i,j})^*(\hat{p}'_1,\hat{p}_1)^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)\otimes_{\hat{Q}_{a,b,c;i,j}}\overline{\mathcal{M}}_{2;a;i}\to$$

$$\to(\hat{\mathcal{E}}_{a,b,c;i,j},\hat{\mathcal{V}}_{a,b,c;i,j})\to(\hat{\theta}'_{2;a,b,c;i,j},\hat{\theta}_{2;a,b,c;i,j})^*(\hat{p}'_{34},\hat{p}_{34})^*(\hat{\mathcal{E}}''_{a;i},\hat{\mathcal{V}}''_{a;i})\to 0, \tag{12.32}$$

where $\overline{\mathcal{M}}_{2;a;i}$ is a locally free sheaf of rank 2 on $\hat{Q}_{a,b,c;i,j}$. Now let us apply the functor

$$\mathcal{H}om_{\pi_{\hat{U}_{a,b,c;i,j}}}\Big(-,(\hat{p}'_1,\hat{p}_1)^*(\hat{\mathcal{Q}}_1,\hat{\mathcal{W}}_1)\Big)$$

to the pullback via $\hat{p}_{34}$ of the exact sequence (12.25). Then we get a long exact sequence as in (12.30); the only differences are that the space $\hat{U}_{a,b,c,d;i,j}$ is replaced by $\hat{U}_{a,b,c;i,j}$ and that the morphisms $\hat{p}_1$ and $\hat{p}_{34}$ now have a different source. By construction of $\hat{U}_{a,b,c;i,j}$ the first sheaf of such a long exact sequence is zero. Moreover, by the previous construction both the second and the third sheaves of that sequence are locally free, so we can rewrite such a sequence as an injective morphism of vector bundles:

$$\overline{\beta^1_{2;a;i}}:\hat{F}_{a,b,c;i,j}\hookrightarrow\hat{E}_{a,b,c;i,j}.$$

So for every $(a,b,c;i,j)$ it makes sense to consider the scheme

$$\hat{R}_{a,b,c;i,j}:=Grass(2,\hat{E}^{\vee}_{a,b,c;i,j})\smallsetminus Grass(2,\hat{F}^{\vee}_{a,b,c;i,j})\subseteq\hat{Q}_{a,b,c;i,j}$$

together with the fibration to $\hat{U}_{a,b,c;i,j}$ given by the restriction of $\hat{\theta}_{2;a,b,c;i,j}$. By lemma 12.1.6, for every point $r\in\hat{R}_{a,b,c;i,j}$ we have that the sequence (12.32) restricts to an exact sequence

$$0\to(Q_1,W_1)^{\oplus 2}\to(E,V)\to(E'',V'')\to 0,$$

where the central object belongs to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, and it has α_c-canonical filtration of type (2,1,1). Then we conclude as usual. □

Proof of proposition 7.5.4. The proof of this proposition is similar to the proof of proposition 7.5.3, so we omit it. We only remark that we don't need an index c since this is equal to a by definition of the set G' in this case; the invariant $a-1$ that replaces $a=c$ in the second grassmannian is a consequence of the second part of lemma 12.1.6. □

12.2 Canonical filtration of type (1,2,1)

In this case the α_c-canonical filtration is given by:

$$0 \subset (E_1, V_1) \subset (E_2, V_2) \subset (E_3, V_3) = (E, V),$$

where $(E_1, V_1) =: (Q_1, W_1)$, $(E_2, V_2)/(E_1, V_1) \simeq (Q_2, W_2) \oplus (Q_3, W_3)$ and $(E, V)/(E_2, V_2) := (Q_4, W_4)$. All the (Q_i, W_i)'s for $i = 1, \cdots, 4$ are α_c-stable coherent systems with the same α_c-slope μ. For the computations of chapter 15 we will only need to restrict to the following 2 subcases:

- $(Q_4, W_4) \not\simeq (Q_i, W_i)$ for $i = 1, 2, 3$ (this will be needed for the case of $G^+(\alpha_c; 4, d, 1)$);
- $(Q_1, W_1) \not\simeq (Q_i, W_i)$ for $i = 2, 3, 4$ (this will be needed for the case of $G^-(\alpha_c; 4, d, 1)$).

12.2.1 First case

In this subsection we will consider the first case. We can associate to every (E, V) that we want to parametrize a pair of exact sequences of the form:

$$0 \to (Q_2, W_2) \oplus (Q_3, W_3) \xrightarrow{\alpha_2} (E'', V'') \xrightarrow{\beta_2} (Q_4, W_4) \to 0; \tag{12.33}$$

$$0 \to (Q_1, W_1) \xrightarrow{\alpha_1} (E, V) \xrightarrow{\beta_1} (E'', V'') \to 0. \tag{12.34}$$

We denote by μ and ν the classes of those 2 exact sequences. If (E, V) has α_c-canonical filtration of type (1,2,1), then it has certainly the following proper α_c-semistable subobjects with α_c-slope μ:

(a) (Q_1, W_1), that is the only α_c-stable one;

(b) an extension of (Q_i, W_i) by (Q_1, W_1) for $i = 2, 3$;

(c) an extension of $(Q_2, W_2) \oplus (Q_3, W_3)$ by (Q_1, W_1).

This is not a complete list, see lemma 12.2.2 If we consider only those subobjects, we get that the following conditions are necessary (but in general not sufficient) in order to have that (E, V) belongs to $G^+(\alpha_c; n, d, k)$:

$$\frac{k_1}{n_1} < \frac{k}{n}, \quad \frac{k_1 + k_i}{n_1 + n_i} < \frac{k}{n} \quad \forall i \in \{2, 3\}, \quad \frac{k_1 + k_2 + k_3}{n_1 + n_2 + n_3} < \frac{k}{n}. \tag{12.35}$$

Analogously, we get that the following conditions are necessary (but in general not sufficient) in order to have that (E, V) belongs to $G^-(\alpha_c; n, d, k)$:

$$\frac{k_1}{n_1} > \frac{k}{n}, \quad \frac{k_1 + k_i}{n_1 + n_i} > \frac{k}{n} \quad \forall i \in \{2, 3\}, \quad \frac{k_1 + k_2 + k_3}{n_1 + n_2 + n_3} > \frac{k}{n}. \tag{12.36}$$

Now let us consider the following long exact sequence obtained by applying the functor $\mathrm{Hom}(-, (Q_1, W_1))$ to (12.33):

$$\cdots \mathrm{Hom}((E'',V''),(Q_1,W_1)) \to \mathrm{Hom}((Q_2,W_2)\oplus(Q_3,W_3),(Q_1,W_1)) \to$$
$$\to \mathrm{Ext}^1((Q_4,W_4),(Q_1,W_1)) \xrightarrow{\overline{\beta_2}}$$
$$\xrightarrow{\overline{\beta_2}} \mathrm{Ext}^1((E'',V''),(Q_1,W_1)) \xrightarrow{\overline{\alpha_2}} \mathrm{Ext}^1((Q_2,W_2)\oplus(Q_3,W_3),(Q_1,W_1)) \to \cdots \tag{12.37}$$

If we apply $\overline{\alpha_2}$ to ν we get a diagram of this form:

$$\begin{tikzcd}
0 \arrow[r] & (Q_1,W_1) \arrow[r, "\alpha_1"] \arrow[d, equal] \arrow[dr, phantom, "\curvearrowright"] & (E,V) \arrow[r, "\beta_1"] \arrow[dr, phantom, "\curvearrowright"] & (E'',V'') \arrow[r] & 0 & \nu \arrow[d, "\overline{\alpha_2}"] \\
0 \arrow[r] & (Q_1,W_1) \arrow[r, "\alpha_1'"] & (E_2,V_2) \arrow[r, "\beta_1'"] \arrow[u, "\delta"] & \oplus_{i=2}^3 (Q_i,W_i) \arrow[r] \arrow[u, "\alpha_2"'] & 0 & \overline{\alpha_2}(\nu).
\end{tikzcd} \tag{12.38}$$

By the snake lemma and (12.33) we get a short exact sequence

$$0 \to (E_2,V_2) \xrightarrow{\delta} (E,V) \xrightarrow{\eta} (Q_4,W_4) \to 0. \tag{12.39}$$

Again by the snake lemma, eventually by replacing η with $\eta\circ\varphi$ for a suitable automorphism φ of (Q_4,W_4) (i.e. $\varphi = \lambda \cdot \mathrm{id}_{(Q_4,W_4)}$ for some $\lambda \in \mathbb{C}^*$), we have that

$$\eta = \beta_2 \circ \beta_1. \tag{12.40}$$

We can identify μ with a pair

$$(\mu_2,\mu_3) \in \bigoplus_{i=1}^{2} \mathrm{Ext}^1\Big((Q_4,W_4),(Q_i,W_i)\Big).$$

For every $i = 2,3$, this identification gives a diagram of the form:

$$\begin{tikzcd}
0 \arrow[r] & \oplus_{l=2}^3 (Q_l,W_l) \arrow[r, "\alpha_2"] \arrow[d, "pr_i"] \arrow[dr, phantom, "\curvearrowright"] & (E'',V'') \arrow[r, "\beta_2"] \arrow[d, "\eta_i"] \arrow[dr, phantom, "\curvearrowright"] & (Q_4,W_4) \arrow[r] \arrow[d, equal] & 0 & \mu \arrow[d, "\overline{pr_i}"] \\
0 \arrow[r] & (Q_i,W_i) \arrow[r, "\alpha_{4i}"] & (E_{4i},V_{4i}) \arrow[r, "\beta_{4i}"] & (Q_4,W_4) \arrow[r] & 0 & \mu_i,
\end{tikzcd} \tag{12.41}$$

where pr_i is the quotient $(Q_2,W_2)\oplus(Q_3,W_3) \twoheadrightarrow (Q_i,W_i)$ for $i = 2,3$. By the snake lemma, for every $i = 2,3$ we get an induced short exact sequence

$$0 \to (Q_j,W_j) \xrightarrow{\delta_i} (E'',V'') \xrightarrow{\eta_i} (E_{4i},V_{4i}) \to 0, \tag{12.42}$$

where j is the index in $\{2,3\}$ different from i. Having fixed all those notations, let us state and prove the following results.

Lemma 12.2.1. *Let us fix any pair of exact sequences as (12.33) and (12.34) and let us denote by μ and ν their classes. Let us suppose that $(Q_4, W_4) \not\simeq (Q_i, W_i)$ for all $i = 1, 2, 3$. Then the following facts are equivalent:*

(a) (E, V) has α_c-canonical filtration of type $(1, 2, 1)$;

(b) for all $i = 2, 3$ and for all morphisms $\gamma_i : (Q_i, W_i) \to (E_2, V_2)$ we have $\beta_1' \circ \gamma_i = 0$; moreover, $\mu \neq 0$.

Proof. Let us assume (b) and let us prove that (a) holds. By diagram (12.38), we get a filtration of (E, V) as follows:

$$0 \subset (Q_1, W_1) =: (E_1, V_1) \subset (E_2, V_2) \subset (E_3, V_3) = (E, V). \tag{12.43}$$

The second line of (12.38) proves that $(E_2, V_2)/(E_1, V_1) = (Q_2, W_2) \oplus (Q_3, W_3)$; moreover, by (12.39) we have $(E, V)/(E_2, V_2) = (Q_4, W_4)$. So for all $i = 1, 2, 3$ the objects of the form $(E_i, V_i)/(E_{i-1}, V_{i-1})$ are α_c-stable or α_c-polystable. Then by proposition 2.1.3 we get that (12.43) is the α_c-canonical filtration of (E, V) if and only if condition (c) of that proposition is satisfied. In our case the index t is equal to 3, $gr_{\alpha_c}(E, V) = \oplus_{i=1}^4 (Q_i, W_i)$ and

$$gr_{\alpha_c}((E, V)/(E_1, V_1)) = gr_{\alpha_c}(E'', V'') = (Q_2, W_2) \oplus (Q_3, W_3) \oplus (Q_4, W_4).$$

So (12.43) is the α_c-canonical filtration of (E, V) if and only if the following two conditions hold:

(i) for all $i = 1, \cdots, 4$ and for all morphisms $\overline{\gamma}_i : (Q_i, W_i) \to (E, V)$ we have $\beta_1 \circ \overline{\gamma}_i = 0$;

(ii) for all $i = 2, 3, 4$ and for all morphisms $\widetilde{\gamma}_i : (Q_i, W_i) \to (E'', V'')$ we have $\beta_2 \circ \widetilde{\gamma}_i = 0$.

Let us consider (ii): if there exists $\widetilde{\gamma}_i$ such that $\beta_2 \circ \widetilde{\gamma}_i \neq 0$, then this is equivalent to have that $\mu = 0$. So (ii) is equivalent to imposing that $\mu \neq 0$.

Now let us consider (i) and let us suppose that there is a morphism $\overline{\gamma_i} : (Q_i, W_i) \to (E, V)$ such that $\beta_1 \circ \overline{\gamma_i} \neq 0$ for some $i = 1, \cdots, 4$. Let us consider the sequence (12.39) and the identity (12.40). If $\eta \circ \overline{\gamma_i} \neq 0$, then $\beta_2 \circ \beta_1 \circ \overline{\gamma_i} \neq 0$, so we get that $\beta_1 \circ \overline{\gamma_i}$ is a splitting of (12.33), so $\mu = 0$. If $\eta \circ \overline{\gamma_i} = 0$, then by exactness of (12.39) there exists a non-zero morphism $\gamma_i : (Q_i, W_i) \to (E_2, V_2)$ such that $\delta \circ \gamma_i = \overline{\gamma}_i$. Now by commutativity of (12.38) and by definition of γ_i, we get:

$$\alpha_2 \circ \beta_1' \circ \gamma_i = \beta_1 \circ \delta \circ \gamma_i = \beta_1 \circ \overline{\gamma_i} \neq 0.$$

Then $\gamma_i : (Q_i, W_i) \to (E_2, V_2)$ is such that $\beta_1' \circ \gamma_i \neq 0$. Since $\beta_1' \circ \gamma_i \neq 0$, then the index i belongs to $\{2, 3\}$. So we have proved that if (b) holds, then (12.43) is the α_c-canonical filtration of (E, V).

Conversely, if $\mu = 0$, then the α_c-canonical filtration of (E, V) can be of type (1,3), (2,2), (3,1) or (4), but not of type (1,2,1). If there exists a morphism $\gamma_i : (Q_i, W_i) \to (E_2, V_2)$ for some $i = 2, 3$ such that $\beta_1' \circ \gamma_i \neq 0$, then by diagram (12.38) we get that (E, V) contains a subobject of the form $(Q_1, W_1) \oplus (Q_i, W_i)$, so the α_c-canonical filtration of (E, V) cannot be of type (1,2,1). So we have proved that (a) and (b) are equivalent. $\square$

Lemma 12.2.2. *Let us fix any pair of exact sequences as (12.33) and (12.34) and let us suppose that*

$$\frac{k_i}{n_i} < \frac{k}{n} \quad \forall\, i \in \{1, 2, 3\}, \tag{12.44}$$

respectively that

$$\frac{k_i}{n_i} > \frac{k}{n} \quad \forall\, i \in \{1, 2, 3\}. \tag{12.45}$$

Let us also suppose that (E, V) *has* α_c*-canonical filtration of type* $(1, 2, 1)$*. Then the following facts are equivalent.*

(a) (E, V) *belongs to* $G^+(\alpha_c; n, d, k)$*, respectively to* $G^-(\alpha_c; n, d, k)$*;*

(b) for all $i = 2, 3$*, there are no quotients* $\zeta_i : (E'', V'') \twoheadrightarrow (Q_i, W_i)$*.*

Proof. Let us suppose that we use conditions (12.44); the other case is completely analogous.

If we assume those conditions, then a direct check proves that conditions (12.35) are satisfied; in particular by using the last part of (12.35) together with (12.44) we get that

$$\frac{k_i}{n_i} < \frac{k}{n} < \frac{k_4}{n_4} \quad \forall\, i = 1, 2, 3.$$

Therefore, $(Q_4, W_4) \not\simeq (Q_i, W_i)$ for all $i = 1, 2, 3$ (so we are in the hypothesis of lemma 12.2.1).

Now let us suppose that there is a quotient $\zeta_i : (E'', V'') \twoheadrightarrow (Q_i, W_i)$ for some $i = 2, 3$. Then by (12.44) we get that the kernel of $\zeta_i \circ \beta_1$ destabilizes (E, V) for α_c^+.

Conversely, if (E, V) is not α_c^+-stable, then there exists a proper subsystem (E', V') of (E, V) that destabilizes it for α_c^+. Such an object is necessarily α_c-semistable and the length r of any α_c-Jordan-Hölder filtration of (E', V') can be equal to $1, 2$ or 3, so we have to handle all these cases. By definition of α_c-canonical filtration of type (1,2,1), we have that (Q_1, W_1) is the only α_c-stable subobject of (E, V), so it is a subobject also of (E', V'). So if $r = 1$, then we have that $(E', V') = (Q_1, W_1)$, so it does not destabilize (E, V) for α_c^+ because of (12.35).

Length of any α_c-Jordan-Hölder filtration of (E', V') equal to 2. If (E', V') is an extension of (Q_i, W_i) by (Q_1, W_1) for some $i = 2, 3$, then it does not destabilize (E, V) for α_c^+

because of conditions (12.35). So let us suppose that (E', V') sits in an exact sequence of the form

$$0 \to (Q_1, W_1) \xrightarrow{\alpha} (E', V') \xrightarrow{\beta} (Q_4, W_4) \to 0 \tag{12.46}$$

and let us denote by γ the inclusion of (E', V') in (E, V). Then let us consider the exact sequence (12.39) and let us distinguish the following 2 cases.

- If $\eta \circ \gamma = 0$, this implies that there exists an embedding $\gamma' : (E', V') \to (E_2, V_2)$ such that $\gamma = \delta \circ \gamma'$. The graded of the first coherent system is $(Q_1, W_1) \oplus (Q_4, W_4)$ and the graded of the second one is $\oplus_{i=1}^3 (Q_i, W_i)$. Since $(Q_4, W_4) \not\simeq (Q_i, W_i)$ for $i = 1, 2, 3$, we get a contradiction, so this case cannot happen in our hypothesis.

- Let us suppose that $\eta \circ \gamma : (E', V') \to (Q_4, W_4)$ is non-zero. Then by (12.40) we get that $\beta_2 \circ \beta_1 \circ \gamma \neq 0$. Since $(Q_1, W_1) \not\simeq (Q_4, W_4)$, then

$$\beta_2 \circ \beta_1 \circ \gamma \circ \alpha = 0.$$

So by exactness of (12.46) we get an induced morphism $\gamma'' : (Q_4, W_4) \to (Q_4, W_4)$ such that $\gamma'' \circ \beta = \beta_2 \circ \beta_1 \circ \gamma \neq 0$. In particular, this implies that $\gamma'' \neq 0$, so it is of the form $\lambda \cdot \mathrm{id}_{(Q_4, W_4)}$ for some $\lambda \in \mathbb{C}^*$. Since $\beta_2 \circ \beta_1 \circ \gamma \circ \alpha = 0$, then by exactness of (12.33) we get an induced injective morphism $\gamma''' : (Q_1, W_1) \to (Q_2, W_2) \oplus (Q_3, W_3)$ such that $\alpha_2 \circ \gamma''' = \beta_1 \circ \gamma \circ \alpha$ (this proves also that (Q_1, W_1) is isomorphic to (Q_2, W_2) or to (Q_3, W_3)). Then we get a commutative diagram with exact rows as follows:

$$\begin{tikzcd}
0 \arrow[r] & (Q_1, W_1) \arrow[r, "\alpha"] \arrow[d, "\gamma'''"] \arrow[dr, phantom, "\curvearrowright"] & (E', V') \arrow[r, "\lambda \cdot \beta"] \arrow[d, "\gamma"] \arrow[dr, phantom, "\curvearrowright"] & (Q_4, W_4) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \oplus_{l=2}^3 (Q_l, W_l) \arrow[r, "\alpha_2"] & (E, V) \arrow[r, "\beta_2"] & (Q_4, W_4) \arrow[r] & 0.
\end{tikzcd}$$

The second line is a representative for μ; the previous diagram proves that μ is in the image of

$$\overline{\gamma'''} : \mathrm{Ext}^1((Q_4, W_4), (Q_1, W_1)) \longrightarrow \mathrm{Ext}^1((Q_4, W_4), (Q_2, W_2) \oplus (Q_3, W_3)).$$

Then if we denote by $pr_i : (Q_2, W_2) \oplus (Q_3, W_3) \twoheadrightarrow (Q_i, W_i)$ the cokernel of γ''' for some $i = 2, 3$, then we get that $\overline{pr_i}(\mu) = 0$, so we have a commutative diagram where the second line is split:

$$\begin{tikzcd}
0 \arrow[r] & \oplus_{l=2}^{3}(Q_l,W_l) \arrow[r, "\alpha_2"] \arrow[d, "pr_i"] \arrow[dr, phantom, "\curvearrowright"] & (E'',V'') \arrow[r, "\beta_2"] \arrow[d, "pr'_i"] \arrow[dr, phantom, "\curvearrowright"] & (Q_4,W_4) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (Q_i,W_i) \arrow[r] & (Q_i,W_i)\oplus(Q_4,W_4) \arrow[r] & (Q_4,W_4) \arrow[r] & 0.
\end{tikzcd}$$

In particular, we get a quotient $(E'',V'') \twoheadrightarrow (Q_i,W_i)$ for some $i = 2,3$, so (b) is not satisfied.

Length of any α_c-Jordan-Hölder filtration of (E',V') equal to 3. In this case the quotient $(\widetilde{E},\widetilde{V}) := (E,V)/(E',V')$ is an α_c-stable coherent system. Since (E',V') contains (Q_1,W_1), then $(\widetilde{E},\widetilde{V})$ is isomorphic to (Q_i,W_i) for some $i = 2,3,4$. Using the last condition of (12.35), if $i = 4$ then (E',V') does not destabilize (E,V) for α_c^+. If i is equal to 2 or 3, then conditions (12.44) imply that (E',V') does destabilize (E,V) for α_c^+. Let us suppose that (E,V) has such a subobject and let us denote by $\zeta_i : (E,V) \twoheadrightarrow (Q_i,W_i)$ the induced quotient for $i = 2$ or 3. Let us consider the exact sequence (12.34): if $\zeta_i \circ \alpha_1 \neq 0$, then $(Q_i,W_i) \simeq (Q_1,W_1)$ and that sequence is split, so we get that

$$(E,V) \simeq (Q_1,W_1) \oplus (E'',V'') \supset (Q_1,W_1) \oplus (Q_2,W_2) \oplus (Q_3,W_3),$$

so the α_c-canonical filtration of (E,V) cannot be of type $(1,2,1)$, so this is impossible by hypothesis. Therefore, $\zeta_i \circ \alpha_1 = 0$, so we get an induced morphism $\zeta'_i : (E'',V'') \twoheadrightarrow (Q_i,W_i)$ such that $\zeta_i = \zeta'_i \circ \beta_1$.

So we have proved that if (a) is not satisfied, neither is (b). Together with the first part of the proof, this is enough to conclude. □

We remark that the second condition of (b) in lemma 12.2.1 (i.e. $\mu \neq 0$) is implicated by condition (b) of lemma 12.2.2 Therefore, as a corollary of those 2 lemmas we get:

Corollary 12.2.3. *Let fix any quadruple $(Q_i,W_i)_{i=1,\cdot,4} \in \prod_{i=1}^4 G_i$ and let us suppose that conditions (12.44), respectively (12.45), are satisfied (automatically, this implies that $(Q_4,W_4) \not\simeq (Q_i,W_i)$ for all $i = 1,2,3$). Then the (E,V)'s that belong to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, that have α_c-canonical filtration of type $(1,2,1)$ and graded $\oplus_{i=1}^4(Q_i,W_i)$ are those induced by pairs of exact sequences as (12.33) and (12.34), such that:*

- *for all $i = 2,3$ there are no quotients $\zeta_i : (E'',V'') \twoheadrightarrow (Q_i,W_i)$;*
- *for all $i = 2,3$ and for all morphisms $\gamma_i : (Q_i,W_i) \to (E_2,V_2)$ we have $\beta'_1 \circ \gamma_i = 0$.*

Now we have to state 2 lemmas according to the relation between (Q_2,W_2) and (Q_3,W_3).

Lemma 12.2.4. *Let us fix any quadruple $(Q_i, W_i)_{i=1,\cdots,4} \in \prod_{i=1}^{4} G_i$ with numerical conditions (12.44), respectively (12.45), and such that:*

$$(Q_2, W_2) \not\simeq (Q_3, W_3)$$

(automatically, this implies that $(Q_4, W_4) \not\simeq (Q_i, W_i)$ for all $i = 1, 2, 3$). Let us denote by μ_i any class of an extension of the form

$$0 \to (Q_i, W_i) \xrightarrow{\alpha_{4i}} (E_{4i}, V_{4i}) \xrightarrow{\beta_{4i}} (Q_4, W_4) \to 0 \tag{12.47}$$

for $i = 2, 3$ and let us denote by μ the class of the extension

$$0 \to (Q_2, W_2) \oplus (Q_3, W_3) \xrightarrow{\alpha_1} (E'', V'') \xrightarrow{\beta_1} (Q_4, W_4) \to 0 \tag{12.48}$$

obtained by μ_2 and μ_3. Having fixed

$$([\mu_2], [\mu_3]) \in \bigoplus_{i=2}^{3} \mathbb{P}(Ext^1((Q_4, W_4), (Q_i, W_i))),$$

let us consider the morphisms

$$Ext^1((E_{4i}, V_{4i}), (Q_1, W_1)) \xrightarrow{\overline{\eta_i}} Ext^1((E'', V''), (Q_1, W_1))$$

for $i = 2, 3$ induced by the morphisms η_i of diagram (12.41). Let us consider the set $M([\mu_2], [\mu_3])$ defined as

$$Ext^1((E'', V''), (Q_1, W_1)) \smallsetminus (Im\ \overline{\eta_2} + Im\ \overline{\eta_3}).$$

Then we have that the set of all the (E, V)'s that belong to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, that have α_c-canonical filtration of type $(1, 2, 1)$ and graded $\oplus_{i=1}^{4}(Q_i, W_i)$ is given by a fibration over

$$\mathbb{P}(Ext^1((Q_4, W_4), (Q_2, W_2))) \times \mathbb{P}(Ext^1((Q_4, W_4), (Q_3, W_3))).$$

The fiber over any point $([\mu_2], [\mu_3])$ with μ_2, μ_3 as before is given by

$$\overline{M}([\mu_2], [\mu_3]) := M([\mu_2], [\mu_3])/\mathbb{C}^*.$$

Moreover, if we consider the morphism $\overline{\beta_2}$ appearing in (12.37), we get that $Im\ \overline{\eta_2} \cap Im\ \overline{\eta_3} = Im\ \overline{\beta_2}$, so

$$dim(Im\ \overline{\eta_2} + Im\ \overline{\eta_3}) = dim\ Im\ \overline{\eta_2} + dim\ Im\ \overline{\eta_3} - Im\ \overline{\beta_2}.$$

In addition, if we write:

$$c := dim\ Ext^1((E'', V''), (Q_1, W_1)), \quad d := dim\ Ext^1((E_{42}, V_{42}), (Q_1, W_1)),$$
$$e := dim\ Ext^1((E_{43}, V_{43}), (Q_1, W_1)), \quad f := dim\ Ext^1((Q_4, W_4), (Q_1, W_1)),$$

then for every $([\mu_2],[\mu_3])$ *we have that*

$$\overline{M}([\mu_2],[\mu_3]) \simeq \mathbb{P}^{c-1} \smallsetminus \mathbb{P}^{d+e-f-1}$$

Proof. To any (E,V) that we want to parametrize we can associate a triple $(\mu_2,\mu_3,\nu)=(\mu,\nu)$, where μ and ν have representatives of the form (12.33), respectively (12.34) and μ_2 and μ_3 are as in diagram (12.41). Then by corollary 12.2.3, we get that (E,V) belongs to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, and it has α_c-canonical filtration of type $(1,2,1)$ if and only if the following conditions hold:

(i) for all $i=2,3$ there are no quotients $\zeta_i:(E'',V'')\twoheadrightarrow(Q_i,W_i)$;

(ii) for all $i=2,3$ and for all morphisms $\gamma_i:(Q_i,W_i)\to(E_2,V_2)$ we have $\beta_1'\circ\gamma_i=0$.

Let us consider the first condition: since $(Q_4,W_4)\not\simeq(Q_i,W_i)$ for $i=2,3$ and since $(Q_2,W_2)\not\simeq(Q_3,W_3))$, then by lemma 3.3.1 we get that (i) is equivalent to saying that μ_2 and μ_3 are both different from zero. So from now on let us simply restrict to the case when this happens.

Now let us consider condition (ii) and let us denote by $\nu'=(\nu_2',\nu_3')$ the object $\overline{\alpha_2}(\nu)$. For every $i=2,3$, this identification is given by a diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & (Q_1,W_1) \arrow[r, "\alpha_1'"] \arrow[d, equal] \arrow[dr, phantom, "\curvearrowright"] & (E_2,V_2) \arrow[r, "\beta_1'"] \arrow[dr, phantom, "\curvearrowright"] & \oplus_{l=2}^3(Q_l,W_l) \arrow[r] & 0 & \nu'=\overline{\alpha_2}(\nu) \arrow[d, "\overline{\varepsilon_i}"] \\
0 \arrow[r] & (Q_1,W_1) \arrow[r, "\alpha^{i1}"] & (E^{i1},V^{i1}) \arrow[r, "\beta^{i1}"] \arrow[u, "\delta^{i1}"'] & (Q_i,W_i) \arrow[r] \arrow[u, "\varepsilon_i"'] & 0 & \nu_i'.
\end{tikzcd} \tag{12.49}$$

Since $(Q_2,W_2)\not\simeq(Q_3,W_3)$, by lemma 3.3.2 we get that (ii) is equivalent to imposing that ν_2' and ν_3' are both non-zero.

Now let us fix any ordering (i,j) of $\{2,3\}$. By construction for every $j=2,3$ we have

$$\nu_j'=\overline{\varepsilon_j}(\nu')=\overline{\varepsilon_j}\circ\overline{\alpha_2}(\nu)=\overline{\alpha_2\circ\varepsilon_j}(\nu).$$

Let us consider the sequence (12.42): the morphism $\alpha_2\circ\varepsilon_j$ is an embedding from (Q_j,W_j) to (E'',V''). Since (Q_j,W_j) is not isomorphic neither to (Q_i,W_i) nor to (Q_4,W_4), then we get that $\eta_i\circ(\alpha_2\circ\varepsilon_j)=0$. So by exactness of that sequence we get that $\alpha_2\circ\varepsilon_j$ coincides with the morphism $\delta_i:(Q_j,W_j)\to(E'',V'')$, up to an automorphism of (Q_j,W_j), i.e. up to multiplication by non-zero scalars. So we can write

$$\nu_j'=\overline{\lambda\cdot\delta_i}(\nu)=\lambda\cdot\overline{\delta_i}(\nu)$$

for some $\lambda \in \mathbb{C}^*$. So ν_j' is different from zero if and only if $\overline{\delta_i}(\nu) \neq 0$.

Now let us consider the long exact sequences induced by applying the functor $\mathrm{Hom}(-,(Q_1, W_1))$ to the exact sequences (12.42) for $i = 2,3$:

$$\cdots \to \mathrm{Hom}((E'',V''),(Q_1,W_1)) \to \mathrm{Hom}((Q_j,W_j),(Q_1,W_1)) \to \mathrm{Ext}^1((E_{4i},V_{4i}),(Q_1,W_1)) \xrightarrow{\overline{\eta_i}}$$

$$\xrightarrow{\overline{\eta_i}} \mathrm{Ext}^1((E'',V''),(Q_1,W_1)) \xrightarrow{\overline{\delta_i}} \mathrm{Ext}^1((Q_j,W_j),(Q_1,W_1)) \to \cdots$$

Let us consider the first term of this sequence and let us suppose that it contains a non-zero morphism ζ. Since the graded of (E'',V'') is $\oplus_{l=2}^4(Q_l,W_l)$, then necessarily $(Q_1,W_1) \simeq (Q_l,W_l)$ for some $l = 2,3,4$. By hypothesis (Q_1,W_1) is not isomorphic to (Q_4,W_4), therefore ζ is a non-zero morphism from (E'',V'') to (Q_l,W_l) for some $l = 2,3$. Since it is non-zero and the target is α_c-stable, then it is surjective, but this is impossible by condition (i). Therefore the first object of the previous long exact sequence is zero. Now if $(Q_j,W_j) \not\simeq (Q_1,W_1)$, then also the second object of such a sequence is zero, so $\overline{\eta_i}$ is injective; in the opposite case $\overline{\eta_i}$ has a kernel of dimension 1.

Now we need to remove from the set of all the ν's in $\mathrm{Ext}^1((E'',V''),(Q_1,W_1))$ those such that either ν_2' or ν_3' are zero, i.e. all those $\nu's$ that are in the image of $\overline{\eta_2} + \overline{\eta_3}$. In order to compute the dimension of such a space we need to describe the subvector space $\mathrm{Im}\ \overline{\eta_2} \cap \mathrm{Im}\ \overline{\eta_3}$. If ν belongs to such a space, then this is equivalent to saying that $\nu' = (\nu_2', \nu_3') = (0,0)$, i.e. $\overline{\alpha_2}(\nu) = 0$. So by exactness of (12.37) we have

$$\mathrm{Im}\ \overline{\eta_2} \cap \mathrm{Im}\ \overline{\eta_3} = \mathrm{Im}\ \overline{\beta_2}.$$

Now let us consider again the sequence (12.37). Also there the first term is zero, so we have that:

- if $(Q_1,W_1) \not\simeq (Q_i,W_i)$ for $i = 2,3$, then $\overline{\beta_2}$ is injective;
- if $(Q_1,W_1) \simeq (Q_i,W_i) \not\simeq (Q_j,W_j)$ for any choice of ordering (i,j) of $\{2,3\}$, then $\overline{\beta_2}$ has a kernel of dimension 1.

Moreover,

- if $(Q_1,W_1) \not\simeq (Q_i,W_i)$ for $i = 2,3$, then both $\overline{\eta_2}$ and $\overline{\eta_3}$ are injective;
- if $(Q_1,W_1) \simeq (Q_i,W_i) \not\simeq (Q_j,W_j)$ for any choice of ordering (i,j) of $\{2,3\}$, then $\overline{\eta_j}$ has a kernel of dimension 1 and $\overline{\eta_i}$ is injective.

Therefore in both cases we get that

$$\dim\ \mathrm{Im}(\overline{\eta_2} + \overline{\eta_3}) = d + e - f = d + e - 1 - (f - 1).$$

Now if we look at the sequence (12.34), we get that the (E'', V'')'s there are parametrized by pairs

$$([\mu_2], [\mu_3]) \in \bigoplus_{i=2}^{3} \mathbb{P}(\mathrm{Ext}^1((Q_i, W_i), (Q_1, W_1))).$$

Moreover, $\mathrm{Aut}(Q_1, W_1) = \mathbb{C}^*$ because (Q_1, W_1) is α_c-stable and also $\mathrm{Aut}(E'', V'') = \mathbb{C}^*$ since $(Q_4, W_4) \not\simeq (Q_i, W_i)$ for $i = 2, 3$. So we conclude that having fixed $([\mu_2], [\mu_3])$, the (E, V)'s that we are interested in are parametrized by

$$\overline{M}([\mu_2], [\mu_3]) := \Big(\mathrm{Ext}^1((E'', V''), (Q_1, W_1)) \smallsetminus \mathrm{Im}(\overline{\eta_2} + \overline{\eta_3})\Big)/\mathbb{C}^*.$$

Then the previous description proves that such a set is isomorphic to $\mathbb{P}^{c-1} \smallsetminus \mathbb{P}^{d+e-f-1}$. $\square$

Lemma 12.2.5. *Let us fix any quadruple $(Q_i, W_i)_{i=1,\cdots,4} \in \prod_{i=1}^4 G_i$ with numerical conditions (12.44), respectively (12.45), and such that:*

$$(Q_2, W_2) \simeq (Q_3, W_3)$$

(automatically, this implies that $(Q_4, W_4) \not\simeq (Q_i, W_i)$ for all $i = 1, 2, 3$). Let us denote by μ_i any class of an extension of the form

$$0 \to (Q_2, W_2) \xrightarrow{\alpha_{4i}} (E_{4i}, V_{4i}) \xrightarrow{\beta_{4i}} (Q_4, W_4) \to 0 \tag{12.50}$$

for $i = 2, 3$ and let us denote by μ the class of the extension

$$0 \to (Q_2, W_2) \oplus (Q_2, W_2) \xrightarrow{\alpha_2} (E'', V'') \xrightarrow{\beta_2} (Q_4, W_4) \to 0 \tag{12.51}$$

obtained by μ_2 and μ_3. Then we have that the set of all the (E, V)'s that belong to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, that have α_c-canonical filtration of type $(1, 2, 1)$ and graded $\oplus_{i=1}^4 (Q_i, W_i)$ is given by a fibration over

$$Grass(2, Ext^1((Q_4, W_4), (Q_2, W_2))).$$

The fiber over any point $< \mu_2, \mu_3 >$ with μ_2, μ_3 as before is given by

$$\overline{M}(< \mu_2, \mu_3 >) := M(< \mu_2, \mu_3 >)/\mathbb{C}^*,$$

where

$$M(< \mu_2, \mu_3 >) := \{\nu \in Ext^1((E'', V''), (Q_1, W_1)) \text{ s.t. } \overline{\varepsilon_2} \circ \overline{\alpha_2}(\nu)$$
$$\text{and } \overline{\varepsilon_3} \circ \overline{\alpha_2}(\nu) \text{ are linearly independent in } Ext^1((Q_2, W_2), (Q_1, W_1))\}.$$

(this set is well defined even if α_2 is not uniquely determined by $< \mu_2, \mu_3 >$, see the proof below).

Proof. To any (E, V) that we want to parametrize we can associate a triple $(\mu_2, \mu_3, \nu) = (\mu, \nu)$, where μ and ν have representatives of the form (12.33), respectively (12.34) and μ_2 and μ_3 are as in diagram (12.41). Then by corollary 12.2.3, we get that (E, V) belongs to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, and it has α_c-canonical filtration of type $(1, 2, 1)$ if and only if the following conditions hold:

(i) there are no quotients $\zeta_2 : (E'', V'') \twoheadrightarrow (Q_2, W_2)$;

(ii) for all morphisms $\gamma_2 : (Q_2, W_2) \to (E_2, V_2)$ we have $\beta_1' \circ \gamma_2 = 0$.

Let us consider the first condition: since $(Q_4, W_4) \not\simeq (Q_2, W_2)$, then by lemma 3.3.1 we get that (i) is equivalent to saying that μ_2 and μ_3 are linearly independent in $\mathrm{Ext}^1((Q_4, W_4), (Q_2, W_2))$.

Now let us consider condition (ii) and let us denote by $\nu' = (\nu_2', \nu_3')$ the object $\overline{\alpha_2}(\nu)$. For every $i = 2, 3$, this identification is given by a diagram as (12.49) with $(Q_2, W_2) = (Q_3, W_3)$. Then by lemma 3.3.2 we get that (ii) is equivalent to imposing that ν_2' and ν_3' are linearly independent in $\mathrm{Ext}^1((Q_2, W_2), (Q_1, W_1))$. By construction for every $j = 2, 3$ we have

$$\nu_j' = \overline{\varepsilon_j}(\nu') = \overline{\varepsilon_j} \circ \overline{\alpha_2}(\nu).$$

Now let us consider the sequence (12.34). The set of the (E'', V'')'s there is in bijection with the set of points

$$< \mu_2, \mu_3 > \in Grass(2, \mathrm{Ext}^1((Q_4, W_4), (Q_2, W_2))).$$

Since $(Q_2, W_2) \not\simeq (Q_4, W_4)$, then $\mathrm{Aut}(E'', V'') = \mathbb{C}^*$; moreover we have also $\mathrm{Aut}(Q_1, W_1) = \mathbb{C}^*$ because (Q_1, W_1) is α_c-stable. Therefore, having fixed a point $< \mu_2, \mu_3 >$, we have that there's a natural action of $\mathbb{C}^*$ on the set

$$M(< \mu_2, \mu_3 >) := \{\nu \in \mathrm{Ext}^1((E'', V''), (Q_1, W_1)) \text{ s.t.}$$
$$\overline{\varepsilon_2} \circ \overline{\alpha_2}(\nu) \text{ and } \overline{\varepsilon_3} \circ \overline{\alpha_2}(\nu) \text{ are linearly independent}\}$$

and the corresponding (E, V)'s are parametrized by

$$\overline{M}(< \mu_2, \mu_3 >) := M(< \mu_2, \mu_3 >)/\mathbb{C}^*.$$

The only thing we have still to prove is that the set $M(< \mu_2, \mu_3 >)$ is well defined. Indeed a priori the condition

$$\text{“}\overline{\varepsilon_2} \circ \overline{\alpha_2}(\nu) \text{ and } \overline{\varepsilon_3} \circ \overline{\alpha_2}(\nu) \text{ linearly independent”}$$

depends on the choice of α_2, i.e. on the representative (12.51) for the point $< \mu_2, \mu_3 >$. So let us suppose that we have chosen another representative $< \mu_2', \mu_3' >$ for $< \mu_2, \mu_3 >$ and let us denote by μ' the class of the extension

$$0 \to (Q_2, W_2) \oplus (Q_2, W_2) \xrightarrow{\alpha_2'} (E'', V'') \xrightarrow{\beta_2'} (Q_4, W_4) \to 0 \tag{12.52}$$

associated to the pair (μ_2', μ_3'). Note that the central term is the same of (12.51) (up to isomorphism) since it depends only on $< \mu_2, \mu_3 >$ and not on its representative. Now since (12.51) is exact and since $(Q_2, W_2) \not\simeq (Q_4, W_4)$, then we get that the morphism α_2' induces an injective morphism

$$A : (Q_2, W_2)^{\oplus_2} \longrightarrow (Q_2, W_2)^{\oplus_2}$$

such that $\alpha_2' = \alpha_2 \circ A$. Since A is injective, then it is also surjective, so it is of the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Aut}((Q_2, W_2)^{\oplus_2}) = GL(2, \mathbb{C}).$$

Now let us fix any

$$\nu \in \mathrm{Ext}^1((E'', V''), (Q_1, W_1))$$

and let us suppose that there exists $(\lambda_2, \lambda_3) \in \mathbb{C}^2 \smallsetminus \{0\}$ such that

$$\lambda_2 \overline{\varepsilon_2} \circ \overline{\alpha_2'}(\nu) + \lambda_3 \overline{\varepsilon_3} \circ \overline{\alpha_2'}(\nu) = 0.$$

By construction of A, this implies that

$$0 = \lambda_2 \cdot \overline{\varepsilon_2} \circ \overline{A} \circ \overline{\alpha_2}(\nu) + \lambda_3 \cdot \overline{\varepsilon_3} \circ \overline{A} \circ \overline{\alpha_2}(\nu) = \lambda_2 \cdot \overline{A \circ \varepsilon_2} \circ \overline{\alpha_2}(\nu) + \lambda_3 \cdot \overline{A \circ \varepsilon_3} \circ \overline{\alpha_2}(\nu) =$$

$$= \lambda_2 \cdot \overline{a\varepsilon_2 + b\varepsilon_3} \circ \overline{\alpha_2}(\nu) + \lambda_3 \cdot \overline{c\varepsilon_2 + d\varepsilon_3} \circ \overline{\alpha_2}(\nu) = (a\lambda_2 + c\lambda_3) \cdot \overline{\varepsilon_2} \circ \overline{\alpha_2}(\nu) + (b\lambda_2 + d\lambda_3) \cdot \overline{\varepsilon_3} \circ \overline{\alpha_2}(\nu).$$

Now

$$\begin{pmatrix} a\lambda_2 + c\lambda_3 \\ b\lambda_2 + d\lambda_3 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix}.$$

Since the matrix A is invertible, so is its transposed; moreover $(\lambda_2, \lambda_3) \in \mathbb{C}^2 \smallsetminus \{0\}$. Therefore, $(a\lambda_2 + c\lambda_3, b\lambda_2 + d\lambda_3) \in \mathbb{C}^2 \smallsetminus \{0\}$. So we conclude that the set $M(< \mu_2, \mu_3 >)$ does not depend on α_2 but only on $< \mu_2, \mu_3 >$.

□

Now we want to give a global parametrization of the objects described before, i.e. we want to describe families of schemes that parametrize various types of (E, V)'s when the graded $\oplus_{i=1}^4 (Q_i, W_i)$ varies over $\prod_{i=1}^4 G_i$. Let us denote by $\oplus_{i=1}^4 (Q_i, W_i)$ a fixed graded with conditions (12.44), respectively (12.45), and such that $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$. If we assume that $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$, then (12.44), respectively (12.45), are equivalent to imposing that

$$\frac{k_1}{n_1} < \frac{k}{n}, \tag{12.53}$$

respectively that

$$\frac{k_1}{n_1} > \frac{k}{n}. \tag{12.54}$$

If $(Q_2, W_2) \not\simeq (Q_3, W_3)$, then by lemma 12.2.4 the corresponding (E, V)'s are parametrized by triples $([\mu_2], [\mu_3], [\nu])$ with $[\mu_i] \in \mathbb{P}(\mathrm{Ext}^1((Q_4, W_4), (Q_i, W_i)))$ for $i = 2, 3$, representative (12.48) for $\mu = (\mu_2, \mu_3)$ and

$$[\nu] \in \overline{M}([\mu_2], [\mu_3]) \subset \mathbb{P}(\mathrm{Ext}^1((E'', V''), (Q_1, W_1))).$$

We are considering the case when the (Q_i, W_i)'s are all of the same type for $i = 1, 2, 3$. Therefore we need to take into account the possible isomorphisms between them. So we need to consider separately the following cases.

(1) If $(Q_1, W_1) \simeq (Q_2, W_2) \not\simeq (Q_3, W_3)$, then the roles of (Q_2, W_2) and of (Q_3, W_3) are not interchangeable, so we need to consider *ordered* pairs $([\mu_2], [\mu_3])$.

(2) If $(Q_i, W_i) \not\simeq (Q_j, W_j)$ for all $i \neq j \in \{1, 2, 3\}$, then the roles of (Q_2, W_2) and of (Q_3, W_3) are interchangeable. Therefore, we need to consider *unordered* pairs $([\mu_2], [\mu_3])$, so we will have to take into account an action of $\mathbb{Z}_2$ on schemes constructed as in (1).

Note that since the order of (Q_2, W_2) and (Q_3, W_3) is not important, we don't need to consider also the case $(Q_1, W_1) \simeq (Q_3, W_3) \not\simeq (Q_2, W_2)$.

If $(Q_2, W_2) \simeq (Q_3, W_3)$, then by lemma 12.2.5 the corresponding (E, V)'s are parametrized by pairs $(< \mu_2, \mu_3 >, [\nu])$ with

$$< \mu_2, \mu_3 > \in Grass(2, \mathrm{Ext}^1((Q_4, W_4), (Q_2, W_2)))$$

and

$$[\nu] \in \overline{M}(< \mu_2, \mu_3 >) \subset \mathbb{P}(\mathrm{Ext}^1((E'', V''), (Q_1, W_1))).$$

We need to consider separately the following cases:

(3) $(Q_1, W_1) \not\simeq (Q_2, W_2) \simeq (Q_3, W_3)$;

(4) $(Q_1, W_1) \simeq (Q_2, W_2) \simeq (Q_3, W_3)$.

Remark 12.2.1. The only case that we are able to describe completely is case (1). In the other 3 cases it is not currently possible to get a global description and/or such a description is not good enough in order to compute Hodge-Deligne polynomials. Therefore, we only give the details for the first case, namely proposition 7.6.1.

Proof of proposition 7.6.1. The first part of this construction is analogous to the construction performed in the proof of proposition 7.1.1. We state anyway all the details since we cannot use exactly the same notations used there.

First of all, we consider a set of data $\mathscr{D}_a^2$ given by:

- $r = 2$, i.e. we are considering a tree with only 2 leaves and an internal node;
- the invariants (n_2, k_2) and (n_4, k_4) associated to the first leaf, respectively to the second leaf;
- any non-negative integer a such that there exists $((Q_2, W_2), (Q_4, W_4)) \in G_2 \times G_4$ with

$$\dim \operatorname{Ext}^1((Q_4, W_4), (Q_2, W_2)) = a.$$

If we use condition (12.53), respectively (12.54), together with the fact that $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$ then we get that $\frac{k_2}{n_2} \neq \frac{k_4}{n_4}$. So by lemma 1.0.4 for every pair of points $(Q_2, W_2) \in G_2$ and $(Q_4, W_4) \in G_4$ we have

$$\operatorname{Hom}((Q_4, W_4), (Q_2, W_2)) = 0.$$

Then by proposition 5.0.5 for $r = 2$ we get the following objects:

- a finite set of indices L_a^2;
- a covering of

$$\hat{U}_a^2 := \{t \in \hat{G}_2 \times \hat{G}_4 \text{ s.t. } \dim \operatorname{Ext}^1((\hat{q}_a^{2'}, \hat{q}_a^2)(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)_t, (\hat{p}_a^{2'}, \hat{p}_a^2)(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2)_t) = a\}$$

  by locally closed subschemes $\hat{U}_{a;i}^2$ with $i \in L_a^2$; we denote by $\hat{p}_{a;i}^2$, $\hat{q}_{a;i}^2$ and $\hat{\pi}_{a;i}^2$ the various projections composed with the corresponding locally closed embeddings; so for example $\hat{p}_{a;i}^2 : \hat{U}_{a;i}^2 \hookrightarrow \hat{G}_2 \times \hat{G}_4 \to \hat{G}_2$;
- for every $i \in L_a^2$, a locally free sheaf on $\hat{U}_{a;i}^2$:

$$\hat{\mathcal{H}}_{a;i}^2 := \mathcal{E}xt^1_{\hat{\pi}_{\hat{U}_{a;i}^2}} \Big((\hat{q}_{a;i}^{2'}, \hat{q}_{a;i}^2)^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4), (\hat{p}_{a;i}^{2'}, \hat{p}_{a;i}^2)^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2) \Big)^{\vee},$$

  where $(\hat{\mathcal{Q}}_l, \hat{\mathcal{W}}_l)$ is the local universal family parametrized by $\hat{G}(\alpha_c; n_l, d_l, k_l)$;
- projective fibrations for every $i \in L_a^2$:

$$\hat{\varphi}_{a;i}^2 : \hat{R}_{a;i}^2 := \mathbb{P}(\hat{\mathcal{H}}_{a;i}^2) \longrightarrow \hat{U}_{a;i}^2$$

  with fibers isomorphic to $\mathbb{P}^{a-1}$;

- universal extensions for every $i \in L^2_a$, parametrized by $\hat{R}^2_{a;i}$

$$
\begin{aligned}
&0 \to (\hat{\varphi}^{2'}_{a;i}, \hat{\varphi}^{2}_{a;i})^*(\hat{p}^{2'}_{a;i}, \hat{p}^{2}_{a;i})^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2) \otimes_{\hat{R}^2_{a;i}} \mathcal{O}_{\hat{R}^2_{a;i}}(1) \to \\
&\to (\hat{\mathcal{E}}^2_{a;i}, \hat{\mathcal{V}}^2_{a;i}) \to (\hat{\varphi}^{2'}_{a;i}, \hat{\varphi}^{2}_{a;i})^*(\hat{q}^{2'}_{a;i}, \hat{q}^{2}_{a;i})^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4) \to 0.
\end{aligned} \tag{12.55}
$$

Analogously, let us fix a set of data $\mathscr{D}^3_b$ as follows:

- $r = 2$, i.e. we are again considering a tree with only 2 leaves and an internal node;
- the invariants (n_3, k_3) and (n_4, k_4) associated to the first leaf, respectively to the second leaf;
- any non-negative integer b such that there exists $((Q_3, W_3), (Q_4, W_4)) \in G_3 \times G_4$ with

$$\dim \operatorname{Ext}^1((Q_4, W_4), (Q_3, W_3)) = b.$$

Then by proposition 5.0.5 we get the following objects:

- a finite set of indices L^3_b;
- a covering of

$$\hat{U}^3_b := \{t \in \hat{G}_2 \times \hat{G}_4 \text{ s.t. } \dim \operatorname{Ext}^1((\hat{q}^{3'}_b, \hat{q}^3_b)(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)_t, (\hat{p}^{3'}_b, \hat{p}^3_b)(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)_t) = b\}$$

  by integral locally closed subschemes $\hat{U}^3_{b;j}$ with $j \in L^3_b$; we denote by $\hat{p}^3_{b;j}$, $\hat{q}^3_{b;j}$ and $\hat{\pi}^3_{b;j}$ the various projections composed with the corresponding locally closed embeddings;
- for every $j \in L^3_b$, a locally free sheaf on $\hat{U}^3_{b;j}$:

$$\hat{\mathcal{H}}^3_{b;j} := \mathcal{E}xt^1_{\hat{\pi}_{\hat{U}^3_{b;j}}} \Big((\hat{q}^{3'}_{b;j}, \hat{q}^3_{b;j})^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4), (\hat{p}^{3'}_{b;j}, \hat{p}^3_{b;j})^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3) \Big)^\vee;$$

- projective fibrations for every $j \in L^3_b$:

$$\hat{\varphi}^3_{b;j} : \hat{R}^3_{b;j} := \mathbb{P}(\hat{\mathcal{H}}^3_{b;j}) \longrightarrow \hat{U}^3_{b;j}$$

  with fibers isomorphic to $\mathbb{P}^{b-1}$;
- universal extensions for every $j \in L^3_b$, parametrized by $\hat{R}^3_{b;j}$:

$$
\begin{aligned}
&0 \to (\hat{\varphi}^{3'}_{b;j}, \hat{\varphi}^{3}_{b;j})^*(\hat{p}^{3'}_{b;j}, \hat{p}^{3}_{b;j})^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3) \otimes_{\hat{R}^3_{b;j}} \mathcal{O}_{\hat{R}^3 b;j}(1) \to \\
&\to (\hat{\mathcal{E}}^3_{b;j}, \hat{\mathcal{V}}^3_{b;j}) \to (\hat{\varphi}^{3'}_{b;j}, \hat{\varphi}^{3}_{b;j})^*(\hat{q}^{3'}_{b;j}, \hat{q}^{3}_{b;j})^*(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4) \to 0.
\end{aligned} \tag{12.56}
$$

Now we fix any $(a, b; i, j)$ and we consider the following cartesian diagram constructed in several steps, starting from (a):

$$\begin{tikzcd}
\hat{Q}_{a,b;i;j} \arrow[r, "\hat{\theta}^3_{b;j}"] \arrow[d, "\hat{\theta}^2_{a;i}"'] \arrow[dr, phantom, "\square(d)"] & \hat{Q}^2_{a,b;i,j} \arrow[r, "\hat{s}^3_{b;j}"] \arrow[d, "\hat{\psi}^2_{a;i}"] \arrow[dr, phantom, "\square(b)"] & \hat{R}^2_{a;i} \arrow[d, "\hat{\varphi}^2_{a;i}"] \\
\hat{Q}^3_{a,b;i,j} \arrow[r, "\hat{\psi}^3_{b;j}"] \arrow[d, "\hat{s}^2_{a;i}"'] \arrow[dr, phantom, "\square(c)"] & \hat{U}_{a,b;i,j} \arrow[r, "\hat{r}^3_{b;j}"] \arrow[d, "\hat{r}^2_{a;i}"] \arrow[dr, phantom, "\square(a)"] & \hat{U}^2_{a;i} \arrow[d, "\hat{q}^2_{a;i}"] \\
\hat{R}^3_{b;j} \arrow[r, "\hat{\varphi}^3_{b;j}"'] & \hat{U}^3_{b;j} \arrow[r, "\hat{q}^3_{b;j}"'] & \hat{G}_4.
\end{tikzcd} \tag{12.57}$$

Then we define the locally closed subscheme of $\hat{U}_{a,b;i,j}$:

$$\hat{V}_{a,b;i,j} := \{t \in \hat{U}_{a,b;i,j} \text{ s.t.}$$
$$\mathrm{Hom}((\hat{r}^{3'}_{b;j}, \hat{r}^3_{b;j})^*(\hat{p}^{2'}_{a;i}, \hat{p}^2_{a;i})^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2)_t, (\hat{r}^{2'}_{a;i}, \hat{r}^2_{a;i})^*(\hat{p}^{3'}_{b;j}, \hat{p}^3_{b;j})^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)_t) = 0\}.$$

and we set:

$$\hat{R}_{a,b;i,j} := \hat{Q}_{a,b;i,j}|_{\hat{V}_{a,b;i,j}}.$$

Now we denote by $(\overline{\mathcal{Q}}_4, \overline{\mathcal{W}}_4)$ the pullback of $(\hat{\mathcal{Q}}_4, \hat{\mathcal{W}}_4)$ from $\hat{G}_4$ to $\hat{R}_{a,b;i,j}$; moreover we set

$$(\overline{\mathcal{Q}}_2, \overline{\mathcal{W}}_2) := (\hat{\theta}^{3'}_{b;j}, \hat{\theta}^3_{b;j})^*(\hat{s}^{3'}_{b;j}, \hat{s}^3_{b;j})^*\Big((\hat{\varphi}^{2'}_{a;i}, \hat{\varphi}^2_{a;i})^*(\hat{p}^{2'}_{a;i}, \hat{p}^2_{a;i})^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2) \otimes_{\hat{R}^2_{a;i}} \mathcal{O}_{\hat{R}^2_{a;i}}(1)\Big)$$

and analogously for $(\overline{\mathcal{Q}}_3, \overline{\mathcal{W}}_3)$. By pullback from $\hat{R}^2_{a;i}$ and from $\hat{R}^3_{b;j}$ (see lemma 3.2.1), the sequences (12.55) and (12.56) give rise to 2 short exact sequences of coherent systems parametrized by $\hat{R}_{a,b;i,j}$:

$$0 \to (\overline{\mathcal{Q}}_2, \overline{\mathcal{W}}_2) \xrightarrow{\alpha_{42}} (\overline{\mathcal{E}}^{42}_{a;i}, \overline{\mathcal{V}}^{42}_{a;i}) \xrightarrow{\beta_{42}} (\overline{\mathcal{Q}}_4, \overline{\mathcal{W}}_4) \to 0, \tag{12.58}$$
$$0 \to (\overline{\mathcal{Q}}_3, \overline{\mathcal{W}}_3) \xrightarrow{\alpha_{43}} (\overline{\mathcal{E}}^{43}_{b;j}, \overline{\mathcal{V}}^{43}_{b;j}) \xrightarrow{\beta_{43}} (\overline{\mathcal{Q}}_4, \overline{\mathcal{W}}_4) \to 0. \tag{12.59}$$

Then we sum these 2 extensions in order to get an extension of the form

$$0 \to (\overline{\mathcal{Q}}_2, \overline{\mathcal{W}}_2) \oplus (\overline{\mathcal{Q}}_3, \overline{\mathcal{W}}_3) \xrightarrow{\alpha_2} (\hat{\mathcal{E}}''_{a,b;i,j}, \hat{\mathcal{V}}''_{a,b;i,j}) \xrightarrow{\beta_2} (\overline{\mathcal{Q}}_4, \overline{\mathcal{W}}_4) \to 0. \tag{12.60}$$

In particular, we get a diagram of the form

$$
\begin{tikzcd}
0 \arrow[r] & \bigoplus_{l=2}^{3}(\overline{\mathcal{Q}}_l,\overline{\mathcal{W}}_l) \arrow[r, "\alpha_2"] \arrow[d, "pr_2"] \arrow[r, phantom, "\curvearrowright", shift right=8] & (\hat{\mathcal{E}}''_{a,b;i,j},\hat{\mathcal{V}}''_{a,b;i,j}) \arrow[r, "\beta_2"] \arrow[d, "\eta_2"] \arrow[r, phantom, "\curvearrowright", shift right=8] & (\overline{\mathcal{Q}}_4,\overline{\mathcal{W}}_4) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\overline{\mathcal{Q}}_2,\overline{\mathcal{W}}_2) \arrow[r, "\alpha_{42}"] & (\overline{\mathcal{E}}^{42}_{a;i},\overline{\mathcal{V}}^{42}_{a;i}) \arrow[r, "\beta_{42}"] & (\overline{\mathcal{Q}}_4,\overline{\mathcal{W}}_4) \arrow[r] & 0,
\end{tikzcd}
\tag{12.61}
$$

where pr_2 is the quotient $(\overline{\mathcal{Q}}_2,\overline{\mathcal{W}}_2)\oplus(\overline{\mathcal{Q}}_2,\overline{\mathcal{W}}_3) \twoheadrightarrow (\overline{\mathcal{Q}}_2,\overline{\mathcal{W}}_2)$. So we have a surjective morphism

$$\eta_2 : (\hat{\mathcal{E}}''_{a,b;i,j},\hat{\mathcal{V}}''_{a,b;i,j}) \twoheadrightarrow (\overline{\mathcal{E}}^{42}_{a;i},\overline{\mathcal{V}}^{42}_{a;i}).$$

Analogously, we get a surjective morphism

$$\eta_3 : (\hat{\mathcal{E}}''_{a,b;i,j},\hat{\mathcal{V}}''_{a,b;i,j}) \twoheadrightarrow (\overline{\mathcal{E}}^{43}_{b;j},\overline{\mathcal{V}}^{43}_{b;j}).$$

Now let us consider the locally closed subscheme of $\hat{R}_{a,b;i,j}$:

$$\hat{U}_{a,b,c,d,e;i,j} := \{t \in \hat{R}_{a,b;i,j} \text{ s.t. } \dim \mathrm{Ext}^1((\hat{\mathcal{E}}''_{a,b;i,j},\hat{\mathcal{V}}''_{a,b;i,j})_t,(\overline{\mathcal{Q}}_2,\overline{\mathcal{W}}_2)_t) = c,$$
$$\dim \mathrm{Ext}^1((\overline{\mathcal{E}}^{42}_{a;i},\overline{\mathcal{V}}^{42}_{a;i})_t,(\overline{\mathcal{Q}}_2,\overline{\mathcal{W}}_2)_t) = d,\ \dim \mathrm{Ext}^1((\overline{\mathcal{E}}^{43}_{b;j},\overline{\mathcal{V}}^{43}_{b;j})_t,(\overline{\mathcal{Q}}_2,\overline{\mathcal{W}}_2)_t) = e\}.$$

By applying several times lemma 4.6.1 we have that there is a finite locally closed disjoint covering $\{\hat{U}_{a,b,c,d,e;i,j,k}\}_k$ of such a scheme such that all the following sheaves are all locally free and commute with base change

$$\hat{\mathcal{H}}_{a,b,c,d,e;i,j,k} := \mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c,d,e;i,j,k}}}\Big((\hat{\mathcal{E}}''_{a,b;i,j},\hat{\mathcal{V}}''_{a,b;i,j}),(\overline{\mathcal{Q}}_2,\overline{\mathcal{W}}_2)\Big),$$
$$\hat{\mathcal{H}}^2_{a,b,c,d,e;i,j,k} := \mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c,d,e;i,j,k}}}\Big((\overline{\mathcal{E}}^{42}_{a;i},\overline{\mathcal{V}}^{42}_{a;i}),(\overline{\mathcal{Q}}_2,\overline{\mathcal{W}}_2)\Big),$$
$$\hat{\mathcal{H}}^3_{a,b,c,d,e;i,j,k} := \mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c,d,e;i,j,k}}}\Big((\overline{\mathcal{E}}^{43}_{b;j},\overline{\mathcal{V}}^{43}_{b;j}),(\overline{\mathcal{Q}}_2,\overline{\mathcal{W}}_2)\Big).$$

Let us fix any point t of $\hat{U}_{a,b,c,d,e;i,j}$ and let us denote by

$$0 \to (Q_2,W_2)\oplus(Q_3,W_3) \to (E'',V'') \to (Q_4,W_4) \to 0$$

the restriction of (12.60) to t. Such a sequence is a representative for a pair $(\mu_{2,t},\mu_{3,t})$ with both $\mu_{2,t}$ and $\mu_{3,t}$ different from zero; moreover by construction $(Q_4,W_4) \not\simeq (Q_2,W_2)$ and $(Q_2,W_2) \not\simeq (Q_3,W_3)$. So by lemma 3.3.1 we get that

$$\mathrm{Hom}((E'',V''),(Q_i,W_i)) = 0 \quad \forall i = 1,2.$$

So by base change the sheaf

$$\mathcal{H}om_{\pi_{\hat{U}_{a,b,c,d,e;i,j,k}}}((\hat{\mathcal{E}}''_{a,b;i,j},\hat{\mathcal{V}}''_{a,b;i,j}),(\overline{\mathcal{Q}}_2,\overline{\mathcal{W}}_2))$$

is zero. So by corollary 4.4.4 we get that for every index k there is a projective bundle

$$\hat{\psi}_{a,b,c,d,e;i,j,k} : \hat{P}_{a,b,c,d,e;i,j,k} := \mathbb{P}((\hat{\mathcal{H}}_{a,b,c,d,e;i,j,k})^{\vee}) \longrightarrow \hat{U}_{a,b,c,d,e;i,j} \subset \hat{R}_{a,b;i,j}$$

with fibers isomorphic to $\mathbb{P}^{c-1}$. Moreover, there exists a universal family of classes of non-split extensions parametrized by $\hat{P}_{a,b,c,d,e;i,j,k}$:

$$\begin{aligned} 0 \to (\hat{\varphi}'_{a,b,c,d,e;i,j,k}, \hat{\varphi}_{a,b,c,d,e;i,j,k})^*(\overline{\mathcal{Q}}_2, \overline{\mathcal{W}}_2) \otimes_{\hat{P}_{a,b,c,d,e;i,j,k}} \mathcal{O}_{\hat{P}_{a,b,c,d,e;i,j,k}}(1) \to \\ \to (\hat{\mathcal{E}}_{a,b,c,d,e;i,j,k}, \hat{\mathcal{V}}_{a,b,c,d,e;i,j,k}) \to \\ \to (\hat{\varphi}'_{a,b,c,d,e;i,j,k}, \hat{\varphi}_{a,b,c,d,e;i,j,k})^*(\hat{\mathcal{E}}''_{a,b;i,j}, \hat{\mathcal{V}}''_{a,b;i,j}) \to 0. \end{aligned} \tag{12.62}$$

Now let us consider the morphism induced by η_2:

$$\overline{\eta_2} : \hat{\mathcal{H}}^2_{a,b,c,d,e;i,j,k} = \mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c,d,e;i,j,k}}}((\overline{\mathcal{E}}^{42}_{a;i}, \overline{\mathcal{V}}^{42}_{a;i}), (\overline{\mathcal{Q}}_2, \overline{\mathcal{W}}_2)) \longrightarrow$$

$$\longrightarrow \mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c,d,e;i,j,k}}}((\hat{\mathcal{E}}''_{a,b;i,j}, \hat{\mathcal{V}}''_{a,b;i,j}), (\overline{\mathcal{Q}}_2, \overline{\mathcal{W}}_2)) = \hat{\mathcal{H}}_{a,b,c,d,e;i,j,k};$$

analogously, we can consider a morphism of the form

$$\overline{\eta_3} : \hat{\mathcal{H}}^3_{a,b,c,d,e;i,j,k} \longrightarrow \hat{\mathcal{H}}_{a,b,c,d,e;i,j,k}.$$

Then let us consider the morphism:

$$\overline{\eta_2} + \overline{\eta}_3 : \hat{\mathcal{H}}^2_{a,b,c,d,e;i,j,k} \oplus \hat{\mathcal{H}}^3_{a,b,c,d,e;i,j,k} \longrightarrow \hat{\mathcal{H}}_{a,b,c,d,e;i,j,k}.$$

By construction all the sheaves in the previous line are locally free and commute with base change. Moreover, by the proof of lemma 12.2.4 and base change we get that the rank of such a morphism is constant. To be more precise, in the notation of lemma 12.2.4, that rank is equal to $d+e-f$; in the case under consideration we are imposing $(Q_1, W_1) \simeq (Q_2, W_2)$, so $f = a$ and the rank of that morphism is therefore $d+e-a$. Therefore the image of $\overline{\eta}_2 + \overline{\eta}_3$ is a locally free subsheaf of $\hat{\mathcal{H}}_{a,b,c,d,e;i,j,k}$; we denote such subsheaf by $\hat{\mathcal{H}}'_{a,b,c,d,e;i,j,k}$. So it makes sense to consider the projective subbundle

$$\hat{Q}_{a,b,c,d,e;i,j,k} := \mathbb{P}((\hat{\mathcal{H}}'_{a,b,c,d,e;i,j,k})^{\vee}) \subset \mathbb{P}((\hat{\mathcal{H}}_{a,b,c,d,e;i,j,k})^{\vee}) = \hat{P}_{a,b,c,d,e;i,j,k}$$

and to define the scheme

$$\hat{R}_{a,b,c,d,e;i,j,k} := \hat{P}_{a,b,c,d,e;i,j,k} \smallsetminus \hat{Q}_{a,b,c,d,e;i,j,k}.$$

Then the proof of lemma 12.2.4 shows that for every point r of $\hat{R}_{a,b,c,d,e;i,j,k}$ the restriction of $(\hat{\mathcal{E}}_{a,b,c,d,e;i,j,k}, \hat{\mathcal{V}}_{a,b,c,d,e;i,j,k})$ to r gives rise to an object of $G^+(\alpha_c; n, d, k)$, respectively of $G^-(\alpha_c; n, d, k)$. Then we conclude as usual. $\square$

12.2.2 Second case

In this subsection we consider the case when (E, V) has α_c-canonical filtration of type (1,2,1) and $(Q_1, W_1) \not\simeq (Q_i, W_i)$ for $i = 2, 3, 4$. We can associate to every (E, V) that we want to parametrize a pair of exact sequences of the form:

$$0 \to (Q_1, W_1) \xrightarrow{\sigma} (E_2, V_2) \xrightarrow{\kappa} (Q_2, W_2) \oplus (Q_3, W_3) \to 0; \tag{12.63}$$

$$0 \to (E_2, V_2) \xrightarrow{\varepsilon} (E, V) \xrightarrow{\delta} (Q_4, W_4) \to 0. \tag{12.64}$$

We denote by μ and ν the classes of those 2 exact sequences. If (E, V) has α_c-canonical filtration of type (1,2,1), then it has certainly the following proper α_c-semistable subobjects with α_c-slope μ:

(a) (Q_1, W_1), that is the only α_c-stable one;

(b) an extension of (Q_i, W_i) by (Q_1, W_1) for $i = 2, 3$;

(c) an extension of $(Q_2, W_2) \oplus (Q_3, W_3)$ by (Q_1, W_1).

So we get that conditions (12.35) are necessary in order to have that (E, V) belongs to $G^+(\alpha_c; n, d, k)$. Analogously, we get that conditions (12.36) are necessary in order to have that (E, V) belongs to $G^-(\alpha_c; n, d, k)$

Let us consider the following long exact sequence obtained by applying the functor $\mathrm{Hom}((Q_4, W_4), -)$ to (12.63):

$$\begin{aligned} \cdots \to \mathrm{Hom}((Q_4, W_4), (E_2, V_2)) \to \mathrm{Hom}((Q_4, W_4), (Q_2, W_2) \oplus (Q_3, W_3)) \to \\ \to \mathrm{Ext}^1((Q_4, W_4), (Q_1, W_1)) \xrightarrow{\overline{\sigma}} \mathrm{Ext}^1((Q_4, W_4), (E_2, V_2)) \xrightarrow{\overline{\kappa}} \\ \xrightarrow{\overline{\kappa}} \mathrm{Ext}^1((Q_4, W_4), (Q_2, W_2) \oplus (Q_3, W_3)) \to \cdots \end{aligned} \tag{12.65}$$

If we apply $\overline{\kappa}$ to ν we get a diagram of this form:

$$\begin{tikzcd} 0 \arrow[r] & (E_2, V_2) \arrow[r, "\varepsilon"] \arrow[d, "\kappa"] \arrow[dr, phantom, "\curvearrowright"] & (E, V) \arrow[r, "\delta"] \arrow[d, "\beta_1"] \arrow[dr, phantom, "\curvearrowright"] & (Q_4, W_4) \arrow[r] \arrow[d, equal] & 0 & \nu \arrow[d, "\overline{\kappa}"] \\ 0 \arrow[r] & \oplus_{i=2}^{3}(Q_i, W_i) \arrow[r, "\alpha_2"] & (E'', V'') \arrow[r, "\beta_2"] & (Q_4, W_4) \arrow[r] & 0 & \overline{\kappa}(\nu). \end{tikzcd} \tag{12.66}$$

By the snake lemma and (12.63), we get an induced short exact sequence

$$0 \to (Q_1, W_1) \xrightarrow{\alpha_1} (E, V) \xrightarrow{\beta_1} (E'', V'') \to 0. \tag{12.67}$$

Again by the snake lemma, eventually by replacing α_1 with $\alpha_1 \circ \varphi$ for a suitable automorphism φ of (Q_1, W_1) (i.e. $\varphi = \lambda \cdot \mathrm{id}_{(Q_1, W_1)}$ for some $\lambda \in \mathbb{C}^*$), we have that

$$\alpha_1 = \varepsilon \circ \sigma. \tag{12.68}$$

We can identify μ with a pair

$$(\mu_2, \mu_3) \in \bigoplus_{i=2}^{3} \operatorname{Ext}^1\Big((Q_i, W_i), (Q_1, W_1)\Big).$$

For every $i = 2, 3$, this identification gives a diagram of the form:

$$\begin{tikzcd}
0 \arrow[r] & (Q_1, W_1) \arrow[r, "\sigma"] \arrow[d, equal] \arrow[dr, phantom, "\curvearrowright"] & (E_2, V_2) \arrow[r, "\kappa"] \arrow[dr, phantom, "\curvearrowright"] & \oplus_{l=2}^{3}(Q_l, W_l) \arrow[r] & 0 & \mu \arrow[d, "\overline{\varepsilon_i}"] \\
0 \arrow[r] & (Q_1, W_1) \arrow[r, "\sigma_{i1}"] & (E_{i1}, V_{i1}) \arrow[r, "\kappa_{i1}"] \arrow[u, "\delta_i"'] & (Q_i, W_i) \arrow[r] \arrow[u, "\varepsilon_i"'] & 0 & \mu_i,
\end{tikzcd} \tag{12.69}$$

where ε_i is the embedding of (Q_i, W_i) in $(Q_2, W_2) \oplus (Q_3, W_3)$ for $i = 2, 3$. By the snake lemma, for every $i = 2, 3$ we get an induced short exact sequence

$$0 \to (E_{i1}, V_{i1}) \xrightarrow{\delta_i} (E_2, V_2) \xrightarrow{\eta_i} (Q_j, W_j) \to 0, \tag{12.70}$$

where j is the index in $\{2, 3\}$ different from i. Having fixed all those notations, we have the following results.

Lemma 12.2.6. *Let us fix any pair of exact sequences as (12.63) and (12.64), let us denote by μ and ν their classes. Then the following facts are equivalent.*

(a) *(E, V) has α_c-canonical filtration of type $(1, 2, 1)$;*

(b) *for all $i = 2, 3$ and for all morphisms $\gamma_i : (Q_i, W_i) \to (E_2, V_2)$ we have $\kappa \circ \gamma_i = 0$; moreover, $\overline{\kappa}(\nu) \neq 0$.*

The proof is analogous to the proof of lemma 12.2.1, so we omit it.

Lemma 12.2.7. *Let us fix any pair of exact sequences as (12.63) and (12.64) and let us suppose that:*

$$\frac{k_i}{n_i} > \frac{k}{n} \quad \forall\, i \in \{2, 3, 4\}, \tag{12.71}$$

respectively that

$$\frac{k_i}{n_i} < \frac{k}{n} \quad \forall\, i \in \{2, 3, 4\}. \tag{12.72}$$

Let us also suppose that (E, V) has α_c-canonical filtration of type $(1, 2, 1)$. Then (E, V) belongs to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$.

Proof. Since (E,V) has α_c-canonical filtration of type (1,2,1), all its proper subobjects that are α_c-semistable with α_c-slope as (E,V) contain (Q_1,W_1). Now let us assume conditions (12.72), the other case is analogous. Those conditions imply that

$$\frac{k_1}{n_1}<\frac{k}{n},\quad \frac{k_1+k_i}{n_1+n_i}<\frac{k}{n}\quad \forall\, i\in\{2,3,4\},\quad \frac{k_1+k_i+k_j}{n_1+n_i+n_j}<\frac{k}{n}\quad \forall\, i\neq j\in\{2,3,4\}$$

So all possible subobjects of (E,V) that are α_c-semistable with α_c-slope as (E,V) do not destabilize (E,V) for α_c^-, so we conclude. □

Corollary 12.2.8. *Let fix any quadruple $(Q_i,W_i)_{i=1,\cdot,4}\in\prod_{i=1}^4 G_i$ and let us suppose that conditions (12.71), respectively (12.72), are satisfied (automatically, we have that $(Q_1,W_1)\not\simeq(Q_i,W_i)$ for all $i=2,3,4$). Then the (E,V)'s that belong to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, that have α_c-canonical filtration of type $(1,2,1)$ and graded $\oplus_{i=1}^4(Q_i,W_i)$ are those induced by pairs μ,ν with representatives (12.63) and (12.64), such that:*

- *for all $i=2,3$ and for all morphisms $\gamma_i:(Q_i,W_i)\to(E_2,V_2)$ we have $\kappa\circ\gamma_i=0$;*
- *$\overline{\kappa}(\nu)\neq 0$.*

Now we will have to state 2 lemmas according to the relation between (Q_2,W_2) and (Q_3,W_3).

Lemma 12.2.9. *Let us fix any quadruple $(Q_i,W_i)_{i=1,\cdots,4}\in\prod_{i=1}^4 G_i$ with numerical conditions (12.71), respectively (12.72), and such that $(Q_2,W_2)\not\simeq(Q_3,W_3)$ (automatically, we have $(Q_1,W_1)\not\simeq(Q_i,W_i)$ for all $i=2,3,4$). Let us denote by μ_i any class of an extension of the form*

$$0\to(Q_1,W_1)\xrightarrow{\sigma_{i1}}(E_{i1},V_{i1})\xrightarrow{\kappa_{i1}}(Q_i,W_i)\to 0 \tag{12.73}$$

for $i=2,3$ and let us denote by μ the class of the extension

$$0\to(Q_1,W_1)\xrightarrow{\sigma}(E_2,V_2)\xrightarrow{\kappa}(Q_2,W_2)\oplus(Q_3,W_3)\to 0 \tag{12.74}$$

obtained by μ_2 and μ_3 (so that we have diagrams of the form (12.69) for $i=2,3$). Then the set of all the (E,V)'s that belong to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, that have α_c-canonical filtration of type $(1,2,1)$ and graded $\oplus_{i=1}^4(Q_i,W_i)$ is given by a fibration over

$$\mathbb{P}(Ext^1((Q_2,W_2),(Q_1,W_1)))\times\mathbb{P}(Ext^1((Q_3,W_3),(Q_1,W_1))).$$

The fiber over any point $([\mu_2],[\mu_3])$ with μ_2,μ_3 as before is given by

$$\overline{M}([\mu_2],[\mu_3]):=\Big(Ext^1((Q_4,W_4),(E_2,V_2))\smallsetminus Im\ \overline{\sigma}\Big)/\mathbb{C}^*,$$

where $\overline{\sigma}$ is as in (12.65). In addition, if we write:

$$c:=dim\ Ext^1((Q_4,W_4),(E_2,V_2)),\quad d:=dim\ Ext^1((Q_4,W_4),(Q_1,W_1)),$$

then for every pair $([\mu_2],[\mu_3])$ we have that

- *if* $(Q_4, W_4) \not\simeq (Q_i, W_i)$ *for all* $i = 2, 3$*, then* $\overline{M}([\mu_2], [\mu_3]) \simeq \mathbb{P}^{c-1} \smallsetminus \mathbb{P}^{d-1}$*;*
- *if* (Q_4, W_4) *is isomorphic to* (Q_2, W_2) *and not to* (Q_3, W_3) *or conversely, then* $\overline{M}([\mu_2], [\mu_3])$ $\simeq \mathbb{P}^{c-1} \smallsetminus \mathbb{P}^{d-2}$*.*

Proof. To any pair (E, V) that we want to parametrize we can associate a triple $(\mu_2, \mu_3, \nu) = (\mu, \nu)$, where μ and ν have representatives of the form (12.63), respectively (12.64), and μ_2, μ_3 are as in diagram (12.69). Then by corollary 12.2.8, we have that (E, V) belongs to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, and it has α_c-canonical filtration of type $(1, 2, 1)$ if and only if the following conditions hold:

(i) for all $i = 2, 3$ and for all morphisms $\gamma_i : (Q_i, W_i) \to (E_2, V_2)$ we have $\kappa \circ \gamma_i = 0$;

(ii) $\overline{\kappa}(\nu) \neq 0$.

Since $(Q_2, W_2) \not\simeq (Q_3, W_3)$, then lemma 3.3.2 proves that (i) is equivalent to imposing that both μ_2 and μ_3 are non-zero.

By exactness of (12.65), condition (ii) is equivalent to saying that ν does not belong to the image of $\overline{\sigma}$, so the objects we are interested in are only those induced by triples (μ_2, μ_3, ν) such that $\mu_i \neq 0$ for $i = 2, 3$ and $\nu \notin \mathrm{Im}\, \overline{\sigma}$. Now if we look at the sequence (12.64), we get that the set of all the (E_2, V_2)'s there is given by

$$\mathbb{P}(\mathrm{Ext}^1((Q_2, W_2), (Q_1, W_1))) \times \mathbb{P}(\mathrm{Ext}^1((Q_3, W_3), (Q_1, W_1))).$$

We have that $\mathrm{Aut}(E_2, V_2) = \mathbb{C}^*$ because $(Q_1, W_1) \not\simeq (Q_i, W_i)$ for $i = 2, 3$; moreover also $\mathrm{Aut}(Q_4, W_4) = \mathbb{C}^*$ because (Q_4, W_4) is α_c-stable. In addition,

$$\mathrm{Hom}((Q_4, W_4), (E_2, V_2)) = 0.$$

Indeed, since the graded of (E_2, V_2) is $\oplus_{i=1}^{3}(Q_i, W_i)$, then if there exists a non-zero morphism γ in that space, then $(Q_4, W_4) \simeq (Q_i, W_i)$ for some $i = 1, 2, 3$. Now by hypothesis $(Q_4, W_4) \not\simeq (Q_1, W_1)$, so we must have that γ is of the form γ_i for some $i = 2, 3$. But condition (i) implies that this is impossible; so the previous space is the zero space.

Then we get that having fixed $([\mu_2], [\mu_3])$, the (E, V)'s we are interested in are parametrized by $\overline{M}([\mu_2], [\mu_3])$. Now let us consider the long exact sequence (12.65): as we just said, the first term in that exact sequence is zero. The second term is zero or $\mathbb{C}$ according to the relations between (Q_4, W_4) and (Q_i, W_i) for $i = 2, 3$. Therefore, we get that $\overline{\sigma}$ is injective if $(Q_4, W_4) \not\simeq (Q_i, W_i)$ for $i = 2, 3$, while it has a kernel of dimension 1 if (Q_4, W_4) is isomorphic to (Q_2, W_2) and not to (Q_3, W_3) or conversely, so we conclude. □

Lemma 12.2.10. *Let us fix any quadruple* $(Q_i, W_i)_{i=1,\cdots,4} \in \prod_{i=1}^{4} G_i$ *with numerical conditions (12.71), respectively (12.72), and such that* $(Q_2, W_2) \simeq (Q_3, W_3)$ *(automatically, we have that* $(Q_1, W_1) \not\simeq (Q_i, W_i)$ *for all* $i = 2, 3, 4$*). Let us denote by* μ_i *any class of an extension of the form*

$$0 \to (Q_1, W_1) \xrightarrow{\sigma_{i1}} (E_{i1}, V_{i1}) \xrightarrow{\kappa_{i1}} (Q_2, W_2) \to 0 \tag{12.75}$$

for $i = 2, 3$ and let us denote by μ the class of the extension

$$0 \to (Q_1, W_1) \xrightarrow{\sigma} (E_2, V_2) \xrightarrow{\kappa} (Q_2, W_2)^{\oplus 2} \to 0$$

obtained by μ_2 and μ_3 (so that we have diagrams of the form (12.69) for $i = 2, 3$). Then the set of all the (E, V)'s that belong to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, that have α_c-canonical filtration of type (1,2,1) and graded $\oplus_{i=1}^4 (Q_i, W_i)$ is given by a fibration over

$$Grass(2, Ext^1((Q_2, W_2), (Q_1, W_1))).$$

The fiber over any point $< \mu_2, \mu_3 >$ with μ_2 and μ_3 as before is given by

$$\overline{M}(< \mu_2, \mu_3 >) := (Ext^1((Q_4, W_4), (E_2, V_2)) \smallsetminus Im\ \overline{\sigma})/\mathbb{C}^*,$$

where $\overline{\sigma}$ is as in (12.65). In addition, if we write:

$$b := dim\ Ext^1((Q_4, W_4), (E_2, V_2)), \quad c := dim\ Ext^1((Q_4, W_4), (Q_1, W_1)),$$

then for every point $< \mu_2, \mu_3 >$ we have that

- *if $(Q_4, W_4) \not\simeq (Q_2, W_2)$ then $\overline{M}(< \mu_2, \mu_3 >) \simeq \mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1}$;*
- *if (Q_4, W_4) is isomorphic to (Q_2, W_2), then $\overline{M}(< \mu_2, \mu_3 >) \simeq \mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-3}$.*

Proof. To any pair (E, V) that we want to parametrize we can associate a triple $(\mu_2, \mu_3, \nu) = (\mu, \nu)$, where μ and ν have representatives of the form (12.63), respectively (12.64), and μ_2, μ_3 are as in diagram (12.69). Then by corollary 12.2.8, we have that (E, V) belongs to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, and it has α_c-canonical filtration of type $(1, 2, 1)$ if and only if the following conditions hold:

(i) for all $i = 2, 3$ and for all morphisms $\gamma_2 : (Q_2, W_2) \to (E_2, V_2)$ we have $\kappa \circ \gamma_2 = 0$;

(ii) $\overline{\kappa}(\nu) \neq 0$.

Since $(Q_2, W_2) \simeq (Q_3, W_3)$, then lemma 3.3.2 proves that (i) is equivalent to imposing that μ_2 and μ_3 are linearly independent in

$$\mathrm{Ext}^1((Q_2, W_2), (Q_1, W_1)).$$

By exactness of (12.65), condition (ii) is equivalent to saying that ν does not belong to the image of $\overline{\sigma}$, so the objects we are interested in are only those induced by triples (μ_2, μ_3, ν) such that μ_2 and μ_3 are linearly independent and $\nu \notin \mathrm{Im}\ \overline{\sigma}$. Now if we look at the sequence (12.64), we get that the set of all the (E_2, V_2)'s there is given by

$$Grass(2, \mathrm{Ext}^1((Q_2, W_2), (Q_1, W_1))).$$

We have that $\mathrm{Aut}(E_2, V_2) = \mathbb{C}^*$ because $(Q_1, W_1) \not\simeq (Q_2, W_2)$; moreover also $\mathrm{Aut}(Q_4, W_4) = \mathbb{C}^*$ because (Q_4, W_4) is α_c-stable. In addition, as in the previous lemma we have:

$$\mathrm{Hom}((Q_4, W_4), (E_2, V_2)) = 0.$$

Then we get that having fixed a point $< \mu_2, \mu_3 >$ in the grassmannian, the (E, V)'s we are interested in are parametrized by $\overline{M}(< \mu_2, \mu_3 >)$. Now let us consider the long exact sequence (12.65): as we just said, the first term in that exact sequence is zero. The second term is zero or $\mathbb{C}^2$ according to the relations between (Q_4, W_4) and (Q_2, W_2). Therefore, we get that $\overline{\sigma}$ is injective if $(Q_4, W_4) \not\simeq (Q_2, W_2)$, while it has a kernel of dimension 2 if (Q_4, W_4) is isomorphic to (Q_2, W_2); so we conclude. □

Now we want to give a global parametrization of the objects described before, i.e. we want to describe families of schemes that parametrize various types of (E, V)'s when the graded varies over $\prod_{i=1}^4 G_i$ and the α_c-canonical filtration is of type (1,2,1). Let us denote by $\oplus_{i=1}^4 (Q_i, W_i)$ a fixed graded with conditions (12.71), respectively (12.72). If we assume that $(n_2, k_2) = (n_3, k_3) = (n_4, k_4)$, then (12.71), respectively (12.72), are equivalent to imposing that

$$\frac{k_2}{n_2} > \frac{k}{n}, \tag{12.76}$$

respectively that

$$\frac{k_2}{n_2} < \frac{k}{n}. \tag{12.77}$$

If $(Q_2, W_2) \not\simeq (Q_3, W_3)$, then by lemma 12.2.9 the corresponding (E, V)'s are parametrized by triples $([\mu_2], [\mu_3], [\nu])$ with $[\mu_i] \in \mathbb{P}(\mathrm{Ext}^1((Q_i, W_i), (Q_1, W_1)))$ and representative (12.74) for $\mu = (\mu_2, \mu_3)$ and

$$[\nu] \in \overline{M}([\mu_2], [\mu_3]) \subset \mathbb{P}(\mathrm{Ext}^1((Q_4, W_4), (E_2, V_2))).$$

We are considering the case when the (Q_i, W_i)'s are all of the same type for $i = 2, 3, 4$. Therefore we need to take into account the possible isomorphisms between them. So we need to consider separately the following cases.

(1) If $(Q_2, W_2) \not\simeq (Q_3, W_3) \simeq (Q_4, W_4)$, then the roles of (Q_2, W_2) and of (Q_3, W_3) are not interchangeable, so we need to consider *ordered* pairs $([\mu_2], [\mu_3])$.

(2) If $(Q_i, W_i) \not\simeq (Q_j, W_j)$ for all $i \neq j \in \{2, 3, 4\}$, then the roles of (Q_2, W_2) and of (Q_3, W_3) are interchangeable, so we need to consider *unordered* pairs $([\mu_2], [\mu_3])$, so we will have to take into account an action of $\mathbb{Z}_2$ on schemes constructed as in (1).

Note that since the order of (Q_2, W_2) and (Q_3, W_3) is not important, we don't need to consider also the case $(Q_2, W_2) \simeq (Q_4, W_4) \not\simeq (Q_3, W_3)$.

If $(Q_2, W_2) \simeq (Q_3, W_3)$, then by lemma 12.2.10 the corresponding (E, V)'s are parametrized by pairs $(<\mu_2, \mu_3>, [\nu])$ with

$$<\mu_2, \mu_3> \in Grass(2, \mathrm{Ext}^1((Q_2, W_2), (Q_1, W_1))),$$

representative (12.2.10) for $\mu = (\mu_2, \mu_3)$ and

$$[\nu] \in \overline{M}(<\mu_2, \mu_3>) \subset \mathbb{P}(\mathrm{Ext}^1((Q_4, W_4), (E_2, V_2))).$$

We need to consider separately the following cases:

(3) $(Q_2, W_2) \simeq (Q_3, W_3) \not\simeq (Q_4, W_4)$;

(4) $(Q_2, W_2) \simeq (Q_3, W_3) \simeq (Q_4, W_4)$.

Remark 12.2.2. In case (2) it is not currently possible to give a global description; the cases that we are able to describe completely are the remaining 3 ones, accounted for by propositions 7.6.2, 7.6.3 and 7.6.4 respectively.

The proof of proposition 7.6.2 is on the same line of the proof of proposition 7.6.1, with the only significant difference that we use lemma 12.2.9 instead of lemma 12.2.4. We remark that we don't need the invariant d defined in lemma 12.2.9: indeed we are assuming that $(Q_3, W_3) \simeq (Q_4, W_4)$, therefore

$$d = \dim\ \mathrm{Ext}^1((Q_4, W_4), (Q_1, W_1)) = \dim\ \mathrm{Ext}^1((Q_3, W_3), (Q_1, W_1)) = b.$$

The proof of proposition 7.6.3 is a direct consequence of lemma 12.2.10, so we omit the details.

The proof of proposition 7.6.4 is also a direct consequence of lemma 12.2.10. We only remark that we don't need the invariant c of that lemma: indeed we are imposing that $(Q_2, W_2) \simeq (Q_4, W_4)$, therefore

$$c = \dim\ \mathrm{Ext}^1((Q_4, W_4), (Q_1, W_1)) = \dim\ \mathrm{Ext}^1((Q_2, W_2), (Q_1, W_1)) = a.$$

12.3 Canonical filtration of type (1,1,2)

In this case the α_c-canonical filtration is given by:

$$0 \subset (E_1, V_1) \subset (E_2, V_2) \subset (E_3, V_3) = (E, V),$$

where $(E_1, V_1) =: (Q_1, W_1)$, $(E_2, V_2)/(E_1, V_1) := (Q_2, W_2)$ and $(E, V)/(E_2, V_2) \simeq (Q_3, W_3) \oplus (Q_4, W_4)$. All the (Q_i, W_i)'s for $i = 1, \cdots, 4$ are α_c-stable coherent systems with the same α_c-slope μ. Then we can associate to every (E, V) that we want to parametrize a pair of exact sequences of the form:

$$0 \to (Q_1, W_1) \xrightarrow{\sigma} (E_2, V_2) \xrightarrow{\kappa} (Q_2, W_2) \to 0; \tag{12.78}$$

$$0 \to (E_2, V_2) \xrightarrow{\varepsilon} (E, V) \xrightarrow{\delta} (Q_3, W_3) \oplus (Q_4, W_4) \to 0. \tag{12.79}$$

We denote by μ and ν the classes of those 2 exact sequences. If (E, V) has α_c-canonical filtration of type (1,1,2), then it has certainly the following proper α_c-semistable subobjects with α_c-slope μ:

- (Q_1, W_1), that is the only α_c-stable one;
- (E_2, V_2), that is an extension of (Q_2, W_2) by (Q_1, W_1);
- for all $i = 3, 4$, an extension of (Q_i, W_i) by (E_2, V_2).

Actually, this is a complete list (see lemma 12.3.2); for the moment we don't prove that, so let us consider this as a partial list. Given that, having fixed any pair of sequences of the form (12.78) and (12.79), the following numerical conditions are necessary in order to have that (E, V) belongs to $G^+(\alpha_c; n, d, k)$

$$\frac{k_1}{n_1} < \frac{k}{n}, \quad \frac{k_1 + k_2}{n_1 + n_2} < \frac{k}{n}, \quad \frac{k_1 + k_2 + k_i}{n_1 + n_2 + n_i} < \frac{k}{n} \quad \forall\, i \in \{3, 4\}, \tag{12.80}$$

where the last condition is equivalent to:

$$\frac{k_i}{n_i} > \frac{k}{n} \quad \forall\, i \in \{3, 4\}. \tag{12.81}$$

Analogously, the following conditions are necessary in order to have that (E, V) belongs to $G^-(\alpha_c; n, d, k)$:

$$\frac{k_1}{n_1} > \frac{k}{n}, \quad \frac{k_1 + k_2}{n_1 + n_2} > \frac{k}{n}, \quad \frac{k_1 + k_2 + k_i}{n_1 + n_2 + n_i} > \frac{k}{n} \quad \forall\, i \in \{3, 4\}, \tag{12.82}$$

where the last condition is equivalent to:

$$\frac{k_i}{n_i} < \frac{k}{n} \quad \forall\, i \in \{3, 4\}. \tag{12.83}$$

Let us consider the long exact sequence obtained by applying the functor $\mathrm{Hom}((Q_3, W_3) \oplus (Q_4, W_4), -)$ to (12.78):

$$\begin{aligned} &\cdots \to \mathrm{Hom}((Q_3, W_3) \oplus (Q_4, W_4), (Q_2, W_2)) \to \mathrm{Ext}^1((Q_3, W_3) \oplus (Q_4, W_4), (Q_1, W_1) \xrightarrow{\overline{\sigma}} \\ &\xrightarrow{\overline{\sigma}} \mathrm{Ext}^1((Q_3, W_3) \oplus (Q_4, W_4), (E_2, V_2) \xrightarrow{\overline{\kappa}} \mathrm{Ext}^1((Q_3, W_3) \oplus (Q_4, W_4), (Q_2, W_2)) \to \cdots \end{aligned} \tag{12.84}$$

If we apply $\overline{\kappa}$ to ν, we get a diagram of this form:

$$\begin{tikzcd} 0 \arrow[r] & (E_2, V_2) \arrow[r, "\varepsilon"] \arrow[d, "\kappa"] \arrow[dr, phantom, "\curvearrowright"] & (E, V) \arrow[r, "\delta"] \arrow[d, "\beta_1"] \arrow[dr, phantom, "\curvearrowright"] & \oplus_{i=3}^4 (Q_i, W_i) \arrow[r] \arrow[d, equal] & 0 & \nu \arrow[d, "\overline{\kappa}"] \\ 0 \arrow[r] & (Q_2, W_2) \arrow[r, "\alpha_2"] & (E'', V'') \arrow[r, "\beta_2"] & \oplus_{i=3}^4 (Q_i, W_i) \arrow[r] & 0 & \overline{\kappa}(\nu). \end{tikzcd} \tag{12.85}$$

By the snake lemma and (12.78) we get an induced exact sequence:

$$0 \to (Q_1, W_1) \xrightarrow{\alpha_1} (E, V) \xrightarrow{\beta_1} (E'', V'') \to 0. \tag{12.86}$$

We can identify ν with a pair

$$(\nu_3, \nu_4) \in \bigoplus_{i=3}^{4} \operatorname{Ext}^1\left((Q_i, W_i), (E_2, V_2)\right).$$

For every $i = 3, 4$, this identification gives a diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & (E_2, V_2) \arrow[r, "\varepsilon"] \arrow[d, equal] \arrow[dr, phantom, "\curvearrowright"] & (E, V) \arrow[r, "\delta"] \arrow[dr, phantom, "\curvearrowright"] & \bigoplus_{l=3}^{4}(Q_l, W_l) \arrow[r] & 0 & \nu \arrow[d, "\overline{\varepsilon_i}"] \\
0 \arrow[r] & (E_2, V_2) \arrow[r, "\varepsilon_{i21}"] & (E_{i21}, V_{i21}) \arrow[r, "\eta_{i21}"] \arrow[u, hook, "\delta_i"'] & (Q_i, W_i) \arrow[r] \arrow[u, hook, "\varepsilon_i"'] & 0 & \nu_i,
\end{tikzcd} \tag{12.87}$$

where ε_i is the embedding $(Q_i, W_i) \hookrightarrow (Q_3, W_3) \oplus (Q_4, W_4)$ for $i = 3, 4$. Having fixed all those notations, let us state and prove the following results.

Lemma 12.3.1. *Let us fix any pair of exact sequences as (12.78) and (12.79), let us denote by μ and $\nu = (\nu_3, \nu_4)$ their classes and let us suppose that $(Q_1, W_1) \not\simeq (Q_i, W_i) \,\forall\, i \in \{2, 3, 4\}$. Then the following facts are equivalent:*

(a) (E, V) has α_c-canonical filtration of type $(1, 1, 2)$;

(b) for all $i = 3, 4$ and for all morphisms $\gamma_i : (Q_i, W_i) \to (E'', V'')$ we have $\beta_2 \circ \gamma_i = 0$; moreover $\mu \neq 0$.

Proof. To any (E, V) that we want to parametrize we can associate a triple $(\mu, \nu_3, \nu_4) = (\mu, \nu)$ with μ and ν represented by (12.78) and (12.79) respectively. By looking at those 2 sequences, we get that (E, V) has a filtration of the form

$$0 = (E_0, V_0) \subset (E_1, V_1) := (Q_1, W_1) \subset (E_2, V_2) \subset (E_3, V_3) = (E, V). \tag{12.88}$$

Here $(E_1, V_1)/(E_0, V_0) = (Q_1, W_1)$ and $(E_2, V_2)/(E_1, V_1) = (Q_2, W_2)$ by (12.78). Moreover, by (12.79) we have that $(E_3, V_3)/(E_2, V_2) \simeq (Q_3, W_3) \oplus (Q_4, W_4)$. Since all the (Q_i, W_i)'s are α_c-stable with the same α_c-slope μ, then we can apply proposition 2.1.3: the filtration (12.88) is the α_c-canonical filtration of (E, V) (and so (E, V) has α_c-canonical filtration of type $(1, 1, 2)$), if and only if condition (c) of that proposition is satisfied. In our case the index t is equal to 3, so we need to consider 2 sequences as in that proposition. It is easy to see that those 2 sequences are exactly the second line of diagram (12.85) and (12.86). So (E, V) has α_c-canonical filtration of type $(1, 1, 2)$ if and only if the following 2 conditions hold:

(i) for all $i = 2,3,4$ and for all non-zero morphisms $\gamma_i : (Q_i, W_i) \to (E'', V'')$ we have $\beta_2 \circ \gamma_i = 0$;

(ii) for all $i = 1, \cdots, 4$ and for all non-zero morphisms $\widetilde{\gamma}_i : (Q_i, W_i) \to (E, V)$ we have $\beta_1 \circ \widetilde{\gamma}_i = 0$.

Let us first consider condition (ii). Since $(Q_1, W_1) \not\simeq (Q_i, W_i)$ for $i = 2,3,4$, then there are no morphisms $\widetilde{\gamma}_1$ with $\beta_1 \circ \widetilde{\gamma}_1 \neq 0$, so we can ignore that possibility in (ii). Now let us suppose that there is a non-zero morphism $\widetilde{\gamma}_i$ for some $i = 2,3,4$. Then we have to consider 2 cases. If $\delta \circ \widetilde{\gamma}_i = 0$; then this induces a non-zero morphism $\widetilde{\gamma}'_i : (Q_i, W_i) \to (E_2, V_2)$ such that $\varepsilon \circ \widetilde{\gamma}'_i = \widetilde{\gamma}_i$. Since $(Q_1, W_1) \not\simeq (Q_i, W_i)$ for $i = 2,3,4$, then this implies that (Q_i, W_i) is isomorphic to (Q_2, W_2) and that $\widetilde{\gamma}'_i$ gives a splitting of (12.78), so $\mu = 0$. If $\delta \circ \widetilde{\gamma}_i \neq 0$, then we write $\gamma_i := \beta_1 \circ \widetilde{\gamma}_i$; by diagram (12.85), we get that

$$\beta_2 \circ \gamma_i = \beta_2 \circ \beta_1 \circ \widetilde{\gamma}_i = \delta \circ \widetilde{\gamma}_i \neq 0.$$

So if condition (ii) is not satisfied, then either $\mu = 0$ or condition (i) is not satisfied.

Now let us consider condition (i): if there is any morphism $\gamma_2 : (Q_2, W_2) \to (E'', V'')$ such that $\beta_2 \circ \gamma_2 \neq 0$, then we have a non-zero morphism from (Q_2, W_2) to $(Q_3, W_3) \oplus (Q_4, W_4)$; therefore $(Q_2, W_2) \simeq (Q_i, W_i)$ for some $i = 3,4$. Therefore the morphism γ_2 is of the form γ_i for some $i = 3,4$. So until now we have proved that if (12.88) is the α_c-canonical filtration of (E, V), then (b) holds.

Conversely, if $\mu = 0$, then the α_c-canonical filtration of (E, V) cannot be of type $(1,1,2)$ because (E, V) contains $(E_2, V_2) \simeq (Q_1, W_1) \oplus (Q_2, W_2)$. If there exists any morphism $\gamma_i : (Q_i, W_i) \to (E'', V'')$ such that $\beta_2 \circ \gamma_i \neq 0$ for some $i = 3,4$, then (E'', V'') contains $(Q_2, W_2) \oplus (Q_i, W_i)$, so the α_c-canonical filtration of (E, V) cannot be of type (1,1,2). Hence we have proved that if (b) does not hold, neither do (a), so we conclude. □

Lemma 12.3.2. *Let us fix any pair of exact sequences as (12.78) and (12.79), let us denote by μ and $\nu = (\nu_3, \nu_4)$ their classes and let us suppose that*

$$(Q_1, W_1) \not\simeq (Q_i, W_i) \quad \forall\, i \in \{2,3,4\}.$$

Moreover, let us suppose that (E, V) has α_c-canonical filtration of type $(1,1,2)$ and let us assume conditions (12.80), respectively (12.82). Then (E, V) belongs to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$.

Proof. Let us assume conditions (12.80); the other case is analogous. If there exists a proper subobject (E', V') that destabilizes (E, V) for α_c^+, then we must have that (E', V') is α_c-semistable with the same α_c-slope as (E, V). Since (E, V) has α_c-canonical filtration of type (1,1,2), then (E', V') contains (Q_1, W_1). Since $\frac{k_1}{n_1} < \frac{k}{n}$, then any (E', V') that is α_c-stable does not destabilize (E, V) for α_c^+. So we need only to prove that if the rank of any α_c-Jordan-Hölder filtration of (E', V') is equal to 2 or 3, then (E', V') does not destabilize (E, V) for α_c^+.

Length of any α_c-Jordan-Hölder filtration of (E', V') equal to 2. If (E', V') is an extension of (Q_2, W_2) by (Q_1, W_1), then this subobject does not destabilize (E, V) because of the second part of conditions (12.80). So let us suppose that we have a (non-split) exact sequence

$$0 \to (Q_1, W_1) \xrightarrow{\alpha} (E', V') \xrightarrow{\beta} (Q_i, W_i) \to 0 \tag{12.89}$$

for some $i = 3, 4$. Let us denote by γ the inclusion of (E', V') in (E, V) and let us consider the exact sequence (12.86). Since $(Q_1, W_1) \not\simeq (Q_j, W_j)$ for $j = 2, 3, 4$, then we have $\beta_1 \circ \gamma \circ \alpha = 0$. Moreover, $\beta_1 \circ \gamma \neq 0$, otherwise we have an induced injective morphism $\gamma' : (E', V') \to (Q_1, W_1)$ such that $\gamma = \alpha_1 \circ \gamma'$, but this is impossible because (E', V') is strictly α_c-semistable, while (Q_1, W_1) is α_c-stable. Then by exactness of (12.89) we have an induced morphism $\gamma_i : (Q_i, W_i) \to (E'', V'')$ such that $\gamma_i \circ \beta = \beta_1 \circ \gamma \neq 0$; in particular, $\gamma_i \neq 0$. Since we are assuming that (E, V) has α_c-canonical filtration of type (1,1,2), then we can use condition (b) of the previous lemma, so $\beta_2 \circ \gamma_i = 0$. By exactness of the second line of (12.85), γ_i induces a non-zero morphism $\gamma_i' : (Q_i, W_i) \to (Q_2, W_2)$ such that $\gamma_i = \alpha_2 \circ \gamma_i'$. Since both (Q_2, W_2) and (Q_i, W_i) are α_c-stable with the same α_c-slope, then this proves that $(Q_i, W_i) \simeq (Q_2, W_2)$. So (E', V') is an extension of (Q_2, W_2) by (Q_1, W_1), so it does not destabilize (E, V) for α_c^+.

Length of any α_c-Jordan-Hölder filtration of (E', V') equal to 3. In this case, let us denote by $(\widetilde{E}, \widetilde{V})$ the quotient $(E, V)/(E', V')$, which is an α_c-stable coherent system. Since $(Q_1, W_1) \subset (E', V')$ and since $(Q_1, W_1) \not\simeq (Q_i, W_i)$ for $i = 2, 3, 4$, then $(\widetilde{E}, \widetilde{V})$ can only be equal to (Q_i, W_i) for some $i = 2, 3, 4$. If $i = 3, 4$, then (E', V') does not destabilize (E, V) because of the last condition of (12.80), so we need to consider only the case when we have a quotient $\zeta_2 : (E, V) \twoheadrightarrow (\widetilde{E}, \widetilde{V}) = (Q_2, W_2)$. If we use (12.86) together with the fact that $(Q_1, W_1) \not\simeq (Q_2, W_2)$, we get that $\zeta_2 \circ \alpha_1 = 0$, so we have an induced morphism $\zeta_2' : (E'', V'') \twoheadrightarrow (Q_2, W_2)$ such that $\zeta_2 = \zeta_2' \circ \beta_1$. Now let us consider the second line of (12.85): if $\zeta_2' \circ \alpha_2 = 0$, then we get an induced morphism $\zeta_2'' : (Q_3, W_3) \oplus (Q_4, W_4) \twoheadrightarrow (Q_2, W_2)$ such that $\zeta_2' = \zeta_2'' \circ \beta_2$. In particular, $\zeta_2'' \neq 0$, so we get that necessarily (Q_2, W_2) is isomorphic to (Q_i, W_i) for some $i = 3, 4$. Then we have $(\widetilde{E}, \widetilde{V}) \simeq (Q_i, W_i)$ for some $i = 3, 4$, so (E', V') does not destabilize (E, V) for α_c^+. If $\zeta_2' \circ \alpha_2 \neq 0$, then it belongs to $\mathrm{Aut}(Q_2, W_2) = \mathbb{C}^*$, so the second line of (12.85) is split. Therefore, there exists a morphism $\theta : (Q_3, W_3) \oplus (Q_4, W_4) \to (E'', V'')$ such that $\beta_2 \circ \theta = \mathrm{id}$. Let us consider the composition:

$$\gamma_3 : (Q_3, W_3) \hookrightarrow (Q_3, W_3) \oplus (Q_4, W_4) \xrightarrow{\theta} (E'', V'').$$

Then $\beta_2 \circ \gamma_3 \neq 0$, so condition (b) of the previous lemma is not satisfied, so we get a contradiction. So if (E, V) has α_c-canonical filtration of type (1,1,2), then $\zeta_2' \circ \alpha_2$ is always zero.

So we have proved that there are no proper subobjects of (E, V) that destabilize it for α_c^+. $\square$

Remark 12.3.1. The previous proof shows also that the only α_c-semistable proper subobjects of (E,V) with the same α_c-slope are those listed at the beginning of this section.

Lemma 12.3.3. *Let us fix any quadruple $(Q_i,W_i)_{i=1,\cdots,4}\in\prod_{i=1}^4 G_i$ with conditions (12.80), respectively (12.82), and such that:*

$$(Q_1,W_1)\not\simeq(Q_2,W_2),\quad (Q_3,W_3)\not\simeq(Q_4,W_4)$$

(because of the numerical conditions assumed, we have automatically that $(Q_1,W_1)\not\simeq(Q_i,W_i)$ for $i=3,4$). Let us denote by μ any class of an extension of the form

$$0\to(Q_1,W_1)\xrightarrow{\sigma}(E_2,V_2)\xrightarrow{\kappa}(Q_2,W_2)\to 0. \tag{12.90}$$

Having fixed $[\mu]\in\mathbb{P}(Ext^1((Q_2,W_2),(Q_1,W_1)))$, let us consider the morphisms

$$Ext^1((Q_i,W_i),(Q_1,W_1))\xrightarrow{\overline{\sigma^i}}Ext^1((Q_i,W_i),(E_2,V_2))\quad for\ i=3,4$$

induced by the morphism σ, so that the morphism $\overline{\sigma}$ of (12.84) coincides with the pair $(\overline{\sigma^3},\overline{\sigma^4})$. Moreover, let us write $M([\mu])$ for the set

$$\left(Ext^1((Q_3,W_3),(E_2,V_2))\smallsetminus Im\ \overline{\sigma^3}\right)\oplus\left(Ext^1((Q_4,W_4),(E_2,V_2))\smallsetminus Im\ \overline{\sigma^4}\right).$$

Such a set has a natural action of $\mathbb{C}^*\times\mathbb{C}^*$ on it, given by multiplication by scalars on the 2 components. Then we have that the set of all the (E,V)'s that belong to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, that have α_c-canonical filtration of type $(1,1,2)$ and graded $\oplus_{i=1}^4(Q_i,W_i)$ is given by a fibration over $\mathbb{P}(Ext^1((Q_2,W_2),(Q_1,W_1)))$. The fiber over any point $[\mu]$ with μ represented by (12.90) is given by $\overline{M}([\mu]):=M([\mu])/(\mathbb{C}^*\times\mathbb{C}^*)$. In addition, if we write:

$$b:=dim\ Ext^1((Q_4,W_4),(E_2,V_2)),\quad c:=dim\ Ext^1((Q_4,W_4),(Q_1,W_1)),$$
$$d:=dim\ Ext^1((Q_3,W_3),(E_2,V_2)),\quad e:=dim\ Ext^1((Q_3,W_3),(Q_1,W_1)),$$

then for every $[\mu]$ we have the following description.

- *If no (Q_i,W_i)'s are isomorphic for $i=2,3,4$, then*

$$\overline{M}([\mu])\simeq(\mathbb{P}^{d-1}\smallsetminus\mathbb{P}^{e-1})\times(\mathbb{P}^{b-1}\smallsetminus\mathbb{P}^{c-1}).$$

- *If $(Q_2,W_2)\simeq(Q_3,W_3)\not\simeq(Q_4,W_4)$, then*

$$\overline{M}([\mu])\simeq(\mathbb{P}^{d-1}\smallsetminus\mathbb{P}^{e-2})\times(\mathbb{P}^{b-1}\smallsetminus\mathbb{P}^{c-1}).$$

- *If $(Q_2,W_2)\simeq(Q_4,W_4)\not\simeq(Q_3,W_3)$, then*

$$\overline{M}([\mu])\simeq(\mathbb{P}^{d-1}\smallsetminus\mathbb{P}^{e-1})\times(\mathbb{P}^{b-1}\smallsetminus\mathbb{P}^{c-2}).$$

Proof. To any (E,V) that we want to parametrize we can associate a triple $(\mu,\nu_3,\nu_4)=(\mu,\nu)$, where μ and ν have representatives (12.78), respectively (12.79), and ν_3,ν_4 are as in diagram (12.87). Then by lemmas 12.3.1 and 12.3.2, the following facts are equivalent

(a) (E,V) belongs to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, and it has α_c-canonical filtration of type $(1,1,2)$;

(b) for all $i=3,4$ and for all morphisms $\gamma_i:(Q_i,W_i)\to(E'',V'')$ we have that $\beta_2\circ\gamma_i=0$; moreover $\mu\neq 0$.

In order to give equivalent conditions to (b), let us consider the 2 long exact sequences obtained by applying the functors $\mathrm{Hom}((Q_i,W_i),-)$ to (12.90) for $i=3,4$:

$$\begin{aligned}\cdots\longrightarrow \mathrm{Hom}((Q_i,W_i),(E_2,V_2))\longrightarrow \mathrm{Hom}((Q_i,W_i),(Q_2,W_2))\longrightarrow\\ \longrightarrow \mathrm{Ext}^1((Q_i,W_i),(Q_1,W_1))\xrightarrow{\overline{\sigma^i}}\mathrm{Ext}^1((Q_i,W_i),(E_2,V_2))\xrightarrow{\overline{\kappa^i}}\\ \xrightarrow{\overline{\kappa^i}}\mathrm{Ext}^1((Q_i,W_i),(Q_2,W_2))\longrightarrow\cdots\end{aligned}\tag{12.91}$$

The morphisms $\overline{\sigma}$ and $\overline{\kappa}$ of (12.84) coincide with $(\overline{\sigma^3},\overline{\sigma^4})$, respectively with $(\overline{\kappa^3},\overline{\kappa^4})$. Now let us consider the second line of (12.85):

$$0\to(Q_2,W_2)\xrightarrow{\alpha_2}(E'',V'')\xrightarrow{\beta_2}(Q_3,W_3)\oplus(Q_4,W_4)\to 0.$$

This is a representative for $\overline{\kappa}(\nu)=(\overline{\kappa^3}(\nu),\overline{\kappa^4}(\nu))$. Since $(Q_3,W_3)\not\simeq(Q_4,W_4)$, then by lemma 3.3.2 we have that the following facts are equivalent:

(i) for all $i=3,4$ and for all morphisms $\gamma_i:(Q_i,W_i)\to(E'',V'')$ we have that $\beta_2\circ\gamma_i=0$;

(ii) $\overline{\kappa^3}(\nu_3)\neq 0$ and $\overline{\kappa^4}(\nu_4)\neq 0$.

By exactness of the previous 2 long exact sequences, (ii) is equivalent to

(iii) $\nu_3\notin \mathrm{Im}\ \overline{\sigma^3}$ and $\nu_4\notin\mathrm{Im}\ \overline{\sigma^4}$.

By substituting in (b), we get that (a) is equivalent to

(c) $\mu\neq 0$, $\nu_3\notin\mathrm{Im}\ \overline{\sigma^3}$ and $\nu_4\notin\mathrm{Im}\ \overline{\sigma^4}$.

So until now we have proved that the objects (E,V)'s that we need to parametrize are those induced by triples (μ,ν_3,ν_4) that satisfy conditions (c). Now let us consider the exact sequence (12.79): the objects of the form (E_2,V_2) are parametrized by $\mathbb{P}(\mathrm{Ext}^1((Q_2,W_2),(Q_1,W_1)))$. Moreover, since $(Q_1,W_1)\not\simeq(Q_2,W_2)$, we have that $\mathrm{Aut}(E_2,V_2)=\mathbb{C}^*$. In addition, since $(Q_3,W_3)\not\simeq(Q_4,W_4)$, we have that $\mathrm{Aut}((Q_3,W_3)\oplus(Q_4,W_4))=\mathbb{C}^*\times\mathbb{C}^*$.

Therefore, having fixed $[\mu]$ in $\mathbb{P}(\mathrm{Ext}^1((Q_2,W_2),(Q_1,W_1)))$, there is a natural action of $\mathbb{C}^*\times\mathbb{C}^*$ on the set of all (ν_3,ν_4)'s as before, i.e. on the set $M([\mu])$ defined as

$$\left(\mathrm{Ext}^1((Q_3,W_3),(E_2,V_2))\smallsetminus \mathrm{Im}\ \overline{\sigma^3}\right)\oplus\left(\mathrm{Ext}^1((Q_4,W_4),(E_2,V_2))\smallsetminus \mathrm{Im}\ \overline{\sigma^4}\right).$$

So having fixed $[\mu]$ as before, the set of all the (E,V)'s that we want to parametrize is in bijection with $M([\mu])/(\mathbb{C}^*\times\mathbb{C}^*)$.

Now let us fix any $i=3,4$ and let us consider the long exact sequence (12.91). Let us suppose that there is

$$0\neq\gamma\in\mathrm{Hom}((Q_i,W_i),(E_2,V_2));$$

since $(Q_1,W_1)\not\simeq(Q_i,W_i)$, then necessarily we have that $(Q_i,W_i)\simeq(Q_2,W_2)$ and that $\mu=0$, but this is impossible by construction. Therefore,

$$\mathrm{Hom}((Q_i,W_i),(E_2,V_2))=0\quad\forall\, i=3,4.$$

If $(Q_2,W_2)\not\simeq(Q_3,W_3)$, then $\overline{\sigma^3}$ is injective, so $\dim(\mathrm{Im}\ \overline{\sigma^3})=e$; if $(Q_2,W_2)\simeq(Q_3,W_3)$, then $\dim(\mathrm{Im}\ \overline{\sigma^3})=e-1$. Analogously, if $(Q_2,W_2)\not\simeq(Q_4,W_4)$, then $\dim(\mathrm{Im}\ \overline{\sigma^4})=c$; in the opposite case $\dim(\mathrm{Im}\ \overline{\sigma^4})=c-1$; so we conclude. □

Lemma 12.3.4. *Let us fix any quadruple $(Q_i,W_i)_{i=1,\cdots,4}\in\prod_{i=1}^4 G_i$ with numerical conditions (12.80), respectively (12.82), and such that:*

$$(Q_1,W_1)\not\simeq(Q_2,W_2)\qquad(Q_3,W_3)\simeq(Q_4,W_4).$$

Let us denote by μ any class of a non-split extension of the form

$$0\to(Q_1,W_1)\xrightarrow{\sigma}(E_2,V_2)\xrightarrow{\kappa}(Q_2,W_2)\to 0. \tag{12.92}$$

Having fixed $[\mu]\in\mathbb{P}(Ext^1((Q_2,W_2),(Q_1,W_1)))$, let us consider the morphisms $\overline{\sigma^3}$ and $\overline{\sigma^4}$ induced by σ as in the previous lemma; since $(Q_3,W_3)\simeq(Q_4,W_4)$, we can identify those 2 morphisms. Then we have that the set of all the (E,V)'s that belong to $G^+(\alpha_c;n,d,k)$, respectively to $G^-(\alpha_c;n,d,k)$, that have α_c-canonical filtration of type (1,1,2) and graded $\oplus_{i=1}^4(Q_i,W_i)$ is given by a fibration over $\mathbb{P}(Ext^1((Q_2,W_2),(Q_1,W_1)))$. The fiber over any point $[\mu]$ with μ represented by (12.92) is given by a fibration $\overline{M}([\mu])$ over $Grass(2,H''[\mu])$ with fibers isomorphic to $H'([\mu])\times H'([\mu])$, where:

$$H'([\mu]):=Im\ \overline{\sigma^3}\subset Ext^1((Q_3,W_3),(E_2,V_2))=:H([\mu])$$

and $H''([\mu])$ is any vector space such that

$$H'([\mu])\oplus H''([\mu])=H([\mu]).$$

If we write

$$b:=dim\ Ext^1((Q_3,W_3),(E_2,V_2)),\quad c:=dim\ Ext^1((Q_3,W_3),(Q_1,W_1)),$$

then we have that:

(i) if $(Q_2, W_2) \not\simeq (Q_3, W_3)$, then $\overline{M}([\mu])$ is a fibration over a grassmannian $Grass(2, b-c)$ with fibers isomorphic to $\mathbb{C}^{2c}$;

(ii) if $(Q_2, W_2) \simeq (Q_3, W_3)$, then $\overline{M}([\mu])$ is a fibration over a grassmannian $Grass(2, b-c+1)$ with fibers isomorphic to $\mathbb{C}^{2c-2}$.

Proof. Using lemmas 12.3.1 and 12.3.2, the following facts are equivalent

(a) (E, V) belongs to $G^+(\alpha_c; n, d, k)$, respectively to $G^-(\alpha_c; n, d, k)$, and it has α_c-canonical filtration of type $(1, 1, 2)$;

(b) for all morphisms $\gamma_3 : (Q_3, W_3) \to (E'', V'')$ we have that that $\beta_2 \circ \gamma_3 = 0$; moreover $\mu \neq 0$.

In order to give a better description of conditions (b), let us consider the exact sequence (12.91) for $i = 3$ and the following diagram for $i = 3, 4$. Here ε_i for $i = 3, 4$ is the embedding of (Q_3, W_3) in the first, respectively the second, component of $(Q_3, W_3)^{\oplus 2}$; the diagram is commutative by naturality of the functors $\mathrm{Ext}^1(-, -)$'s.

$$\begin{tikzcd}
\mathrm{Ext}^1((Q_3,W_3)^{\oplus 2},(E_2,V_2)) \arrow[r, "\overline{\varepsilon_i}"] \arrow[d, "\overline{\kappa}"] \arrow[dr, phantom, "\curvearrowright"] & \mathrm{Ext}^1((Q_3,W_3),(E_2,V_2)) \arrow[d, "\overline{\kappa^i}"] \\
\mathrm{Ext}^1((Q_3,W_3)^{\oplus 2},(Q_2,W_2)) \arrow[r, "\overline{\varepsilon_i'}"] & \mathrm{Ext}^1((Q_3,W_3),(Q_2,W_2)).
\end{tikzcd}$$

Then by (12.87) we have that

$$\overline{\kappa^3}(\nu_i) = \overline{\kappa^3}(\overline{\varepsilon_i}(\nu)) = \overline{\varepsilon_i'}(\overline{\kappa}(\nu)).$$

We recall that the second line of (12.85) is a representative for $\overline{\kappa}(\nu)$, so for $i = 3, 4$ we have a commutative diagram with exact lines of the form:

$$\begin{tikzcd}
0 \arrow[r] & (Q_2,W_2) \arrow[r, "\alpha_2"] \arrow[d, equal] \arrow[dr, phantom, "\curvearrowright"] & (E'',V'') \arrow[r, "\beta_2"] \arrow[dr, phantom, "\curvearrowright"] & (Q_3,W_3)^{\oplus 2} \arrow[r] & 0 & \overline{\kappa}(\nu) \arrow[d, "\overline{\varepsilon_i'}"] \\
0 \arrow[r] & (Q_2,W_2) \arrow[r, "\alpha_2^i"] & (E_i'',V_i'') \arrow[r, "\beta_2^i"] \arrow[u, hook, "\xi_i"] & (Q_3,W_3) \arrow[r] \arrow[u, hook, "\varepsilon_i"] & 0 & \overline{\kappa^3}(\nu_i).
\end{tikzcd} \tag{12.93}$$

The first line is a representative of

$$\overline{\kappa}(\nu) = (\overline{\kappa^3}(\nu_3), \overline{\kappa^4}(\nu_4)) = (\overline{\kappa^3}(\nu_3), \overline{\kappa^3}(\nu_4))$$

and we are imposing that for all morphisms $\gamma_3 : (Q_3, W_3) \to (E'', V'')$ we have that $\beta_2 \circ \gamma_3 = 0$. By lemma 3.3.2, we get that this is equivalent to imposing that $\overline{\kappa^3}(\nu_3)$ and $\overline{\kappa^3}(\nu_4)$ are linearly independent in $\mathrm{Ext}^1((Q_3, W_3), (Q_2, W_2))$. This is equivalent to saying that there are no pairs of scalars $(\lambda_3, \lambda_4) \in \mathbb{C}^2 \smallsetminus \{0\}$ such that

$$\lambda_3 \cdot \overline{\kappa^3}(\nu_3) + \lambda_4 \cdot \overline{\kappa^3}(\nu_4) = 0.$$

By linearity of $\overline{\kappa^3}$ and by exactness of (12.91), this is equivalent to saying that:

$$\forall\, (\lambda_3, \lambda_4) \in \mathbb{C}^2 \smallsetminus \{0\} \text{ we have } \lambda_3 \cdot \nu_3 + \lambda_4 \cdot \nu_4 \notin \mathrm{Im}\ \overline{\sigma^3} =: H'([\mu]). \tag{12.94}$$

Now if we look at the sequence (12.79), we get that $\mathrm{Aut}((Q_3, W_3)^{\oplus_2}) = GL(2, \mathbb{C})$ and $\mathrm{Aut}(E_2, V_2) = \mathbb{C}^*$, so there is a natural action of $GL(2, \mathbb{C})$ on the set $M([\mu])$ of all the pairs (ν_3, ν_4) that satisfy (12.94). The set $M([\mu])$ is contained in the set $N([\mu])$ of all the pairs (ν_3, ν_4) that are linearly independent in $H([\mu])$. There is a natural action of $GL(2, \mathbb{C})$ also on $N([\mu])$ and the quotient by such an action is $\overline{N}([\mu]) = Grass(2, H([\mu]))$. It is easy to see that $M([\mu])$ is an open invariant subset, so this gives a scheme structure to its quotient $\overline{M}([\mu]) \subset \overline{N}([\mu]))$.

Let us denote by $H''([\mu])$ any complement of $H'([\mu])$ in $H([\mu])$ (it is not unique, but this gives no problems in the following lines); then we can write $h' = (h', 0)$ for every $h' \in H'([\mu])$ and

$$\nu_i = (\nu_i', \nu_i'') \quad \text{for } i = 3, 4.$$

Now let us denote by q the quotient $H([\mu]) \twoheadrightarrow H([\mu])/H'([\mu]) \simeq H''([\mu])$; if we fix any point $< \nu_3, \nu_4 >$ in $\overline{M}([\mu])$, then by definition of $M([\mu])$ we have that $q(\nu_3)$ and $q(\nu_4)$ are linearly independent in $H''(\nu)$, so it makes sense to consider the 2-plane $< q(\nu_3), q(\nu_4) >$ in $Grass(2, H''([\mu]))$. One can prove easily that the induced morphism

$$\overline{M}([\mu]) \longrightarrow Grass(2, H''([\mu]))$$

is well defined and surjective.

Now let us fix any object $< \nu_3'', \nu_4'' >$ in $Grass(2, H''([\mu]))$. Then for all pairs (ν_3', ν_4') in $H'([\mu]) \times H'([\mu])$, the vectors $\nu_3 := (\nu_3', \nu_3'')$ and $\nu_4 := (\nu_4', \nu_4'')$ in $H(\mu) = H'(\mu) \oplus H''(\mu)$ are linearly independent, so the point $< \nu_3, \nu_4 >$ sits in the preimage of $< \nu_3'', \nu_4'' >$ in $\overline{M}([\mu])$. Moreover, a direct check proves that having fixed $< \nu_3'', \nu_4'' >$, any 2 different pairs (ν_3', ν_4') and $(\widetilde{\nu}_3', \widetilde{\nu}_4')$ in $H'([\mu]) \times H'([\mu])$ give rise to different points of $\overline{M}([\mu])$ in the fiber over $< \nu_3'', \nu_4'' >$. Therefore the fibration $\overline{M}([\mu]) \to Grass(2, H''([\mu]))$ has fibers isomorphic to $H'([\mu]) \times H'([\mu])$.

The dimension of $H'([\mu])$, and therefore of $H''([\mu])$ is computed as in the previous lemma, so we omit the details. □

Now we give a global parametrization of the objects described before, i.e. we describe families of schemes that parametrize various types of (E, V)'s when the graded $\oplus_{i=1}^4 (Q_i, W_i)$ varies over $\prod_{i=1}^4 G_i$ and the α_c-canonical filtration is of type (1,1,2). Since the α_c-canonical filtration is of type (2,1,1), then the order of (Q_3, W_3) and of (Q_4, W_4) is not important. As we said in remark 12.0.2, we will state only the global results for the case when $(n_2, k_2) = (n_3, k_3) = (n_4, k_4)$; the cases when this condition does not hold are actually simpler to manage and they are not needed for computing the Hodge-Deligne polynomials of the moduli spaces $G(\alpha; 4, d, 1)$.

Let us denote by $\oplus_{i=1}^4 (Q_i, W_i)$ a fixed graded with conditions (12.80), respectively (12.82). If $(n_1, k_1) = (n_2, k_2) = (n_3, k_3)$ imposing (12.80), respectively (12.82), is equivalent to imposing that:

$$\frac{k_2}{n_2} > \frac{k}{n}, \tag{12.95}$$

respectively that

$$\frac{k_2}{n_2} < \frac{k}{n}. \tag{12.96}$$

If $(Q_3, W_3) \not\simeq (Q_4, W_4)$, then by lemma 12.3.3 the corresponding (E, V)'s are parametrized by triples $([\mu], [\nu_3], [\nu_4])$ with

$$[\mu] \in \mathbb{P}(\mathrm{Ext}^1((Q_2, W_2), (Q_1, W_1))),$$

representative (12.90) for μ and

$$([\nu_3], [\nu_4]) \in \overline{M}([\mu]) \subset \prod_{i=3}^{4} \mathbb{P}(\mathrm{Ext}^1((Q_i, W_i), (E_2, V_2))).$$

We are considering the case when the (Q_i, W_i)'s are all of the same type for $i = 2, 3, 4$. Therefore we need to take into account the possible isomorphisms between them. So, having fixed $[\mu]$, we need to consider separately the following cases.

(1) If $(Q_2, W_2) \simeq (Q_3, W_3) \not\simeq (Q_4, W_4)$, then the roles of (Q_3, W_3) and of (Q_4, W_4) are not interchangeable, so we need to consider *ordered* pairs $([\nu_3], [\nu_4])$.

(2) If $(Q_i, W_i) \not\simeq (Q_j, W_j)$ for all $i \neq j \in \{2, 3, 4\}$, then the roles of (Q_3, W_3) and of (Q_4, W_4) are interchangeable, so we need to consider *unordered* pairs $([\nu_3], [\nu_4])$.

Note that since the order of (Q_3, W_3) and (Q_4, W_4) is not important, we don't need to consider also the case $(Q_2, W_2) \simeq (Q_4, W_4) \not\simeq (Q_3, W_3)$.

If $(Q_3, W_3) \simeq (Q_4, W_4)$, then by lemma 12.3.4 the corresponding (E, V)'s are parametrized by pairs $([\mu], < \nu_3, \nu_4 >)$ with

$$[\mu] \in \mathbb{P}(\mathrm{Ext}^1((Q_2, W_2), (Q_1, W_1))),$$

representative (12.92) for μ and

$$< \nu_3, \nu_4 > \in \overline{M}([\mu]) \subset Grass(2, \mathrm{Ext}^1((Q_3, W_3), (E_2, V_2))).$$

So, having fixed $[\mu]$, we need to consider separately the following cases.

(3) If $(Q_2, W_2) \not\simeq (Q_3, W_3) \simeq (Q_4, W_4)$, then the corresponding (E, V)'s are parametrized by a fibration described as in lemma 12.3.4 (i).

(4) If $(Q_2, W_2) \simeq (Q_3, W_3) \simeq (Q_4, W_4)$, then the corresponding (E, V)'s are parametrized by a fibration described as in lemma 12.3.4 (ii).

These 4 cases are taken into account by propositions 7.7.1, 7.7.2, 7.7.3 and 7.7.4 respectively.

The proof of proposition 7.7.1 is the dual of the construction of the scheme $\hat{R}^1_{a,b,c,d;i,j}$ in proposition 7.5.1 by using lemma 12.3.3 instead of lemma 12.1.5, so we omit all the details. We only remark that if we use lemma 12.3.3 with the condition that $(Q_2, W_2) \simeq (Q_3, W_3)$, then we get that

$$e = \dim\ \mathrm{Ext}^1((Q_3, W_3), (Q_1, W_1)) = \dim\ \mathrm{Ext}^1((Q_2, W_2), (Q_1, W_1)) = a,$$

so we only need the indices a, b, c, d and not the index e of that lemma.

The proof of proposition 7.7.2 is on the same line of the construction of the schemes of the form $R^1_{a,b,c,d,e;i,j}$ for $(b, c) < (d, e)$ and of the schemes of the form $R^1_{a,b,c,b,c;i,j}/\mathbb{Z}_2$ in proposition 7.5.2. The only significant difference is that we use lemma 12.3.3 instead of lemma 12.1.5; so we omit the details.

Proof of proposition 7.7.3. The construction of these spaces follows the lines of the proof of proposition 7.7.1 in order to get a family of fibrations $\{\hat{\varphi}_{a;i} : \hat{R}_{a;i} \to \hat{U}_{a;i}\}_i$ and for every i a universal family of non-split extensions parametrized by $\hat{R}_{a;i}$:

$$\begin{aligned} 0 \to (\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1) \otimes_{\hat{R}_{a;i}} \mathcal{O}_{\hat{R}_{a;i}}(1) \xrightarrow{\sigma_{a;i}} \\ \xrightarrow{\sigma_{a;i}} (\hat{\mathcal{E}}_{2;a;i}, \hat{\mathcal{V}}_{2;a;i}) \xrightarrow{\kappa_{a;i}} (\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_2, \hat{p}_2)^*(\hat{\mathcal{Q}}_2, \hat{\mathcal{W}}_2) \to 0, \end{aligned} \tag{12.97}$$

where $\hat{p}_1$ and $\hat{p}_2$ are the projections from $\hat{U}_{a;i} \subset \hat{G}_1 \times \hat{G}_2$ to its factors. Now let us fix any index i, let us consider the projections

$$\hat{p}_3 : \hat{R}_{a;i} \times \hat{G}_3 \longrightarrow \hat{G}_3, \quad \hat{p}_{12} : \hat{R}_{a;i} \times \hat{G}_3 \longrightarrow \hat{R}_{a;i}$$

and let us define the following scheme

$$\hat{U}_{a,b,c;i} := \{t \in \hat{R}_{a;i} \times \hat{G}_3 \text{ s.t. } \dim\ \mathrm{Ext}^1((\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)_t, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\mathcal{E}}_{2;a;i}, \hat{\mathcal{V}}_{2;a;i})_t) = b,$$

$$\dim\ \mathrm{Ext}^1((\hat{p}'_3, \hat{p}_3)^*(\hat{\mathcal{Q}}_3, \hat{\mathcal{W}}_3)_t, (\hat{p}'_{12}, \hat{p}_{12})^*(\hat{\varphi}'_{a;i}, \hat{\varphi}_{a;i})^*(\hat{p}'_1, \hat{p}_1)^*(\hat{\mathcal{Q}}_1, \hat{\mathcal{W}}_1)_t) = c,$$

$$\mathrm{Hom}((\hat{p}'_{12},\hat{p}_{12})^*(\hat{\varphi}'_{a;i},\hat{\varphi}_{a;i})^*(\hat{p}'_2,\hat{p}_2)^*(\hat{\mathcal{Q}}_2,\hat{\mathcal{W}}_2)_t,(\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)_t)=0\}.$$

By proposition 1.0.5, this is a locally closed subscheme of $\hat{R}_{a;i}\times\hat{G}_3$. Moreover, by applying lemma 4.6.1, we get that $\hat{U}_{a,b,c;i}$ has a finite disjoint locally closed covering $\{\hat{U}_{a,b,c;i,j}\}_j$ such that for all j the sheaf

$$\hat{E}_{a,b,c;i,j}:=\mathcal{E}xt^1_{\pi_{\hat{U}_{a,b,c;i,j}}}\Big((\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3),(\hat{p}'_{12},\hat{p}_{12})^*(\hat{\mathcal{E}}_{2;a;i},\hat{\mathcal{E}}_{2;a;i})\Big)$$

is locally free on $\hat{U}_{a,b,c;i,j}$ and commutes with base change. By construction of $\hat{U}_{a,b,c;i,j}$ and of $\hat{U}_{a;i}$, for every point t of $\hat{U}_{a,b,c;i,j}$ we have that:

$$\mathrm{Hom}\Big((\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)_t,(\hat{p}'_{12},\hat{p}_{12})^*(\hat{\mathcal{E}}_{2;a;i},\hat{\mathcal{V}}_{2;a;i})_t\Big)=0$$

Therefore, by proposition 4.5.1 and corollary 4.5.6 there exists a grassmannian fibration

$$\hat{\theta}_{2;a,b,c;i,j}:\hat{Q}_{a,b,c;i,j}:=Grass(2,\hat{E}^{\vee}_{a,b,c;i,j})\longrightarrow\hat{U}_{a,b,c;i,j}$$

and a universal extension (in the sense of corollary 4.5.4) parametrized by $\hat{Q}_{a,b,c;i,j}$

$$0\to(\hat{\theta}'_{2;a,b,c;i,j},\hat{\theta}_{2;a,b,c;i,j})^*(\hat{p}'_{12},\hat{p}_{12})^*(\hat{\mathcal{E}}_{2;a;i},\hat{\mathcal{V}}_{2;a;i})\to(\hat{\mathcal{E}}_{a,b,c;i,j},\hat{\mathcal{V}}_{a,b,c;i,j})\to$$
$$\to(\hat{\theta}'_{2;a,b,c,;i,j},\hat{\theta}_{2;a,b,c;i,j})^*(\hat{p}'_3,\hat{p}_3)^*(\hat{\mathcal{Q}}_3,\hat{\mathcal{W}}_3)\otimes_{\hat{Q}_{a,b,c;i,j}}\overline{\mathcal{M}}_{2;a,b,c;i,j}\to0,\tag{12.98}$$

where $\overline{\mathcal{M}}_{2;a,b,c;i,j}$ is a locally free sheaf of rank 2 on $\hat{Q}_{a,b,c;i,j}$. For simplicity, we rewrite such an exact sequence as

$$0\to(\overline{\mathcal{E}}_{2;a,i},\overline{\mathcal{V}}_{2;a,i})\to(\hat{\mathcal{E}}_{a,b,c;i,j},\hat{\mathcal{V}}_{a,b,c;i,j})\to(\overline{\mathcal{Q}}_3,\overline{\mathcal{W}}_3)\otimes_{\hat{Q}_{a,b,c;i,j}}\overline{\mathcal{M}}_{2;a,b,c;i,j}\to0.\tag{12.99}$$

Moreover, we consider the morphism

$$\hat{Q}_{a,b,c;i,j}\stackrel{\hat{\theta}_{2;a,b,c;i,j}}{\longrightarrow}\hat{U}_{a,b,c;i,j}\hookrightarrow\hat{R}_{a;i}\times\hat{G}_3\twoheadrightarrow\hat{R}_{a;i}$$

and we denote by

$$0\to(\overline{\mathcal{Q}}_1,\overline{\mathcal{W}}_1)\stackrel{\widetilde{\sigma}_{a;i}}{\longrightarrow}(\overline{\mathcal{E}}_{2;a,i},\overline{\mathcal{V}}_{2;a,i})\stackrel{\widetilde{\kappa}_{a;i}}{\longrightarrow}(\overline{\mathcal{Q}}_2,\overline{\mathcal{W}}_2)\to0.$$

the pullback of (12.97) via that morphism. Now let us consider the functor

$$\overline{\kappa}_{a;i}:\mathrm{Ext}^1\Big((\overline{\mathcal{Q}}_3,\overline{\mathcal{W}}_3)\otimes_{\hat{Q}_{a,b,c;i,j}}\overline{\mathcal{M}}_{2;a,b,c;i,j},(\overline{\mathcal{E}}_{2;a,i},\overline{\mathcal{V}}_{2;a,i})\Big)\longrightarrow$$
$$\longrightarrow\mathrm{Ext}^1\Big((\overline{\mathcal{Q}}_3,\overline{\mathcal{W}}_3)\otimes_{\hat{Q}_{a,b,c;i,j}}\overline{\mathcal{M}}_{2;a,b,c;i,j},(\overline{\mathcal{Q}}_2,\overline{\mathcal{W}}_2)\Big)$$

induced by $\widetilde{\kappa}_{a;i}$. Then it makes sense to apply $\overline{\kappa}_{a;i}$ to the extension represented by (12.99). So we get a commutative diagram with exact lines as follows:

$$
\begin{tikzcd}
0 \arrow[r] & (\overline{\mathcal{E}}_{2;a,i}, \overline{\mathcal{V}}_{2;a,i}) \arrow[r] \arrow[d, "\widetilde{\kappa}_{a;i}"] \arrow[dr, phantom, "\curvearrowright"] & (\hat{\mathcal{E}}_{a,b,c;i,j}, \hat{\mathcal{V}}_{a,b,c;i,j}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\curvearrowright"] & (\overline{\mathcal{Q}}_3, \overline{\mathcal{W}}_3) \otimes_{\hat{Q}_{a,b,c;i,j}} \overline{\mathcal{M}}_{2;a,b,c;i,j} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\overline{\mathcal{Q}}_2, \overline{\mathcal{W}}_2) \arrow[r, "\alpha"] & (\hat{\mathcal{E}}''_{a,b,c;i,j}, \hat{\mathcal{V}}''_{a,b,c;i,j}) \arrow[r, "\beta"] & (\overline{\mathcal{Q}}_3, \overline{\mathcal{W}}_3) \otimes_{\hat{Q}_{a,b,c;i,j}} \overline{\mathcal{M}}_{2;a,b,c;i,j} \arrow[r] & 0.
\end{tikzcd}
$$

Now let us fix any point t of $\hat{Q}_{a,b,c;i,j}$ and let us denote by

$$0 \to (Q_2, W_2) \xrightarrow{\alpha_t} (E'', V'') \xrightarrow{\beta_t} (Q_3, W_3)^{\oplus 2} \to 0 \tag{12.100}$$

the restriction of the second line of such a diagram to t. Let us denote by $(\nu'_{3,t}, \nu'_{4,t})$ the class of (12.100). Then by lemma 3.3.2 we have that $\nu'_{3,t}$ and $\nu'_{4,t}$ are linearly independent in $\mathrm{Ext}^1((Q_3, W_3), (Q_2, W_2))$ if and only if for all morphisms

$$\gamma_{3,t} : (Q_3, W_3) := (\overline{\mathcal{Q}}_3, \overline{\mathcal{W}}_3)_t \longrightarrow (\hat{\mathcal{E}}''_{a,b,c;i,j}, \hat{\mathcal{V}}''_{a,b,c;i,j})_t =: (E'', V'')$$

we have that $\beta_t \circ \gamma_{3,t} = 0$. By construction of $\hat{Q}_{a,b,c;i,j}$ we have that $(Q_2, W_2) \not\simeq (Q_3, W_3)$. Therefore, given any $\gamma_{3,t}$ as before, by exactness of (12.100) we have that $\beta_t \circ \gamma_{3,t}$ is non-zero if and only if $\gamma_{3,t}$ is non-zero. Therefore, $\nu'_{3,t}$ and $\nu'_{4,t}$ are linearly independent if and only if

$$\mathrm{Hom}((Q_3, W_3), (E'', V'')) = 0.$$

So the set of all the points t of $\hat{Q}_{a,b,c;i,j}$ such that $\nu'_{3,t}$ and $\nu'_{4,t}$ are linearly independent coincides with the set

$$\hat{R}_{a,b,c;i,j} := \{t \in \hat{Q}_{a,b,c;i,j} \text{ s.t. } \mathrm{Hom}((\overline{\mathcal{Q}}_3, \overline{\mathcal{W}}_3)_t, (\hat{\mathcal{E}}''_{a,b,c;i,j}, \hat{\mathcal{V}}''_{a,b,c;i,j})_t) = 0\}.$$

By lemma 1.0.4, this is a locally closed subscheme of $\hat{Q}_{a,b,c;i,j}$. We denote by

$$\hat{\varphi}_{a,b,c;i,j} : \hat{R}_{a,b,c;i,j} \longrightarrow \hat{U}_{a,b,c;i,j}$$

the restriction of $\hat{\theta}_{2;a,b,c;i,j}$ to $\hat{R}_{a,b,c;i,j}$. Then the proof of lemma 12.3.4 shows that the fiber of $\hat{\varphi}_{a,b,c;i,j}$ over any point of $\hat{U}_{a,b,c;i,j}$ is isomorphic to $\mathbb{C}^{2c} \times Grass(2, b-c)$. Now let us fix any point r of $\hat{R}_{a,b,c;i,j}$ and let us denote by r' its image in $\hat{R}_{a;i}$ via $\hat{p}_{12} \circ \hat{\varphi}_{a,b,c;i,j}$. Then let us consider the restriction of (12.97) to r' and of (12.99) to r:

$$0 \to (Q_1, W_1) \xrightarrow{\sigma} (E_2, V_2) \xrightarrow{\kappa} (Q_2, W_2) \to 0,$$

$$0 \to (E_2, V_2) \xrightarrow{\varepsilon} (E, V) \xrightarrow{\delta} (Q_3, W_3)^{\oplus 2} \to 0.$$

Let us denote by μ_r and $\nu_r = (\nu_{3,r}, \nu_{4,r})$ the corresponding classes. By construction of $\hat{R}_{a;i}$ we have that $\mu_r \neq 0$. By construction of $\hat{Q}_{a,b,c;i,j}$ we have that $\nu_{3,r}$ and $\nu_{4,r}$ are linearly independent. In addition, by construction of $\hat{R}_{a,b,c;i,j}$ we have that $\nu'_{3,r}$ and $\nu'_{4,r}$ are linearly independent.

So if we assume conditions (12.80), by the proof of lemma 12.3.4 we conclude that (E,V) is an object of $G^+(\alpha_c;n,d,k)$. Now by using the family $(\hat{\mathcal{E}}_{a,b,c;i,j},\hat{\mathcal{V}}_{a,b,c;i,j})$ restricted from $\hat{Q}_{a,b,c;i,j}$ to $\hat{R}_{a,b,c;i,j}$, we get an induced morphism:

$$\hat{\omega}_{a,b,c;i,j}:\hat{R}_{a,b,c;i,j}\longrightarrow G^+(\alpha_c;n,d,k).$$

Then we conclude as usual. □

Proof of proposition 7.7.4. The proof is on the same line of the proof of the previous proposition, with c omitted or replaced by a whenever it is necessary, so we skip the details. The only significant difference is in the definition of $\hat{R}_{a,b;i,j}$. Also in this case we need to describe necessary and sufficient conditions such that the sequence (12.100) is a representative of a pair of linearly independent vectors. In this case such a sequence has the form

$$0\to(Q_3,W_3)\xrightarrow{\alpha_t}(E'',V'')\xrightarrow{\beta_t}(Q_3,W_3)^{\oplus 2}\to 0, \qquad (12.101)$$

so $\dim \mathrm{Hom}((Q_3,W_3),(E'',V''))\geq 1$. Actually, by exactness of (12.101) we have that the following facts are equivalent

- for all morphisms $\gamma_{3,t}:(Q_3,W_3)\to(E'',V'')$ we have $\beta_t\circ\gamma_{3,t}=0$;
- $\dim \mathrm{Hom}((Q_3,W_3),(E'',V''))=1$.

Therefore in this case the correct definition for the scheme $\hat{R}_{a,b;i,j}$ is the following:

$$\hat{R}_{a,b;i,j}:=\{t\in\hat{Q}_{a,b;i,j}\text{ s.t. }\dim \mathrm{Hom}((\overline{\mathcal{Q}}_3,\overline{\mathcal{W}}_3)_t,(\hat{\mathcal{E}}''_{a,b,c;i,j},\hat{\mathcal{V}}''_{a,b,c;i,j})_t)=1\}.$$

The rest of the proof is essentially unchanged, so we omit the details. □

Chapter 13

Case n=2, k=1

In this and in the next chapters we compute the Hodge-Deligne polynomials of some moduli spaces of coherent systems.

In particular, in this chapter we consider the case when $(n,d,k)=(2,d,1)$. By [BGMN, §2 and proposition 4.2] the non-zero virtual critical values for such a triple are all in the set

$$\left\{\frac{nd'-n'd}{n'k-nk'} \text{ s.t. } 0\leq k'\leq k, \quad 0<n'<n, \quad n'k\neq nk', \quad d'\in\mathbb{Z}\right\}\cap\left]0,\frac{d}{n-k}\right[.$$

In our case, this gives

$$\left\{\frac{2d'-d}{1-2k'} \text{ s.t. } k'=0,1, \quad d'\in\mathbb{Z}\right\}\cap]0,d[,$$

that is

$$\left\{2d'-d,\ \frac{d}{2}<d'<d\right\}\cup\left\{d-2d',\ 0<d'<\frac{d}{2}\right\}$$

where the first set corresponds to destabilizing subsystems of the form $(1,d',0)$ and the second one corresponds to destabilizing subsystems of the form $(1,d',1)$. Actually, the 2 sets coincide both with the set

$$\left\{\alpha(j):=d-2j, \quad 0<j<\frac{d}{2}\right\}.$$

Since we will also be interested in the moduli space $G_L(2,d,1)=G(d-\varepsilon;2,d,1)$, we will also consider $\alpha(0)=d$ as a critical value, so that $G(d-\varepsilon;2,d,1)=G(\alpha(0)^-;2,d,1)$. In other words, we are considering all the values:

$$0<\alpha\left(\left\lfloor\frac{d-1}{2}\right\rfloor\right)<\alpha\left(\left\lfloor\frac{d-1}{2}\right\rfloor-1\right)<\cdots<\alpha(1)<\alpha(0)=d.$$

All these values will be actual critical values, as we will see below (anyway, even if some $\alpha(j)$ is not an actual critical value, we can consider it as a critical value, such that when we cross it we neither add nor remove anything from our moduli space).

13.1 The moduli spaces $G^+(\alpha(j);2,d,1)$

Let $\alpha(j)$ be any critical value for $0 \leq j < d/2$ and let us suppose that (E,V) belongs to $G^+(\alpha(j);2,d,1)$. Then by lemma 1.0.6, we get that (E,V) appears in a non-split exact sequence:

$$0 \to (Q_1,W_1) \to (E,V) \to (Q_2,W_2) \to 0 \tag{13.1}$$

where:

(a) (Q_1,W_1) and (Q_2,W_2) are both $\alpha(j)^+$-stable with $\frac{k_1}{n_1} < \frac{k}{n} = \frac{1}{2} < \frac{k_2}{n_2}$;

(b) (Q_1,W_1) and (Q_2,W_2) are both $\alpha(j)$-semistable with the same $\alpha(j)$-slope as (E,V).

Conversely, every such (E,V) is actually a point of $G^+(\alpha(j);2,d,1)$. Indeed it is $\alpha(j)$-semistable by (b) and proposition 1.0.1 and it is $\alpha(j)$-stable because its only proper coherent subsystem is (Q_1,W_1), which does not destabilize (E,V) for $\alpha(j)^+$ and destabilizes it for $\alpha(j)^-$ because of conditions (a) and (b). Moreover, every such (E,V) is completely determined by the class of the non-split extension (13.1), up to multiplication by invertible scalars.

Condition (a) implies that $n_1 = n_2 = 1$ and that $k_1 = 0$, so $k_2 = 1$; condition (b) implies that $d_1 = (d+\alpha(j))/2 = d-j$. By the conditions on j, this proves that $d_1 = d-j$ and $d_2 = j$ are both non-negative integers. Now

$$(Q_1,W_1) \in G(1,d-j,0) = J^{d-j}C =: G_1, \quad (Q_2,W_2) \in G(1,j,1) =: G_2. \tag{13.2}$$

For every pair of objects (Q_1,W_1), (Q_2,W_2) in those 2 spaces, using proposition 1.0.7 we have that

$$\dim \operatorname{Ext}^1((Q_2,W_2),(Q_1,W_1)) = C_{21} + \dim \mathbb{H}^0_{21} + \dim \mathbb{H}^2_{21}.$$

Now both (Q_1,W_1) and (Q_2,W_2) are $\alpha(j)^+$-stable and $\mu_{\alpha(j)^+}(Q_2,W_2) > \mu_{\alpha(j)^+}(Q_1,W_1)$ (as a consequence of properties (a) and (b)). Therefore by lemma 1.0.4, $\mathbb{H}^0_{21} = 0$. Moreover, by [BGMN, lemma 3.3], we have that also $\mathbb{H}^2_{21} = 0$. Therefore

$$\begin{aligned}\dim \operatorname{Ext}^1((Q_2,W_2),(Q_1,W_1)) = C_{21} =\\ = n_1n_2(g-1) - d_1n_2 + d_2n_1 + k_2d_1 - k_2n_1(g-1) - k_1k_2 =\\ = (g-1) - (d-j) + j + (d-j) - (g-1) = j.\end{aligned}$$

Now the moduli spaces G_1 and G_2 are both smooth and irreducible, therefore also $G_1 \times G_2$ is so. On both G_i's there are universal families of coherent systems $(\mathcal{Q}_i, \mathcal{W}_i)$ of coherent systems (because of [BGMMN, proposition A.8]), so we can apply proposition 5.0.5 for $r = 2$ and we get that there is a projective bundle

$$\varphi_j : R_j \longrightarrow G_1 \times G_2$$

with fibers isomorphic to $\mathbb{P}^{j-1}$ and an injective morphism to $G(\alpha(j)^+; 2, d, 1)$, such that the image coincides with $G^+(\alpha(j); 2, d, 1)$. Therefore, the Hodge-Deligne polynomial of $G^+(\alpha(j); 2, d, 1)$ is given by

$$p^j := \mathcal{HD}\Big(G^+(d-2j; 2, d, 1)\Big) =$$
$$= \mathcal{HD}(J^{d-j}C)\mathcal{HD}(G(1, j, 1))\mathcal{HD}(\mathbb{P}^{j-1}) =$$
$$= (1+u)^g(1+v)^g \frac{1-(uv)^j}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}. \tag{13.3}$$

Note that when $j = 0$, we get that a term of the form $\mathcal{HD}(\mathbb{P}^{-1}) = \mathcal{HD}(\varnothing) = 0$, so $p^0 = 0$, as it should be; indeed in this case we know a priori that $G^+(d; 2, d, 1) = G(d+\varepsilon; 2, d, 1) = \varnothing$ since $G(\alpha; 2, d, 1)$ is non-empty only if $\alpha \in]0, d[$.

13.2 The moduli spaces $G^-(\alpha(j); 2, d, 1)$

Let us turn to $G^-(\alpha(j); 2, d, 1)$. By applying again proposition 1.0.7 we get that every $(E, V) \in G^-(\alpha(j); 2, d, 1)$ appears in a non-split exact sequence (13.1) where:

(a') (Q_1, W_1) and (Q_2, W_2) are both $\alpha(j)^-$-stable with $\frac{k_1}{n_1} > \frac{k}{n} = \frac{1}{2} > \frac{k_2}{n_2}$;

(b') (Q_1, W_1) and (Q_2, W_2) are both $\alpha(j)$-semistable with the same $\alpha(j)$-slope as (E, V).

Conversely, as before it is easy to show that every such (E, V) is actually a point of $G^-(\alpha(j); 2, d, 1)$; moreover any such (E, V) is uniquely associated to a non-split exact sequence (13.1) with conditions (a') and (b'), up to multiplication by invertible scalars.

Condition (a') implies that $n_1 = n_2 = 1$ and that $k_1 = 1$, so $k_2 = 0$. Moreover, condition (b') implies that $d_1 + \alpha(j) = (d + \alpha(j))/2$, so $d_1 = (d - \alpha(j))/2 = j$. By the conditions on j, this proves that $d_1 = j$ and $d_2 = d - j$ are both positive integers. Now

$$(Q_1, W_1) \in G(1, j, 1) := G_1, \quad (Q_2, W_2) \in G(1, d-j, 0) = J^{d-j}C := G_2. \tag{13.4}$$

For every pair of objects (Q_1, W_1), (Q_2, W_2) in those 2 spaces, we have that $\mu_{\alpha(j)^-}(Q_2, W_2) > \mu_{\alpha(j)^-}(Q_1, W_1)$ as a consequence of properties (a') and (b'). Therefore, by lemma 1.0.4 there are no morphisms from (E_2, V_2) to (E_1, V_1), so $\mathbb{H}^0_{21} = 0$. Moreover, by [BGMN, equation (11)], we have that also $\mathbb{H}^2_{21} = 0$. Therefore

$$\dim \operatorname{Ext}^1((Q_2,W_2),(Q_1,W_1)) = C_{21} =$$
$$= n_1n_2(g-1) - d_1n_2 + d_2n_1 + k_2d_1 - k_2n_1(g-1) - k_1k_2 =$$
$$= (g-1) - j + d - j = g + d - 1 - 2j.$$

Now as in the previous section we get that the space $G^-(\alpha(j); 2, d, 1)$ is given by a projective bundle over $G_1 \times G_2$ with fibers isomorphic to $\mathbb{P}^{g+d-2-2j}$.
Therefore, the Hodge-Deligne polynomial of $G^-(\alpha(j); 2, d, 1)$ is given by

$$q^j := \mathcal{HD}\Big(G^-(d-2j; 2, d, 1)\Big) =$$
$$= \mathcal{HD}(J^{d-j}C)\mathcal{HD}(G(1,j,1))\mathcal{HD}(\mathbb{P}^{g+d-2-2j}) =$$
$$= (1+u)^g(1+v)^g \frac{1-(uv)^{g+d-1-2j}}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}. \tag{13.5}$$

Remark 13.2.1. This formula makes sense also when $j = 0$. In this case we know that $G^-(d; 2, d, 1) = G(d - \varepsilon; 2, d, 1)$ since

$$G(d-\varepsilon; 2, d, 1) \smallsetminus G^-(d; 2, d, 1) = G(d+\varepsilon; 2, d, 1) \smallsetminus G^+(d; 2, d, 1) = \varnothing.$$

Obviously in this case there is a shorter way to compute the Hodge-Deligne polynomial of $G(d-\varepsilon; 2, d, 1) = G_L(2, d, 1) = G(\alpha(0)^-; 2, d, 1)$. Since $(n-k, d) = (1, d)$, we can apply the last part of theorem 5.4 in [BGMN] and we get that G_L is a fibration over $M(1, d) = J^dC$ with fiber the grassmannian

$$\operatorname{Grass}(k, d + (n-k)(g-1)) = \operatorname{Grass}(1, g+d-1) = \mathbb{P}^{g+d-2}.$$

Therefore,

$$\mathcal{HD}(G(\alpha(1)^+; 2, d, 1) = \mathcal{HD}(G(\alpha(0)^-; 2, d, 1) = \mathcal{HD}(G_L(2, d, 1)) =$$
$$= \mathcal{HD}(J^dC)\mathcal{HD}(\mathbb{P}^{d+g-2}) = (1+u)^g(1+v)^g \frac{1-(uv)^{g+d-1}}{1-uv}. \tag{13.6}$$

Actually a direct check proves that (13.6) coincides with (13.5) for $j = 0$ once one observes that the last part of (13.5) is equal to 1 since the function depending on x is actually holomorphic around $x = 0$. Even if (13.6) is simpler to write, we will use anyway (13.5) also in this case, since it will be simpler to sum all the various terms coming from crossing the critical values. We remark also that if we evaluate the previous polynomial in $u = v =: t$, we get exactly the Poincaré polynomial of [BGMMN, proposition 8.4].

13.3 The polynomials for $G(\alpha(k)^-;2,d,1)$

Theorem 13.3.1. *For every smooth projective curve C of genus $g\geq 2$, for every $d>0$ and for every critical value*

$$\alpha(k)=d-2k,\quad 0\leq k<\frac{d}{2}$$

we have that:

$$\mathcal{HD}(G(\alpha(k)^-;2,d,1))= \tag{13.7}$$
$$=\frac{(1+u)^g(1+v)^g}{1-uv}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\left[\frac{(uv)^kx^{-k}}{1-x(uv)^{-1}}-\frac{(uv)^{g+d-1-2k}x^{-k}}{1-x(uv)^2}\right].$$

Remark 13.3.1. Let us compare this result with proposition 8.0.7. If we set $d_1:=d$, $d_2:=0$ and $d_0=d_1-k$, then we get that

$$\mathcal{HD}(\mathcal{N}_\sigma(2,1,d,0))=(1+u)^g(1+v)^g\mathcal{HD}(G(\alpha(k)^-;2,d,1)).$$

This makes sense since we can consider every α-stable coherent system (E,V) of type $(2,d,1)$ as a σ-stable triple (E_1,E_2,ϕ) of type $(2,1,d,0)$ with the additional restriction that $E_2=\mathcal{O}=\mathcal{O}\otimes V$, instead of E_2 being any point of the Jacobian J^0C, whose Hodge-Deligne polynomial is exactly $(1+u)^g(1+v)^g$.

Proof. By combining (13.3) and (13.5), we have that for every $\alpha(j)$:

$$\mathcal{HD}(G(\alpha(j)^-;2,d,1))-\mathcal{HD}(G(\alpha(j)^+;2,d,1))=q^j-p^j=$$
$$=\frac{(1+u)^g(1+v)^g}{1-uv}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^gx^{-j}}{(1-x)(1-uvx)}[(uv)^j-(uv)^{g+d-1-2j}]. \tag{13.8}$$

Now let us fix any critical value $\alpha(k)$ and let us use formulae (13.8) for all $0\leq j\leq k$. We recall that $G(\alpha(0)^+;2,d,1)=\varnothing$. Therefore,

$$\mathcal{HD}(G(\alpha(k)^-;2,d,1))=\mathcal{HD}(G(\alpha(k)^-;2,d,1))-\mathcal{HD}(G(\alpha(0)^+;2,d,1))=$$
$$=\frac{(1+u)^g(1+v)^g}{1-uv}\sum_{0\leq j\leq k}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^gx^{-j}}{(1-x)(1-uvx)}[(uv)^j-(uv)^{g+d-1-2j})]=$$
$$=\frac{(1+u)^g(1+v)^g}{1-uv}\sum_{0\leq j\leq k}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\left[\left(\frac{uv}{x}\right)^j-(uv)^{g+d-1}\left(\frac{1}{x(uv)^2}\right)^j\right]=$$
$$=\frac{(1+u)^g(1+v)^g}{1-uv}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\left[\left(1-\left(\frac{uv}{x}\right)^{k+1}\right)\left(1-\frac{uv}{x}\right)^{-1}+\right.$$
$$\left.-(uv)^{g+d-1}\left(1-\left(\frac{1}{x(uv)^2}\right)^{k+1}\right)\left(1-\frac{1}{x(uv)^2}\right)^{-1}\right]=$$
$$=\frac{(1+u)^g(1+v)^g}{1-uv}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\left[\frac{x^{k+1}-(uv)^{k+1}}{x^k(x-uv)}+\right.$$

$$-(uv)^{g+d-1-2k}\frac{x^{k+1}(uv)^{2k+2}-1}{x^k(x(uv)^2-1)}\Bigg] =$$

$$= \frac{(1+u)^g(1+v)^g}{1-uv}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\left[\frac{(uv)^k x^{-k}}{1-x(uv)^{-1}}-\frac{(uv)^{g+d-1-2k}x^{-k}}{1-x(uv)^2}\right].$$

□

Remark 13.3.2. Formula (13.8) evaluated in $u = v =: t$ agrees with Thaddeus's formula in [T, proof of formula (4.1)]. The only difference is an additional multiplicative term $(1+t)^{2g}$ given by the fact that in [T] the determinant of the vector bundle E of a coherent system (E, V) is fixed, while in our case this is free to vary over J^dC, whose Poincaré polynomial is exactly $(1+t)^{2g}$.

Remark 13.3.3. Let us verify that Poincaré duality holds for the polynomial obtained in the previous theorem. This amounts to substituting u with u^{-1} and v with v^{-1} in the previous formula and verifying that identity (8.3) holds. We recall that whenever $f(x, u, v)$ is a meromorphic function in x around $x = 0$ and u, v are indeterminate, then by considering the expansion of $f(x)$ in a Laurent series, we can compute

$$\operatorname*{coeff}_{x^0} f(x) = \operatorname*{coeff}_{cx^0} f(x)$$

for every non-zero constant c. In particular, since u and v are indeterminate, we can use $c = uv$. So if we write

$$f(x,u,v) := \frac{(1+ux)^g(1+vx)^g x^{-k}}{(1-x)(1-uvx)}\left[\frac{(uv)^k}{1-x(uv)^{-1}}-\frac{(uv)^{g+d-1-2k}}{1-x(uv)^2}\right]$$

and $g(u, v) := \operatorname{coeff}_{x^0} f(x, u, v)$, then

$$g(u^{-1},v^{-1}) = \operatorname*{coeff}_{x^0} f(uvx,u^{-1},v^{-1}) =$$

$$= \operatorname*{coeff}_{x^0}\frac{(1+vx)^g(1+ux)^g(uv)^{-k}x^{-k}}{(1-uvx)(1-x)}\left[\frac{(uv)^{-k}}{1-(uv)^2x}-\frac{(uv)^{2k+1-g-d}}{1-x(uv)^{-1}}\right] =$$

$$= \operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-k}}{(1-x)(1-uvx)}\cdot\frac{(uv)^{1+k-g-d}}{(uv)^k}\left[\frac{(uv)^{g+d-1-2k}}{1-(uv)^2x}-\frac{(uv)^k}{1-x(uv)^{-1}}\right] =$$

$$= -(uv)^{1-g-d}g(u,v).$$

So if we denote by $p(u, v) := \mathcal{HD}(G(\alpha(k)^-; 2, d, 1))$, then:

$$p(u^{-1},v^{-1}) = -\frac{(uv)^{-g}(1+u)^g(1+v)^g}{(uv)^{-1}(uv-1)}(uv)^{1-g-d}g(u,v) =$$

$$= (uv)^{2-2g-d}\frac{(1+u)^g(1+v)^g}{(1-uv)}g(u,v) = (uv)^{-(2g+d-2)}p(u,v).$$

Now by [BGMN, propositions 3.3 and 3.4], we get that all the moduli spaces $G(\alpha; 2, d, 1)$ for α non-critical are smooth. Moreover, by [BGMN, lemma 3.5] their dimension coincides with the expected dimension

$$\beta(n,d,k) = n^2(g-1) + 1 - k(k-d+n(g-1)) =$$
$$= 4g - 4 + 1 - (1 - d + 2g - 2) = 2g + d - 2.$$

Therefore, the polynomials of all the moduli spaces $G(\alpha; 2, d, 1)$ for α non-critical satisfy Poincaré duality.

By combining theorem 13.3.1 with (13.5) (with j replaced by k), we get the following corollary, that will be useful in the computations for the cases $n = 3, k = 1$ and $n = 4, k = 1$

Corollary 13.3.2. *For every curve C as before and for every critical value*

$$\alpha(k) = d - 2k, \quad 0 \leq k < \frac{d}{2}$$

the Hodge-Deligne polynomial of

$$G^s(\alpha(k); 2, d, 1) \simeq G(\alpha(k)^+; 2, d, 1) \smallsetminus G^+(\alpha(k); 2, d, 1) \simeq$$
$$\simeq G(\alpha(k)^-; 2, d, 1) \smallsetminus G^-(\alpha(k); 2, d, 1)$$

is given by:

$$\mathcal{HD}(G^s(\alpha(k); 2, d, 1)) = \frac{(1+u)^g(1+v)^g}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} \cdot$$
$$\cdot \left[\frac{(uv)^k x^{-k}}{1 - x(uv)^{-1}} - \frac{(uv)^{g+d-1-2k} x^{-k}}{1 - x(uv)^2} - x^{-k} + (uv)^{g+d-2k-1} x^{-k} \right] =$$
$$= \frac{(1+u)^g(1+v)^g}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} \left[\frac{(uv)^k x^{-k}}{1 - x(uv)^{-1}} - \frac{(uv)^{g+d+1-2k} x^{1-k}}{1 - x(uv)^2} - x^{-k} \right].$$

Remark 13.3.4. Note that when $k = 0$, then $G^{\mathrm{s}}(d; 2, d, 1) = \varnothing$, so we will get that the previous polynomial is zero. This holds since (13.6) coincides with (13.5) for $j = 0$, as we already said before.

Remark 13.3.5. As in remark 13.3.1, this formula coincides with that given in [M, proposition 5.4] for the moduli spaces if stable triples, up a the multiplicative term $(1+u)^g(1+v)^g$, once we set in that formula $d_1 := d$ $d_2 := 0$, $\overline{d}_M := d - k$.

13.4 A more explicit formula for α small and d large

The formula of theorem 13.3.1 holds for every value of k and for every value of d, that is for all α that are non-critical for $(2, d, 1)$. We would like to have a more explicit formula, at least for some values of d and k. In order to do that, let us first state the following lemma, taken from [MOVG, proof of proposition 8.1].

Lemma 13.4.1. *Let $a, b, c \in \mathbb{C}$ be all distinct and different from zero and let $f(x)$ be any holomorphic function in $\mathbb{C} \setminus \{0\}$ such that the function*

$$g(x) := \frac{f(x)}{(1-ax)(1-bx)(1-cx)}$$

has no residue at ∞. Let us define

$$A := \frac{a}{(a-b)(a-c)}, \quad B := \frac{b}{(b-a)(b-c)}, \quad C := \frac{c}{(c-a)(c-b)}.$$

Then:

$$\operatorname*{coeff}_{x^0} \frac{xf(x)}{(1-ax)(1-bx)(1-cx)} = Af\left(\frac{1}{a}\right) + Bf\left(\frac{1}{b}\right) + Cf\left(\frac{1}{c}\right).$$

In particular, let us fix $a = 1$, $b = uv$, $c = (uv)^{-1}$ and $f(x) = (uv)^k(1+ux)^g(1+vx)^g x^{-k-1}$. Then we have to ensure that the function $g(x)$ has no residue at ∞. In order to do that, let us consider

$$\operatorname*{Res}_{x=\infty} g(x)dx = \operatorname*{Res}_{z=0} -\frac{g(z^{-1})}{z^2}dz =$$

$$= \operatorname*{Res}_{z=0} -\frac{(uv)^k}{z^2}\left(\frac{(z+u)^g}{z^g}\frac{(z+v)^g}{z^g}z^{k+1}\right)\left(\frac{(z-a)(z-b)(z-c)}{z^3}\right)^{-1} dz =$$

$$= \operatorname*{Res}_{z=0} \frac{-(uv)^k(z+u)^g(z+v)^g}{(z-a)(z-b)(z-c)} z^{k-2g+2}dz.$$

Now we are working with u, v as variables, therefore we can always find infinitely many values of them so that the first part of this expression has no zeros or poles at $z = 0$, so the zeros or poles at $z = 0$ are completely determined by z^{k-2g+2}. In particular, we don't have residues at ∞ if $k \geq 2g - 2$. Now we recall that $0 \leq k < d/2$. So in order to apply the lemma we must have:

$$d > 4g - 4 \quad \text{and} \quad 2g - 2 \leq k < d/2. \tag{13.9}$$

Remark 13.4.1. In terms of critical values, we have that $0 < \alpha(k) = d - 2k \leq d - 4g + 4 =: \bar{\alpha}$. If d is even, then $\bar{\alpha} \geq 2$, while if d is odd, then $\bar{\alpha} \geq 1$; we recall that for d even the smallest critical value is $\alpha = 2$, while for d odd the smallest critical value is $\alpha = 1$. Therefore, in both cases if $d > 4g - 4$ we have that our results will apply to a non-empty set of critical values containing the smallest critical one.

Now, having fixed f as before,

$$Af\left(\frac{1}{a}\right) = \frac{1}{(1-uv)(1-(uv)^{-1})}(uv)^k(1+u)^g(1+v)^g = \frac{-(uv)^{k+1}}{(1-uv)^2}(1+u)^g(1+v)^g;$$

$$Bf\left(\frac{1}{b}\right) = \frac{uv}{(uv-1)(uv-(uv)^{-1})}(uv)^k(1+v^{-1})^g(1+u^{-1})^g(uv)^{k+1} =$$

$$= \frac{(uv)^2}{(1-uv)(1-(uv)^2)}(uv)^{2k+1}\left(\frac{v+1}{v}\right)^g\left(\frac{u+1}{u}\right)^g = \frac{(uv)^{2k+3-g}}{(1-uv)(1-(uv)^2)}(1+u)^g(1+v)^g;$$

$$Cf\left(\frac{1}{c}\right) = \frac{(uv)^{-1}}{((uv)^{-1}-1)((uv)^{-1}-uv)}(uv)^k(1+u^2v)^g(1+uv^2)^g(uv)^{-k-1} =$$

$$= \frac{uv}{(1-uv)(1-(uv)^2)}(uv)^k(1+u^2v)^g(1+uv^2)^g(uv)^{-k-1} = \frac{(1+u^2v)^g(1+uv^2)^g}{(1-uv)(1-(uv)^2)}.$$

Therefore if conditions (13.9) are satisfied, then:

$$\operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} \cdot \frac{(uv)^k x^{-k}}{1-x(uv)^{-1}} = \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g(uv)^k x^{-k-1}x}{(1-x)(1-uvx)(1-x(uv)^{-1})} =$$

$$= \operatorname*{coeff}_{x^0} \frac{xf(x)}{(1-ax)(1-bx)(1-cx)} =$$

$$= \frac{-(uv)^{k+1}(1+u)^g(1+v)^g}{(1-uv)^2} + \frac{(uv)^{2k+3-g}(1+u)^g(1+v)^g + (1+u^2v)^g(1+uv^2)^g}{(1-uv)(1-(uv)^2)}.$$

Now let us fix $a=1$, $b=uv$, $c=(uv)^2$ and $f(x) = (uv)^{g+d-1-2k}(1+ux)^g(1+vx)^g x^{-k-1}$. Then:

$$Af\left(\frac{1}{a}\right) = \frac{(uv)^{g+d-1-2k}(1+u)^g(1+v)^g}{(1-uv)(1-(uv)^2)};$$

$$Bf\left(\frac{1}{b}\right) = \frac{uv}{(uv-1)(uv-(uv)^2)}(uv)^{g+d-1-2k}(1+v^{-1})^g(1+u^{-1})^g(uv)^{k+1} =$$

$$= \frac{-1}{(1-uv)^2}(uv)^{g+d-k}\left(\frac{v+1}{v}\right)^g\left(\frac{u+1}{u}\right)^g = \frac{-1}{(1-uv)^2}(uv)^{d-k}(1+u)^g(1+v)^g;$$

$$Cf\left(\frac{1}{c}\right) = \frac{(uv)^2}{((uv)^2-1)((uv)^2-uv)}(uv)^{g+d-1-2k}(1+(uv^2)^{-1})^g(1+(u^2v)^{-1})^g(uv)^{2k+2} =$$

$$= \frac{uv}{(1-uv)(1-(uv)^2)}(uv)^{g+d+1}\left(\frac{1+uv^2}{uv^2}\right)^g\left(\frac{1+u^2v}{u^2v}\right)^g =$$

$$= \frac{(uv)^{d+2-2g}}{(1-uv)(1-(uv)^2)}(1+uv^2)^g(1+u^2v)^g.$$

As before, in order to apply the lemma we need conditions (13.9) to be satisfied. If such conditions hold, then:

$$\operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} \cdot \frac{(uv)^{g+d-1-2k}x^{-k}}{1-(uv)^2x} =$$

$$= \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g(uv)^{g+d-1-2k}x^{-k-1}x}{(1-x)(1-uvx)(1-(uv)^2x)} = \operatorname*{coeff}_{x^0} \frac{xf(x)}{(1-ax)(1-bx)(1-cx)} =$$

$$= \frac{(uv)^{g+d-1-2k}(1+u)^g(1+v)^g + (uv)^{d+2-2g}(1+u^2v)^g(1+uv^2)^g}{(1-uv)(1-(uv)^2)} - \frac{(uv)^{d-k}(1+u)^g(1+v)^g}{(1-uv)^2}.$$

So by using theorem 13.3.1, we get the following formula.

Corollary 13.4.2. *If conditions (13.9) are satisfied, then*

$$\mathcal{HD}(G(\alpha(k)^-;2,d,1) = \frac{(1+u)^g(1+v)^g}{1-uv}\left\{\frac{-(uv)^{k+1}(1+u)^g(1+v)^g}{(1-uv)^2}+\right.$$

$$+\frac{(uv)^{2k+3-g}(1+u)^g(1+v)^g+(1+u^2v)^g(1+uv^2)^g}{(1-uv)(1-(uv)^2)}+$$

$$-\frac{(uv)^{g+d-1-2k}(1+u)^g(1+v)^g+(uv)^{d+2-2g}(1+u^2v)^g(1+uv^2)^g}{(1-uv)(1-(uv)^2)}+$$

$$\left.+\frac{(uv)^{d-k}(1+u)^g(1+v)^g}{(1-uv)^2}\right\}=$$

$$=\frac{(1+u)^g(1+v)^g}{(1-uv)^2}\left\{\frac{[(uv)^{d-k}-(uv)^{k+1}](1+u)^g(1+v)^g}{1-uv}+\right.$$

$$+\frac{[(uv)^{2k+3-g}-(uv)^{g+d-1-2k}](1+u)^g(1+v)^g}{1-(uv)^2}+$$

$$\left.+\frac{[1-(uv)^{d+2-2g}](1+u^2v)^g(1+uv^2)^g}{1-(uv)^2}\right\}=$$

$$=\frac{(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\left\{\left[(1+uv)((uv)^{d-k}-(uv)^{k+1})+(uv)^{2k+3-g}+\right.\right.$$

$$\left.\left.-(uv)^{g+d-1-2k}\right]\cdot(1+u)^g(1+v)^g+[1-(uv)^{d+2-2g}](1+u^2v)^g(1+uv^2)^g\right\}.$$

Remark 13.4.2. Let us verify that Poincaré duality holds also for such a polynomial. This amounts to substituting u with u^{-1} and v with v^{-1} in the previous formula and verifying that identity (8.3) holds. If we write for simplicity p for the previous polynomial, then:

$$p(u^{-1},v^{-1})=\frac{(1+u)^g(1+v)^g(uv)^{-g}}{(1-uv)^2((uv)^2-1)(uv)^{-4}}\left\{\left[\frac{1+uv}{uv}\cdot\frac{(uv)^{k+1}-(uv)^{d-k}}{(uv)^{d+1}}+\right.\right.$$

$$\left.+\frac{(uv)^{g+d-1-2k}-(uv)^{2k+3-g}}{(uv)^{d+2}}\right]\cdot\frac{(1+u)^g(1+v)^g}{(uv)^g}+$$

$$\left.+\left[\frac{(uv)^{d+2-2g}-1}{(uv)^{d+2-2g}}\right]\cdot\frac{(1+u^2v)^g(1+uv^2)^g}{(uv)^{3g}}\right\}=$$

$$=-\frac{1}{(uv)^{d+2+g}}\cdot\frac{(1+u)^g(1+v)^g(uv)^{4-g}}{(1-uv)^2((uv)^2-1)}\left\{\left[(1+uv)((uv)^{d-k}-(uv)^{k+1})+\right.\right.$$

$$\left.+(uv)^{2k+3-g}-(uv)^{g+d-1-2k}\right](1+u)^g(1+v)^g+$$

$$\left.+\left[1-(uv)^{d+2-2g}\right](1+u^2v)^g(1+uv^2)^g\right\}=(uv)^{-(2g+d-2)}p(u,v).$$

As we already stated before, the dimension of $G(\alpha;2,d,1)$ for α non-critical is $2g+d-2$, so the polynomial written in the previous corollary satisfies Poincaré duality.

Using remark 13.4.1, provided that d is big enough, we can apply the corollary for the first critical value, namely $\alpha(\lfloor (d-1)/2 \rfloor)$, i.e. $\alpha = 2$ if d even and $\alpha = 1$ if d is odd.

If d is odd, then $k = \frac{d-1}{2}$. In this case, $d-k = \frac{d+1}{2} = k+1$; moreover, $2k+3-g = d+2-g$ and $g+d-1-2k = g$. Therefore we get:

Corollary 13.4.3. *Let C be any curve as before; if d is odd and $d > 4g-4$, then*

$$\mathcal{HD}(G_0(2,d,1)) = \frac{(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \left\{ \left[(uv)^{d+2-g} - (uv)^g \right] \cdot \right.$$
$$\left. \cdot (1+u)^g(1+v)^g + [1-(uv)^{d+2-2g}](1+u^2v)^g(1+uv^2)^g \right\}.$$

Remark 13.4.3. We can rewrite this polynomial as

$$\frac{(1+u)^g(1+v)^g(1-(uv)^{d+2-2g})}{(1-uv)^2(1-(uv)^2)} \cdot \{[(1+u^2v)^g(1+uv^2)^g - -(uv)^g(1+u)^g(1+v)^g]\} =$$

$$= \mathcal{HD}(M(2,d)) \cdot \frac{1-(uv)^{d+2-2g}}{1-uv} = \mathcal{HD}(M(2,d)) \cdot \mathcal{HD}(\mathbb{P}^{d+1-2g}).$$

So this agrees with the known fact that if d is odd and $d > n(2g-2) = 4g-4$, then $G_0(2,d,1)$ is a grassmannian fibration over the moduli space $M(2,d)$ of stable rank 2 bundles of degree d, with fiber over any vector bundle E given by

$$\mathrm{Grass}(1,\chi(E)) = \mathrm{Grass}(1,H^0(E)) = \mathrm{Grass}(1,d+2(1-g)) = \mathbb{P}^{d+1-2g}.$$

Here the first identity comes from the fact that $H^1(E) = 0$ for $d > 4g-4$, while the second one is Riemann-Roch. For the Hodge-Deligne polynomial of $M(2,d)$, see (8.10).

If d is even, then $k = \frac{d}{2} - 1$; so $d-k = \frac{d}{2}+1$, $k+1 = \frac{d}{2}$, $2k+3-g = d+1-g$ and $g+d-1-2k = g+1$, so we get:

Corollary 13.4.4. *Let C be any curve as before; if d is even and $d > 4g-4$, then*

$$\mathcal{HD}(G_0(2,d,1)) = \frac{(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \left\{ [1-(uv)^{d+2-2g}](1+u^2v)^g(1+uv^2)^g + \right.$$
$$\left. +[(1+uv)(uv)^{\frac{d}{2}}(uv-1) + (uv)^{d+1-g} - (uv)^{g+1}](1+u)^g(1+v)^g \right\} =$$
$$= \frac{(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \left\{ [1-(uv)^{d+2-2g}](1+u^2v)^g(1+uv^2)^g + \right.$$
$$\left. -[(uv)^{\frac{d}{2}}(1-(uv)^2) + (uv)^{g+1}(1-(uv)^{d-2g})](1+u)^g(1+v)^g \right\}.$$

We remark that the leading term coincides with the leading term of (8.11) times $\mathcal{HD}(\mathbb{P}^{d+1-2g})$.

13.5 Comparison with the cases $g = 0$ and $g = 1$

Let us compare the previous results (that a priori are valid only under the hypothesis $g \geq 2$) with the literature about $g = 0$ and $g = 1$. When $g = 0$, the Hodge-Deligne polynomials of the moduli spaces $G(\alpha; n, d, 1)$ for all n and for all α non-critical for $(n, d, 1)$ were computed in [LN]. In that paper the following notation is used:

$$d = na - t \quad \text{ s.t. } a \in \mathbb{N}, \quad 0 \leq t < n$$

(this is formula (1) in that paper). According to that notation, G_e denotes any moduli space $G(\alpha; n, d, 1)$ for α in the range $en + t < \alpha < (e + 1)n + t$ whenever $0 \leq en + t < d/(n - 1)$. In our case, $n = 2$, so:

- t is 0 or 1 according to the parity of d;
- e is any natural number such that $0 \leq 2e + t < d$;
- $a = \lceil (d + t)/2 \rceil$; therefore both for d even and for d odd, we get $a = (d + t)/2$.

In order to compare the results of [LN] with ours, we want to put $G_e = G(\alpha(k)^-; 2, d, 1)$ for a suitable value of k. For this we need

$$2(e + 1) + t = \alpha(k) = d - 2k,$$

so

$$e = \frac{d - t}{2} - k - 1. \tag{13.10}$$

Since t has the same parity of d, then e is an integer. Therefore, under the identification of (13.10),

$$G_e = G((e + 1)n + t - \varepsilon; 2, d, 1) = G(\alpha(k)^-; 2, d, 1).$$

Now according to [LN, proposition 6.8], whenever $g = 0$ we have

$$\mathcal{HD}(G_e) = \frac{(1 - (uv)^{a-t-e})(1 - (uv)^{a-t-e+1})(1 - (uv)^{2e+t+1})}{(1 - uv)^2(1 - (uv)^2)}.$$

Therefore, if we set $a = (d + t)/2$ and $e = (d - t)/2 - k - 1$, then we can write

$$a - t - e = \frac{d + t}{2} - t - \frac{d - t}{2} + k + 1 = k + 1, \quad a - t - e + 1 = k + 2,$$
$$2e + t + 1 = d - t - 2 - 2k + t + 1 = d - 1 - 2k$$

and

$$\mathcal{HD}(G(\alpha(k)^-; 2, d, 1) = \frac{(1 - (uv)^{k+1})(1 - (uv)^{k+2})(1 - (uv)^{d-1-2k})}{(1 - uv)^2(1 - (uv)^2)} =$$

$$= \frac{(1-(uv)^{k+2}-(uv)^{k+1}+(uv)^{2k+3})(1-(uv)^{d-1-2k})}{(1-uv)^2(1-(uv)^2)} =$$

$$= \frac{1}{(1-uv)^2(1-(uv)^2)}\Big\{1-(uv)^{d-1-2k}-(uv)^{k+2}+(uv)^{d+1-k}+$$

$$-(uv)^{k+1}+(uv)^{d-k}+(uv)^{2k+3}-(uv)^{d+2}\Big\}. \qquad (13.11)$$

Now let us consider the formula of corollary 13.4.2 with $g = 0$. This gives

$$\frac{(1+uv)((uv)^{d-k}-(uv)^{k+1})+(uv)^{2k+3}-(uv)^{d-1-2k}+1-(uv)^{d+2}}{(1-uv)^2(1-(uv)^2)} =$$

$$= \frac{1}{(1-uv)^2(1-(uv)^2)}\Big\{(uv)^{d-k}-(uv)^{k+1}+(uv)^{d-k+1}-(uv)^{k+2}+$$

$$+(uv)^{2k+3}-(uv)^{d-1-2k}+1-(uv)^{d+2}\Big\},$$

which coincides with (13.11).

Let us also consider the case $g = 1$. According to [LN2, theorem 6.7], we have the following formula:

$$\mathcal{HD}(G_i) = (1+u)(1+v)\frac{1-(uv)^d}{1-uv} + \frac{(1+u)^2(1+v)^2(1-(uv)^{(d-\gamma)/2-i})}{(1-(uv)^2(1-(uv)^2)}\cdot$$

$$\cdot(uv-(uv)^{\gamma+2i})(1-(uv)^{(d-\gamma)/2-i+1})$$

where:

- γ is 1 if d is odd and 2 if d is even;
- $G_i = G_i(2+ad, d, 1) = G(\alpha_{i+1}-\varepsilon; 2+ad, d, 1)$;
- a is any non-negative integer; in particular we will be interested in $a = 0$;
- if $a = 0$, by looking at the proof of [LN2, lemma 6.1], the critical values are of the form $\alpha_i = d - 2d_1$, with $d_1 = \lfloor (d-1)/2 \rfloor + 1 - i$, so by substituting we get that $\alpha_i = 2i - 1$ if d is odd, respectively $\alpha_i = 2i$ if d is even. Then α_{i+1} is equal to $2i+1$, respectively $2i+2$. So in both cases $\alpha_{i+1} = 2i + \gamma$ and we set:

$$2i+\gamma = \alpha_{i+1} = \alpha(k) = d-2k \quad \Leftrightarrow \quad i = \frac{d-\gamma}{2} - k.$$

Therefore we have that if $g = 1$ and we set $i := (d-\gamma)/2 - k$, then

$$G_i = G(2i+\gamma-\varepsilon; 2, d, 1) = G(\alpha(k)^-; 2, d, 1).$$

So

$$\mathcal{HD}(G(\alpha(k)^-; 2, d, 1)) = \mathcal{HD}(G_i) =$$

$$= (1+u)(1+v)\frac{1-(uv)^d}{1-uv} + \frac{(1+u)^2(1+v)^2(1-(uv)^k)}{(1-uv)^2(1-(uv)^2)} \cdot (uv-(uv)^{d-2k})(1-(uv)^{k+1}) =$$

$$= \frac{(1+u)(1+v)}{(1-uv)^2(1-(uv)^2)} \Big\{(1-(uv)^d)(1-uv-(uv)^2+(uv)^3)+$$

$$+(1+u+v+uv)\cdot[(uv-(uv)^{d-2k}-(uv)^{k+1}+(uv)^{d-k}]\cdot[1-(uv)^{k+1}]\Big\} =$$

$$= \frac{(1+u)(1+v)}{(1-uv)^2(1-(uv)^2)} \Big\{1-uv-(uv)^2+(uv)^3-(uv)^d+(uv)^{d+1}+(uv)^{d+2}-(uv)^{d+3}+$$

$$+(1+u+v+uv)[uv-(uv)^{k+2}-(uv)^{d-2k}+(uv)^{d-k+1}-(uv)^{k+1}+(uv)^{2k+2}+$$

$$+(uv)^{d-k}-(uv)^{d+1}]\Big\} =$$

$$= \frac{(1+u)(1+v)}{(1-uv)^2(1-(uv)^2)} \Big\{1-uv-(uv)^2+(uv)^3-(uv)^d+(uv)^{d+1}+(uv)^{d+2}-(uv)^{d+3}+$$

$$+uv-(uv)^{k+2}-(uv)^{d-2k}+(uv)^{d-k+1}-(uv)^{k+1}+(uv)^{2k+2}+(uv)^{d-k}-(uv)^{d+1}+(u+v)\cdot$$

$$\cdot[uv-(uv)^{k+2}-(uv)^{d-2k}+(uv)^{d-k+1}-(uv)^{k+1}+(uv)^{2k+2}+(uv)^{d-k}-(uv)^{d+1}]+$$

$$+(uv)^2-(uv)^{k+3}-(uv)^{d-2k+1}+(uv)^{d-k+2}-(uv)^{k+2}+$$

$$(uv)^{2k+3}+(uv)^{d-k+1}-(uv)^{d+2}\Big\} =$$

$$= \frac{(1+u)(1+v)}{(1-uv)^2(1-(uv)^2)} \Big\{1+(uv)^3-(uv)^d-(uv)^{d+3}-(uv)^{k+2}-(uv)^{d-2k}$$

$$+(uv)^{d-k+1}-(uv)^{k+1}+(uv)^{2k+2}+(uv)^{d-k}+(u+v)[uv-(uv)^{k+2}-(uv)^{d-2k}+$$

$$+(uv)^{d-k+1}-(uv)^{k+1}+(uv)^{2k+2}+(uv)^{d-k}-(uv)^{d+1}]-(uv)^{k+3}-(uv)^{d-2k+1}+$$

$$+(uv)^{d-k+2}-(uv)^{k+2}+(uv)^{2k+3}+(uv)^{d-k+1}\Big\}. \tag{13.12}$$

Now let us consider the formula of corollary 13.4.2 with $g=1$. This gives:

$$\frac{(1+u)(1+v)}{(1-uv)^2(1-(uv)^2)} \Big\{\Big[(1+uv)((uv)^{d-k}-(uv)^{k+1})+(uv)^{2k+2}-(uv)^{d-2k}\Big]\cdot$$

$$\cdot(1+u)(1+v)+[1-(uv)^d](1+u^2v)(1+uv^2)\Big\} =$$

$$= \frac{(1+u)(1+v)}{(1-uv)^2(1-(uv)^2)} \Big\{\Big[(uv)^{d-k}-(uv)^{k+1}+(uv)^{d-k+1}+$$

$$-(uv)^{k+2}+(uv)^{2k+2}-(uv)^{d-2k}\Big]\cdot(1+u+v+uv)+$$

$$+[1-(uv)^d](1+(u+v)uv+(uv)^3)\Big\} =$$

$$= \frac{(1+u)(1+v)}{(1-uv)^2(1-(uv)^2)} \Big\{(uv)^{d-k}-(uv)^{k+1}+(uv)^{d-k+1}-(uv)^{k+2}+$$

$$+(uv)^{2k+2}-(uv)^{d-2k}+(u+v)[(uv)^{d-k}-(uv)^{k+1}+(uv)^{d-k+1}-(uv)^{k+2}+$$

$$+(uv)^{2k+2} - (uv)^{d-2k}] + (uv)^{d-k+1} - (uv)^{k+2} + (uv)^{d-k+2} - (uv)^{k+3} + (uv)^{2k+3} +$$
$$-(uv)^{d-2k+1} + 1 - (uv)^d + (u+v)[uv - (uv)^{d+1}] + (uv)^3 - (uv)^{d+3}\Big\} .$$

Then a direct check proves that this expression coincides with (13.12).

We observe that if $g = 0, 1$, then conditions (13.9) are automatically satisfied for all positive integers d and for all integers k corresponding to actual critical values $\alpha(k)$ for $(2, d, 1)$. Therefore, we can restate corollary 13.4.2 as follows:

Corollary 13.5.1. *For every genus $g \geq 0$ and for every pair d, k such that*

$$d > 4g - 4 \quad \textit{and} \quad 2g - 2 \leq k < d/2,$$

the Hodge-Deligne polynomial of $G(\alpha(k)^-; 2, d, 1)$ is given by

$$\mathcal{HD}(G(\alpha(k)^-; 2, d, 1) = \frac{(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \cdot$$
$$\cdot \Big\{ \Big[(1+uv)((uv)^{d-k} - (uv)^{k+1}) + (uv)^{2k+3-g} - (uv)^{g+d-1-2k}\Big] \cdot$$
$$\cdot (1+u)^g(1+v)^g + [1 - (uv)^{d+2-2g}](1+u^2v)^g(1+uv^2)^g \Big\} .$$

Chapter 14

Case n=3, k=1

First of all, let us compute the critical values for the triple $(3, d, 1)$. By [BGMN, §2 and proposition 4.2], the non-zero virtual critical values are all in the set

$$\left\{\frac{nd' - n'd}{n'k - nk'} \text{ s.t. } 0 \leq k' \leq k, \quad 0 < n' < n, \quad n'k \neq nk', \quad d' \in \mathbb{Z}\right\} \cap \left]0, \frac{d}{n-k}\right[.$$

In our case, this gives

$$\left\{\frac{3d' - n'd}{n' - 3k'} \text{ s.t. } k' = 0, 1, \quad n' = 1, 2, \quad n' \neq 3k' \quad d' \in \mathbb{Z}\right\} \cap \left]0, \frac{d}{2}\right[.$$

So we have the following 4 types of non-zero critical values:

- if $n' = 1, k' = 0$, $\{3d' - d \text{ s.t. } \frac{d}{3} < d' < \frac{d}{2}\}$;
- if $n' = 1, k' = 1$, $\{\frac{3d'-d}{1-3}\} \cap]0, \frac{d}{2}[= \{\frac{d}{2} - \frac{3}{2}d' \text{ s.t. } \quad 0 < d' < \frac{d}{3}\}$;
- if $n' = 2, k' = 0$, $\{\frac{3d'-2d}{2}\} \cap]0, \frac{d}{2}[= \{\frac{3}{2}d' - d \text{ s.t. } \frac{2d}{3} < d' < d\}$;
- if $n' = 2, k' = 1$, $\{\frac{3d'-2d}{2-3}\} \cap]0, \frac{d}{2}[= \{2d - 3d' \text{s.t. } \frac{d}{2} < d' < \frac{2d}{3}\}$.

Now it is easy to see that the first set coincides with the last one, so we write both as

$$\left\{2d - 3k \text{ s.t. } \frac{d}{2} < k < \frac{2d}{3}\right\}; \tag{14.1}$$

moreover, the second and the third set coincide, so we write both as:

$$\left\{\alpha(j) := \frac{d - 3j}{2} \text{ s.t. } \quad 0 < j < \frac{d}{3}\right\}. \tag{14.2}$$

Now:

$$2d - 3k = \frac{d}{2} - \frac{3}{2}j \Leftrightarrow j = 2k - d;$$

moreover, by setting $j := 2k - d$ and by using the conditions on k, we get exactly the conditions on j. Therefore, the set (14.1) is contained in the set (14.2). In general, it is strictly contained because we obtain only those values of j that are even or odd according to the parity of d. So from now all the (virtual) critical values will be labeled as in (14.2). Since we will also need to cross the value $d/2$, we will consider also $\alpha(0) = d/2$ as a critical value. We will see in the following that every value $\alpha(j)$ is an actual critical value.

14.1 The moduli spaces $G^+(\alpha(j); 3, d, 1)$

Let us fix any value

$$\alpha(j) = \frac{d-3j}{2}, \quad 0 \leq j < \frac{d}{3}$$

and let us consider any object $(E, V) \in G^+(\alpha(j); 3, d, 1)$ (if $j = 0$, we will obtain the empty set and the zero polynomial, so this will not give any problem for our computation). Since $n = 3$, then all the (E, V)'s have length r of the filtration equal to 2 or 3 (it cannot be equal to one, since this will imply that the coherent system is stable also at α_c). So let us consider the 2 different cases.

14.1.1 Case $r = 2$

By applying lemma 1.0.6, this gives a non-split exact sequence:

$$0 \to (Q_1, W_1) \to (E, V) \to (Q_2, W_2) \to 0 \tag{14.3}$$

with conditions (a)-(b). Then condition (a) implies that $k_1 = 0$, but a priori n_1 can be either equal to 1 or to 2.

(1) On the one hand, if $n_1 = 2$, then $n_2 = 1$; therefore, condition (b) implies that $d_1 = d - j$. Therefore, the previous conditions on j prove that both $d_1 = d - j$ and $d_2 = j$ are non-negative integers. Since $r = 2$, we must impose that both $(Q_1, W_1) = (Q_1, 0)$ and (Q_2, W_2) are $\alpha(j)$-stable. Since there are no critical values for $(2, d_1, 0)$ and $(1, d_2, 1)$, this simply means that we are considering all pairs $(Q_1, 0), (Q_2, W_2)$ such that:

$$(Q_1, W_1) \in G(2, d-j, 0) = M^s(2, d-j) := G_1, \quad (Q_2, W_2) \in G(1, j, 1) := G_2.$$

Since $\mathbb{H}^0_{21} = \mathbb{H}^2_{21} = 0$, by proposition 1.0.7 we get

$$\begin{gathered}\dim \operatorname{Ext}^1((Q_2, W_2), (Q_1, 0)) = C_{21} = \\ = n_1 n_2 (g-1) - d_1 n_2 + d_2 n_1 + k_2 d_1 - k_2 n_1 (g-1) - k_1 k_2 = \\ = 2(g-1) - (d-j) + 2j + (d-j) - 2(g-1) = 2j.\end{gathered}$$

Moreover, we can apply proposition 5.0.5 for $r = 2$: both G_1 and G_2 are smooth, so all the connected components of $G_1 \times G_2$ are irreducible. If $d - j$ is odd we can work at the moduli space level, otherwise we need to work at the Quot scheme level since there are no universal families on $M^s(2,d-j)$ for $d-j$ even. In both cases, we get that for every critical value $\alpha(j)$ we have a contribution to $G^+(\alpha(j);3,d,1)$ by a projective bundle over $G_1 \times G_2$ with fibers isomorphic to $\mathbb{P}^{2j-1}$. So we get the polynomial:

$$p_1^j := \mathcal{HD}(M^s(2,d-j))\frac{1-(uv)^{2j}}{1-uv}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}.$$

If $j=0$, this is the zero polynomial, as it should be.

For the Hodge-Deligne polynomials of $M^s(2,d-j)$ for $d-j$ even and odd, see chapter 8; we recall that such polynomials depend only on the parity of $d-j$. We denote by $p_1^{j\equiv_2 d}$ and $p_1^{j\not\equiv_2 d}$ the corresponding polynomials.

(2) On the other hand, if $n_1=1$, then $n_2=2$. Moreover, condition (b) implies that $d_1=(d-j)/2$, so *this case is possible only if $d-j$ is even.* Then $d_2=d-d_1=(d+j)/2$ and both d_1 and d_2 are non-negative integers. We remark that the space $G(\alpha(j);2,d_2,1)$ is not empty if and only if $\alpha(j)<d_2$, but this condition is automatically satisfied by definition of $\alpha(j)$.

Then we have to verify if $\alpha(j)$ is critical for $(2,d_2,1)$. According to chapter 13, this holds if and only if $\alpha(j)=d_2-2k$ for some $0\le k<d_2/2$. So this gives:

$$\frac{d-3j}{2}=\alpha(j)=d_2-2k=\frac{d+j}{2}-2k \quad\Leftrightarrow\quad j=k.$$

So $\alpha(j)$ is critical for $(2,d_2,1)$ if and only if $0\le j<d_2/2=(d+j)/4$. The second inequality holds if and only if $j<d/3$, which is exactly the condition we already put on j. Therefore, for every admissible value of j (i.e. $0\le j\le d/3$) such that $d-j$ is even, we have that $\alpha(j)$ is critical for $(2,d_2,1)$.

Since we are assuming that the Jordan-Hölder filtration of (E,V) at α_c has length 2, we need to consider only those (Q_2,W_2)'s that are strictly $\alpha(j)$-stable (if they are only semistable, then the filtration would have length 3), so we have to parametrize classes of non-split extensions with $(Q_1,0)\in J^{(d-j)/2}C =: G_1$ and

$$(Q_2,W_2)\in G(\alpha(j)^+;2,d_2,1)\smallsetminus G^+(\alpha(j);2,d_2,1)=G^{\mathrm{s}}(\alpha(j);2,d_2,1):=G_2.$$

Since $\mathbb{H}^0_{21}=\mathbb{H}^2_{21}=0$, we get:

$$\begin{aligned}\dim \operatorname{Ext}^1((Q_2,W_2),(Q_1,0)) = C_{21} = \\ = n_1n_2(g-1)-d_1n_2+d_2n_1+k_2d_1-k_2n_1(g-1)-k_1k_2 = \\ = 2(g-1)-2\tfrac{d-j}{2}+\tfrac{d+j}{2}+\tfrac{d-j}{2}-(g-1)=g-1+j.\end{aligned} \tag{14.4}$$

By proceeding as before, for every critical value $\alpha(j)$ such that $d-j$ is even, we get a contribution to $G^+(\alpha(j);3,d,1)$ by a projective bundle over $G_1\times G_2$ with fibers isomorphic to $\mathbb{P}^{g-2+j}$. For simplicity, we write

$$r_j := \mathcal{HD}(G_2) = \mathcal{HD}\left(G^{\mathrm{s}}\left(\frac{d-3j}{2};2,\frac{d+j}{2},1\right)\right). \tag{14.5}$$

So this case gives a contribution of the form:

$$p_2^{j\equiv_2 d} := r_j \cdot (1+u)^g(1+v)^g\frac{1-(uv)^{g-1+j}}{1-uv}.$$

We will see in section 3 that if $j=0$, then r_j is equal to zero, so p_2 is the zero polynomial, as it should be.

14.1.2 Case $r = 3$

In this case the graded of (E,V) is necessarily made of 3 objects of the form $(Q_1,0)$, $(Q_2,0)$, (Q_3,W_3) (a priori not necessarily in this order), where all the Q_i's are line bundles. The $\alpha(j)$-slopes of these 3 objects must be equal to the $\alpha(j)$-slope of (E,V), therefore we get

$$d_1 = d_2 = \frac{d+\alpha(j)}{3} \quad \Rightarrow \quad d_1 = d_2 = \frac{d-j}{2}$$

and

$$d_3 = d - 2d_1 = j.$$

Therefore *the case $r=3$ is possible only if $d-j$ is even.* So in this case:

$$(Q_1,0),(Q_2,0) \in J^{(d-j)/2}C = G_1 = G_2, \quad (Q_3,W_3) \in G(1,j,1) = G_3.$$

Now the possible $\alpha(j)$-canonical filtrations that we have to take into account are the following.

(1) If the length of the $\alpha(j)$-canonical filtration is $s=3=r$, then the $\alpha(j)$-Jordan-Hölder filtration is unique and we have to parametrize objects (E,V) that sit in non-split exact sequences of the form:

$$0 \to (E',0) \to (E,V) \to (Q_3,W_3) \to 0$$

where E' sits in a nontrivial extension of 2 line bundles Q_1, Q_2 of the same degree. Here the object (Q_3,W_3) must be the last object of the graded; if not, this would imply that we have a quotient of the form $(E,V) \to (Q_i,0)$ (for $i=1$ or $i=2$), but this contradicts the $\alpha(j)^+$-stability of (E,V). Now

$$\mathrm{Hom}\left((Q_3,W_3),(Q_2,0)\right) = 0$$

because of lemma 1.0.4. Then we have to consider two subcases as follows.

(1a) If we suppose that $(Q_1,0) \not\simeq (Q_2,0)$, then we can apply proposition 6.1.2 in order to parametrize all the corresponding (E,V)'s. In this case $\mathrm{Hom}((Q_2,0),(Q_1,0)) = 0$, so the invariant a can only assume the value

$$a = \dim \operatorname{Ext}^1((Q_2, 0), (Q_1, 0)) = C_{21} = n_1 n_2 (g-1) - d_1 n_2 + d_2 n_1 = g - 1.$$

on the set $U_a = G_1 \times G_2 \smallsetminus \Delta_{12}$. So we will get a projective bundle R_a over U_a with fibers isomorphic to $\mathbb{P}^{a-1} = \mathbb{P}^{g-2}$. If we write $E_2 = (E_2, 0)$ for any extension of Q_2 by Q_1, we get that $N_2 = 2$ and $D_2 = 2d_1 = d - j$. Moreover, $\operatorname{Ext}^2((Q_3, W_3), (E_2, 0)) = 0$ because $k_3 = 1$ and also $\operatorname{Hom}(-,-) = 0$; therefore we get that the invariant b can assume only the value:

$$b = \dim \operatorname{Ext}^1((Q_3, W_3), (E_2, 0)) = N_2 n_3 (g-1) - D_2 n_3 + d_3 N_2 + k_3 D_2 - k_3 N_2 (g-1) =$$

$$= 2(g-1) - D_2 + 2d_3 + D_2 - 2(g-1) = 2d_3 = 2j.$$

Moreover, the invariant c can only assume the value:

$$c = \dim \operatorname{Ext}^1((Q_3, W_3), (Q_1, 0)) = C_{31} = n_1 n_3 (g-1) - d_1 n_3 + d_3 n_1 + k_3 d_1 - k_3 n_1 (g-1) =$$

$$= (g-1) - d_1 + d_3 + d_1 - (g-1) = d_3 = j.$$

Therefore, we get that $U_{a,b,c} = P_a \times G_3$ and we get a bundle $R_{a,b,c}$ over $U_{a,b,c}$ with fibers isomorphic to $\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1} = \mathbb{P}^{2j-1} \smallsetminus \mathbb{P}^{j-1}$. The bundle $R_{a,b,c}$ parametrizes all the (E, V)'s in $G^+(\alpha(j); 3, d, 1)$ with unique $\alpha(j)$-Jordan-Hölder filtration and $Q_1 \not\simeq Q_2$. We recall that $G_1 = G_2 = J^{(d-j)/2}C$ and $G_3 = G(1, j, 1)$; then we get the Hodge-Deligne polynomial

$$p_3^{j \equiv_2 d} := \mathcal{HD}(R_{a,b,c}) = \Big(\mathcal{HD}(\mathbb{P}^{2j-1}) - \mathcal{HD}(\mathbb{P}^{j-1})\Big)\mathcal{HD}(G_3)\mathcal{HD}(R_a) =$$

$$= \frac{(uv)^j - (uv)^{2j}}{1 - uv} \cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g (1+vx)^g x^{-j}}{(1-x)(1-uvx)} \cdot \mathcal{HD}(\mathbb{P}^{g-2}) \cdot \mathcal{HD}(G_1 \times G_2 \smallsetminus \Delta_{12}) =$$

$$= \frac{(uv)^j - (uv)^{2j}}{1 - uv} \cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g (1+vx)^g x^{-j}}{(1-x)(1-uvx)} \cdot \frac{1 - (uv)^{g-1}}{1 - uv} \mathcal{HD}(G_1)(\mathcal{HD}(G_1) - 1) =$$

$$= \frac{(uv)^j - (uv)^{2j}}{(1 - uv)^2} \cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g (1+vx)^g x^{-j}}{(1-x)(1-uvx)} (1+u)^g (1+v)^g \cdot$$

$$\cdot (1 - (uv)^{g-1})((1+u)^g (1+v)^g - 1).$$

Also in this case, if $j = 0$, then we get the zero polynomial.

(1b) If we suppose that $(Q_1, 0) \simeq (Q_2, 0)$, then we can apply proposition 6.1.4 in order to parametrize all the corresponding (E, V)'s. In this case we need to compute the invariants:

$$a = \dim \operatorname{Ext}^1((Q_2, 0), (Q_1, 0)) = C_{21} + 1 = g$$

and

$$b = \dim \operatorname{Ext}^1((Q_3, W_3), (Q_1, 0)) = C_{31} = j.$$

Therefore, we get a projective bundle R_a over G_1 with fibers isomorphic to $\mathbb{P}^{g-1}$; the (E, V)'s we are interested in are parametrized by a bundle $R_{a,b}$ over $R_a \times G_3$ with fibers isomorphic to $\mathbb{C}^{j-1} \times \mathbb{P}^{j-1}$. Therefore, we get the polynomial:

$$p_4^{j \equiv_2 d} := \mathcal{HD}(G_1)\mathcal{HD}(G_3)\mathcal{HD}(\mathbb{C}^{j-1})\mathcal{HD}(\mathbb{P}^{j-1})\mathcal{HD}(\mathbb{P}^{g-1}) =$$

$$= (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}(uv)^{j-1}\frac{1-(uv)^j}{1-uv} \cdot \frac{1-(uv)^g}{1-uv} =$$

$$= \frac{(uv)^{j-1}-(uv)^{2j-1}}{(1-uv)^2}(1+u)^g(1+v)^g(1-(uv)^g) \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}.$$

If $j = 0$, then $p_4^{j \equiv_2 d} = 0$.

(2) If the length of the $\alpha(j)$-canonical filtration is $s = 2$, by using the same argument used before we get that the only possible $\alpha(j)$-canonical filtration of (E, V) is given by

$$0 \subset (Q_1, 0) \oplus (Q_2, 0) \subset (E, V)$$

with $(E, V)/((Q_1, 0) \oplus (Q_2, 0)) = (Q_3, W_3)$. We have to consider 2 subcases as follows.

(2a) Let us suppose that $Q_1 \not\simeq Q_2$; since Q_1 and Q_2 are of the same type, then we can use proposition 7.2.2 in order to have a global parametrization. In this case the invariant a can assume only the value

$$a = \dim \operatorname{Ext}^1((Q_3, W_3), (Q_1, 0)) = C_{31} =$$

$$= n_1 n_3(g-1) - d_1 n_3 + d_3 n_1 + k_3 d_1 - k_3 n_1(g-1) = d_3 = j$$

and analogously also b can only assume the value $b = j$. Therefore the scheme U_a^1 coincides with $G_1 \times G_3$ and $U_b^2 = G_2 \times G_3$. Since all the G_i's are irreducible, then we get that $U_{a,b;i,j} = G_1 \times G_2 \times G_3$. So the only significant case in proposition 7.2.2 is case (d). Using the last part of that proposition, we get that the (E, V)'s we are interested in are parametrized by a scheme $M/\mathbb{Z}_2$ and from the point of view of Hodge-Deligne polynomials we can assume that M is the scheme

$$(G_1 \times G_2 \smallsetminus \Delta_{12}) \times G_3 \times \mathbb{P}^{j-1} \times \mathbb{P}^{j-1},$$

where $\mathbb{Z}_2$ acts by:

$$\Big(Q_1, Q_2, (Q_3, W_3), \mu_1, \mu_2\Big) \mapsto \Big(Q_2, Q_1, (Q_3, W_3), \mu_2, \mu_1\Big).$$

Let us write $M' = G_1 \times \mathbb{P}^{j-1} = J^{(d-j)/2}C \times \mathbb{P}^{j-1}$; then

$$\mathcal{HD}(M')(u, v) = \frac{1-(uv)^j}{1-uv}(1+u)^g(1+v)^g.$$

Therefore we can compute:

$$A := \mathcal{HD}\Big((M'\times M')/\mathbb{Z}_2\Big)(u,v)=$$

$$=\frac{1}{2}\Big((\mathcal{HD}(M')(u,v))^2+\mathcal{HD}(M')(-u^2,-v^2)\Big)=$$

$$=\frac{1}{2}\left(\frac{(1-(uv)^j)^2}{(1-uv)^2}(1+u)^{2g}(1+v)^{2g}+\frac{1-(uv)^{2j}}{1-(uv)^2}(1-u^2)^g(1-v^2)^g\right)$$

and

$$B := \mathcal{HD}\Big((\Delta_{12}\times\mathbb{P}^{j-1}\times\mathbb{P}^{j-1})/\mathbb{Z}_2\Big)=\mathcal{HD}(\Delta_{12})\cdot\mathcal{HD}\Big((\mathbb{P}^{j-1}\times\mathbb{P}^{j-1})/\mathbb{Z}_2\Big)=$$

$$=\frac{1}{2}(1+u)^g(1+v)^g\left(\frac{(1-(uv)^j)^2}{(1-uv)^2}+\frac{1-(uv)^{2j}}{1-(uv)^2}\right).$$

Finally, we can compute:

$$p_5^{j\equiv_2 d} := \mathcal{HD}(M/\mathbb{Z}_2)=\mathcal{HD}(G_3)\cdot(A-B)=\frac{1}{2}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}\cdot$$

$$\cdot\left[\frac{(1-(uv)^j)^2}{(1-uv)^2}(1+u)^g(1+v)^g\Big((1+u)^g(1+v)^g-1\Big)+\right.$$

$$\left.+\frac{1-(uv)^{2j}}{1-(uv)^2}\Big((1-u^2)^g(1-v^2)^g-(1+u)^g(1+v)^g\Big)\right].$$

Also in this case, if $j=0$, we get $p_5^{j\equiv_2 d}=0$.

(2b) If $Q_1\simeq Q_2$, then the corresponding (E,V)'s are parametrized using proposition 7.2.3. Also in this case, there is only one value for the invariant a, namely $a=j$ as in (2a). Moreover, there is a single index i, that therefore we ignore. So the (E,V)'s we are interested in are parametrized by a grassmannian $\mathrm{Grass}(2,R_a)$ where R_a is a vector bundle over $U_a=G_1\times G_3=J^{(d-j)/2}C\times C^{(j)}$ with fibers isomorphic to $\mathbb{C}^j$. So we get:

$$p_6^{j\equiv_2 d} := \mathcal{HD}(\mathrm{Grass}(2,R_a))=\mathcal{HD}\Big(\mathrm{Grass}(2,j)\Big)\cdot\mathcal{HD}(J^{(d-j)/2}C)\cdot\mathcal{HD}(C^{(j)})=$$

$$=\frac{(1-(uv)^{j-1})(1-(uv)^j)}{(1-uv)(1-(uv)^2)}(1+u)^g(1+v)^g\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}.$$

Also in this case, we get that if $j=0$, then $p_6=0$.

By putting everything together, we get that if $0\le j<d/3$,

- if $d-j$ is odd, then $\mathcal{HD}(G^+(\alpha(j);3,d,1))=p_1^{j\not\equiv_2 d}$;

- if $d-j$ is even, then

$$\mathcal{HD}(G^+(\alpha(j);3,d,1)) = p_1^\bullet + p_2^\bullet + p_3^\bullet + p_4^\bullet + p_5^\bullet + p_6^\bullet.$$

where $\bullet$ stands for $j \equiv_2 d$.

Both expressions are actually zero if $j=0$.

14.2 The moduli spaces $G^-(\alpha(j);3,d,1)$

Also in this case the length r of the filtration of any $(E,V) \in G^-(\alpha(j);3,d,1)$ can only be equal to 2 or 3. So let us consider the 2 different cases.

14.2.1 Case $r=2$

In this case lemma 1.0.6 implies that necessarily (E,V) sits in a non-split exact sequence

$$0 \to (Q_1,W_1) \to (E,V) \to (Q_2,W_2) \to 0 \tag{14.6}$$

with $k_1=1$ and $k_2=0$.

(1) On the one hand, if $n_1=1$, then $n_2=2$; therefore, condition (b') implies that $d_2=d-j$. Therefore, the previous conditions on j prove that both $d_2=d-j$ and $d_1=j$ are non-negative integers. Since $r=2$, we must impose that both (Q_1,W_1) and $(Q_2,W_2)=(Q_2,0)$ are $\alpha(j)$-stable. Since there are no critical values for $(1,d_1,1)$ and $(2,d_2,0)$, this simply means that we are considering all pairs $(Q_1,W_1),(Q_2,0)$ such that:

$$(Q_1,W_1) \in G(\alpha(1,j,1) =: G_1, \quad (Q_2,0) \in G(2,d-j,0) = M^s(2,d-j) =: G_2.$$

As before, $\mathbb{H}^0_{21} = \mathbb{H}^2_{21} = 0$, so

$$\begin{aligned}\dim \operatorname{Ext}^1((Q_2,0),(Q_1,W_1)) = C_{21} = \\ = n_1n_2(g-1) - d_1n_2 + d_2n_1 = \\ = 2(g-1) - 2j + (d-j) = 2g-2+d-3j.\end{aligned}$$

Now we can apply proposition 5.0.5 for $r=2$. So for every critical value $\alpha(j)$ we get a contribution to $G^-(\alpha(j);3,d,1)$ by a projective bundle over $G_1 \times G_2$ with fibers isomorphic to $\mathbb{P}^{2g-3+d-3j}$. So we get the polynomial:

$$q_1^j := \mathcal{HD}(M^s(2,d-j))\frac{1-(uv)^{2g-2+d-3j}}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}.$$

According to the notation used in the previous section, we denote by $q_1^{j\not\equiv_2 d}$ and $q_1^{j\equiv_2 d}$ the polynomial q_1^j according to the parity of $d-j$.

(2) On the other hand, if $n_1=2$, then $n_2=1$. Moreover, condition (b) implies that $d_2=(d-j)/2$, so *this case is possible only if $d-j$ is even.* Then $d_1=d-d_2=(d+j)/2$ and both d_1 and d_2 are positive integers. As in the previous section, we get that the numerical conditions on j imply that the space $G(\alpha(j);2,(d+j)/2,1)$ is not empty; moreover, $\alpha(j)$ is a critical value for $(2,(d+j)/2,1)$. Since we are assuming that the Jordan-Hölder filtration of (E,V) at α_c has length 2, we need to consider only those (Q_i,W_i)'s that are strictly $\alpha(j)$-stable. Therefore, in this case we have to parametrize extensions with $(Q_2,0)\in J^{(d-j)/2}C=:G_2$ and

$$(Q_1,W_1)\in G(\alpha(j)^+;2,d_1,1)\smallsetminus G^+(\alpha(j);2,d_1,1)=G^{\mathrm{s}}(\alpha(j);2,d_1,1)=:G_2.$$

Since $\mathbb{H}^0_{21}=\mathbb{H}^2_{21}=0$, we get:

$$\begin{aligned}\dim \operatorname{Ext}^1((Q_2,0),(Q_1,W_1)) &= C_{21}=n_1n_2(g-1)-d_1n_2+d_2n_1=\\ &=2(g-1)-\tfrac{d+j}{2}+2\tfrac{d-j}{2}=2g-2+\tfrac{d-3j}{2}.\end{aligned}\tag{14.7}$$

So for every critical value $\alpha(j)$ such that $d-j$ is even, we get a contribution to $G^-(\alpha(j);3,d,1)$ by a projective bundle over $J^{(d-j)/2}C\times G^{\mathrm{s}}(\alpha(j)^+;2,d_1,1)$ with fibers isomorphic to $\mathbb{P}^{2g-3+(d-3j)/2}$. For simplicity, we use the polynomial r_j defined in (14.5). So this case gives a contribution of the form:

$$q_2^{j\equiv_2 d}:=r_j\cdot(1+u)^g(1+v)^g\frac{1-(uv)^{2g-2+(d-3j)/2}}{1-uv}.$$

14.2.2 Case $r=3$

In this case the graded of (E,V) is necessarily made of 3 objects of the form (Q_1,W_1), $(Q_2,0)$, $(Q_3,0)$ (a priori not necessarily in this order), where all the Q_i's are line bundles. The $\alpha(j)$-slopes of these 3 objects must be equal to the $\alpha(j)$-slope of (E,V), therefore we get

$$d_2=d_3=\frac{d+\alpha(j)}{3}\quad\Rightarrow\quad d_2=d_3=\frac{d-j}{2}$$

and

$$d_1=d-2d_2=j.$$

Therefore *the case $r=3$ is possible only if $d-j$ is even.* So in this case:

$$(Q_1,W_1)\in G(1,j,1)=:G_1,\quad (Q_2,0),(Q_3,0)\in J^{(d-j)/2}C=:G_2=G_3.$$

Now the possible $\alpha(j)$-canonical filtrations that we have to take into account are the following.

(1) If the length of the $\alpha(j)$-canonical filtration is $s = 3 = r$, then the $\alpha(j)$-Jordan-Hölder filtration is unique and we have to parametrize all the (E, V)'s that sit in non-split exact sequences of the form:

$$0 \to (Q_1, W_1) \to (E, V) \to (E'', 0) \to 0$$

where E'' sits in a nontrivial extension of 2 line bundles Q_2, Q_3 of the same degree. Here the object (Q_1, W_1) must be the first object of the graded; if not, this would contradict the $\alpha(j)^-$-stability of (E, V). Now

$$\text{Hom}\left((Q_2, 0), (Q_1, W_1)\right) = 0$$

by lemma 1.0.4. Then we have to consider two subcases as follows.

(1a) If we suppose that $(Q_2, 0) \not\simeq (Q_3, 0)$, then we can apply proposition 6.1.6 in order to parametrize all the corresponding (E, V)'s. In this case $\text{Hom}((Q_3, 0), (Q_2, 0)) = 0$, so the invariant a can only assume the value

$$a = \dim \text{Ext}^1((Q_3, 0), (Q_2, 0)) = C_{32} = n_2 n_3 (g-1) - d_2 n_3 + d_3 n_2 = g - 1$$

on the set $U_a = G_2 \times G_3 \smallsetminus \Delta_{23}$. In this case we will get a projective bundle R_a over U_a, with fibers isomorphic to $\mathbb{P}^{a-1}$, If we write $E'' = (E'', 0)$ for any extension of Q_3 by Q_2 we get that $N'' = 2$ and $D'' = 2d_2 = d - j$. Moreover, $\text{Ext}^2((E'', 0), (Q_1, W_1)) = 0$ because $K'' = 0$ and also $\text{Hom}(-, -) = 0$; therefore we get that the invariant b can assume only the value:

$$b = \dim \text{Ext}^1((E'', 0), (Q_1, W_1)) = n_1 N''(g-1) - d_1 N'' + d'' n_1 =$$
$$= 2(g-1) - 2j + d - j = 2g - 2 + d - 3j.$$

Moreover, the invariant c can only assume the value:

$$c = \dim \text{Ext}^1((Q_3, 0), (Q_1, W_1)) = C_{31} = n_1 n_3 (g-1) - d_1 n_3 + d_3 n_1 =$$
$$= g - 1 - j + \frac{d-j}{2} = g - 1 + \frac{d - 3j}{2}.$$

Therefore, we get that $U_{a,b,c} = G_1 \times R_a$; moreover, we get a bundle $R_{a,b,c}$ over $U_{a,b,c}$ with fibers isomorphic to $\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1} = \mathbb{P}^{2g-3+d-3j} \smallsetminus \mathbb{P}^{g-2+(d-3j)/2}$. We recall that $G_1 = G(1, j, 1)$ and $G_2 = G_3 = J^{(d-j)/2}$, so we get the Hodge-Deligne polynomial

$$q_3^{j \equiv_2 d} := \mathcal{HD}(R_{a,b,c}) = \Big(\mathcal{HD}(\mathbb{P}^{2g-3+d-3j}) - \mathcal{HD}(\mathbb{P}^{g-2+(d-3j)/2})\Big)\mathcal{HD}(G_1)\mathcal{HD}(R_a) =$$
$$= \frac{(uv)^{g-1+(d-3j)/2} - (uv)^{2g-2+d-3j}}{1 - uv} \cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g (1+vx)^g x^{-j}}{(1-x)(1-uvx)} \cdot$$
$$\cdot \mathcal{HD}(\mathbb{P}^{g-2}) \cdot \mathcal{HD}(G_2 \times G_3 \smallsetminus \Delta_{23}) =$$
$$= \frac{(uv)^{g-1+(d-3j)/2} - (uv)^{2g-2+d-3j}}{1 - uv} \cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g (1+vx)^g x^{-j}}{(1-x)(1-uvx)} \cdot$$

$$\cdot\frac{1-(uv)^{g-1}}{1-uv}\mathcal{HD}(G_2)(\mathcal{HD}(G_2)-1)=$$

$$=\frac{(uv)^{g-1+(d-3j)/2}-(uv)^{2g-2+d-3j}}{(1-uv)^2}\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}(1+u)^g(1+v)^g\cdot$$

$$\cdot(1-(uv)^{g-1})((1+u)^g(1+v)^g-1).$$

(1b) If we suppose that $(Q_2,0)\simeq(Q_3,0)$, then we can apply proposition 6.1.8 in order to parametrize all the corresponding (E,V)'s. In this case we need to compute the invariants:

$$a=\dim\ \mathrm{Ext}^1((Q_3,0),(Q_2,0))=C_{32}+1=g$$

and

$$b=\dim\ \mathrm{Ext}^1((Q_3,0),(Q_1,W_1))=C_{31}=g-1+\frac{d-3j}{2}.$$

Therefore, we get a projective bundle R_a over G_1 with fibers isomorphic to $\mathbb{P}^{g-1}$; the (E,V)'s we are interested in are parametrized by a bundle $R_{a,b}$ over $P_a\times G_3$ with fibers isomorphic to $\mathbb{C}^{g-2+(d-3j)/2}\times\mathbb{P}^{g-2+(d-3j)/2}$. Therefore, we get the polynomial:

$$q_4^{j\equiv_2 d}:=\mathcal{HD}(G_1)\mathcal{HD}(G_2)\mathcal{HD}(\mathbb{C}^{g-2+(d-3j)/2})\mathcal{HD}(\mathbb{P}^{g-2+(d-3j)/2})\mathcal{HD}(\mathbb{P}^{g-1})=$$

$$=(1+u)^g(1+v)^g\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}(uv)^{g-2+(d-3j)/2}\cdot$$

$$\cdot\frac{1-(uv)^{g-2+(d-3j)/2}}{1-uv}\cdot\frac{1-(uv)^g}{1-uv}=$$

$$=\frac{(uv)^{g-2+(d-3j)/2}-(uv)^{2g-3+d-3j}}{(1-uv)^2}(1+u)^g(1+v)^g(1-(uv)^g)\cdot$$

$$\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}.$$

(2) If the length of the $\alpha(j)$-canonical filtration is $s=2$, by using the same argument used before we get that the only possible canonical filtration of (E,V) is given by

$$0\subset(Q_1,W_1)\subset(E,V)$$

with $(E,V)/(Q_1,W_1)=(Q_2,0)\oplus(Q_3,0)$. We have to consider 2 subcases as follows.

(2a) Let us suppose that $Q_2\not\simeq Q_3$; since Q_2 and Q_3 are of the same type, then we can use proposition 7.1.2 in order to have a global parametrization. In this case the invariant a can assume only the value

$$a=\dim\ \mathrm{Ext}^1\left((Q_2,0),(Q_1,W_1)\right)=C_{21}=$$

$$=n_1n_2(g-1)-d_1n_2+d_2n_1=g-1-j+\frac{d-j}{2}=g-1+\frac{d-3j}{2}$$

and analogously also $b = g - 1 + (d - 3j)/2$. Therefore $U_a^2 = G_1 \times G_2$ and $U_b^3 = G_1 \times G_3$. Since both G_1 and $G_2 = G_3$ are irreducible, then we get that $U_{a,b;i,j} = G_1 \times G_2 \times G_3$. So the only significant case in proposition 7.1.2 is case (d). Using the last part of that proposition, we get that the (E, V)'s we are interested in are parametrized by a scheme $M/\mathbb{Z}_2$; from the point of view of Hodge-Deligne polynomials we can assume that M is the scheme

$$G_1 \times (G_2 \times G_3 \smallsetminus \Delta_{23}) \times \mathbb{P}^{g-2+(d-3j)/2} \times \mathbb{P}^{g-2+(d-3j)/2},$$

where $\mathbb{Z}_2$ acts by:

$$\Big((Q_1, W_1), Q_2, Q_3, \mu_2, \mu_3\Big) \mapsto \Big((Q_1, W_1), Q_3, Q_2, \mu_3, \mu_2\Big).$$

Let us write $M' = G_2 \times \mathbb{P}^{g-2+(d-3j)/2} = J^{(d-j)/2}C \times \mathbb{P}^{g-2+(d-3j)/2}$; then

$$\mathcal{HD}(M')(u,v) = \frac{1-(uv)^{g-1+(d-3j)/2}}{1-uv}(1+u)^g(1+v)^g.$$

Therefore we can compute:

$$A := \mathcal{HD}\Big((M' \times M')/\mathbb{Z}_2\Big)(u,v) = \frac{1}{2}\Big((\mathcal{HD}(M')(u,v))^2 + \mathcal{HD}(M')(-u^2,-v^2)\Big) =$$

$$= \frac{1}{2}\left(\frac{(1-(uv)^{g-1+(d-3j)/2})^2}{(1-uv)^2}(1+u)^{2g}(1+v)^{2g} + \right.$$

$$\left. + \frac{1-(uv)^{2g-2+d-3j}}{1-(uv)^2}(1-u^2)^g(1-v^2)^g\right)$$

and

$$B := \mathcal{HD}\Big((\Delta_{23} \times \mathbb{P}^{g-2+(d-3j)/2} \times \mathbb{P}^{g-2+(d-3j)/2})/\mathbb{Z}_2\Big) =$$

$$= \mathcal{HD}(\Delta_{23}) \cdot \mathcal{HD}\Big((\mathbb{P}^{g-2+(d-3j)/2} \times \mathbb{P}^{g-2+(d-3j)/2})/\mathbb{Z}_2\Big) =$$

$$= \frac{1}{2}(1+u)^g(1+v)^g\left(\frac{(1-(uv)^{g-1+(d-3j)/2})^2}{(1-uv)^2} + \frac{1-(uv)^{2g-2+d-3j}}{1-(uv)^2}\right).$$

Finally, we can compute:

$$q_5^{j \equiv_2 d} := \mathcal{HD}(M/\mathbb{Z}_2) = \mathcal{HD}(G_1) \cdot (A - B) = \frac{1}{2} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)} \cdot$$

$$\cdot \left[\frac{(1-(uv)^{g-1+(d-3j)/2})^2}{(1-uv)^2}(1+u)^g(1+v)^g\Big((1+u)^g(1+v)^g - 1\Big) + \right.$$

$$\left. + \frac{1-(uv)^{2g-2+d-3j}}{1-(uv)^2}\Big((1+u^2)^g(1+v^2)^g - (1+u)^g(1+v)^g\Big)\right].$$

(2b) If $Q_2 \simeq Q_3$, then the corresponding (E, V)'s are parametrized using proposition 7.1.3. Also in this case, there is only one value for the invariant a, namely $a = g - 1 + (d - 3j)/2$ as in (2a). Moreover, there is a single index i, that therefore we ignore. So the (E,V)'s we are interested in are parametrized by a grassmannian $\mathrm{Grass}(2,R_a)$ where R_a is a vector bundle over $G_1\times G_2=C^{(j)}\times J^{(d-j)/2}C$ with fibers isomorphic to $\mathbb{C}^{g-1+(d-3j)/2}$. So we get:

$$q_6^{j\equiv_2 d}:=\mathcal{HD}(\mathrm{Grass}(2,R_a))=$$
$$=\mathcal{HD}\Big(\mathrm{Grass}(2,g-1+(d-3j)/2)\Big)\cdot\mathcal{HD}(J^{(d-j)/2}C)\cdot\mathcal{HD}(C^{(j)})=$$
$$=\frac{(1-(uv)^{g-2+(d-3j)/2})(1-(uv)^{g-1+(d-3j)/2})}{(1-uv)(1-(uv)^2)}(1+u)^g(1+v)^g\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}.$$

By putting everything together we get 2 cases:

- if $d-j$ is odd, then $\mathcal{HD}(G^-(\alpha(j);3,d,1))=q_1^{j\not\equiv_2 d}$;
- if $d-j$ is even, then

$$\mathcal{HD}(G^-(\alpha(j);3,d,1))=q_1^\bullet+q_2^\bullet+q_3^\bullet+q_4^\bullet+q_5^\bullet+q_6^\bullet$$

  where $\bullet$ stands for $j\equiv_2 d$.

14.3 Crossing a critical value $\alpha(j)$

14.3.1 $d-j$ odd, $0\le j<d/3$

As we said before, if $d-j$ is odd, then the only significant contribution is from p_1 and q_1, so we get:

$$\mathcal{HD}(j,\mathrm{odd}):=\mathcal{HD}(G(\alpha(j)^-;3,d,1))-\mathcal{HD}(G(\alpha(j)^+;3,d,1))=q_1^{j\not\equiv_2 d}-p_1^{j\not\equiv_2 d}=$$
$$=\mathcal{HD}(M^s(2,\mathrm{odd}))\frac{(uv)^{2j}-(uv)^{2g-2+d-3j}}{1-uv}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}=$$
$$=\frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^{2g}(1+v)^{2g}}{(1-uv)^2(1-(uv)^2)}\cdot$$
$$\cdot\Big((uv)^{2j}-(uv)^{2g-2+d-3j}\Big)\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}.\tag{14.8}$$

We remark that

$$\mathcal{HD}(j,\mathrm{odd})=-\frac{C_n}{(1+u)^g(1+v)^g}$$

where C_n is defined in [M, proposition 6.3].

14.3.2 $d-j$ even, $0 \leq j < d/3$

If $d-j$ is even, we compute the following quantities:

$$
\begin{aligned}
& q_1^{j\equiv_2 d} - p_1^{j\equiv_2 d} = \\
&= \mathcal{HD}(M^s(2,\text{even}))\frac{(uv)^{2j}-(uv)^{2g-2+d-3j}}{1-uv}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)} = \\
&= \frac{(uv)^{2j}-(uv)^{2g-2+d-3j}}{2(1-uv)^2(1-(uv)^2)}\Big(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g + \\
&-(1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1}-u^2v^2)-(1-u^2)^g(1-v^2)^g(1-uv)^2\Big)\cdot \\
&\cdot \operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)};
\end{aligned}
$$

$$q_2^{j\equiv_2 d} - p_2^{j\equiv_2 d} = r_j \cdot (1+u)^g(1+v)^g\frac{(uv)^{g-1+j}-(uv)^{2g-2+(d-3j)/2}}{1-uv}.$$

We recall that

$$r_j := \mathcal{HD}\left(G^{\mathrm{s}}\left(\frac{d-3j}{2};2,\frac{d+j}{2},1\right)\right) = \mathcal{HD}\left(G^{\mathrm{s}}(d'-2j;2,d',1)\right),$$

where $d' := \frac{d+j}{2}$. So according to corollary 13.3.2 with k substituted by j and d by $d'=(d+j)/2$ (so that $g+d-1-2k$ is replaced by $g-1+(d-3j)/2$), we get:

$$r_j = \frac{(1+u)^g(1+v)^g}{1-uv}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}\left[\frac{(uv)^j}{1-(uv)^{-1}x}-\frac{(uv)^{g+1+(d-3j)/2}x}{1-(uv)^2x}-1\right].$$

This can be done since $k=j\geq 0$, so $d'-2j$ is less or equal than d', which is the last critical value for $(2,d',1)$. We recall that for $j=0$ we will get $r_j=0$, even if this is not a priori obvious from the way it is written here, see remark 13.3.4. Also in this case, we will use this complicated notation simply because it will help in order to sum all the various contributions given by crossings the various critical values. Therefore for every $0 \leq j < d/3$ such that $d-j$ is even, we have:

$$
\begin{aligned}
q_2^{j\equiv_2 d} - p_2^{j\equiv_2 d} &= \frac{(1+u)^{2g}(1+v)^{2g}}{(1-uv)^2}\left((uv)^{g-1+j}-(uv)^{2g-2+(d-3j)/2}\right)\cdot \\
&\cdot \operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}\left[\frac{(uv)^j}{1-(uv)^{-1}x}-\frac{(uv)^{g+1+(d-3j)/2}x}{1-(uv)^2x}-1\right] = \\
&= \frac{(uv)^j-(uv)^{g-1+(d-3j)/2}}{(1-uv)^2}(uv)^{g-1}(1+u)^{2g}(1+v)^{2g}\cdot \\
&\cdot \operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}\left[\frac{(uv)^j}{1-(uv)^{-1}x}-\frac{(uv)^{g+1+(d-3j)/2}x}{1-(uv)^2x}-1\right].
\end{aligned}
$$

Moreover, we can compute:

$$q_3^{j\equiv_2 d}-p_3^{j\equiv_2 d}=\frac{(uv)^{g-1+(d-3j)/2}-(uv)^{2g-2+d-3j}-(uv)^j+(uv)^{2j}}{(1-uv)^2}\cdot$$

$$\cdot(1-(uv)^{g-1})((1+u)^g(1+v)^g-1)(1+u)^g(1+v)^g\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)};$$

$$q_4^{j\equiv_2 d}-p_4^{j\equiv_2 d}=\frac{(uv)^{g-2+(d-3j)/2}-(uv)^{2g-3+d-3j}-(uv)^{j-1}+(uv)^{2j-1}}{(1-uv)^2}\cdot$$

$$\cdot(1+u)^g(1+v)^g(1-(uv)^g)\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)};$$

$$q_5^{j\equiv_2 d}-p_5^{j\equiv_2 d}=\frac{1}{2}\Bigg[\frac{(uv)^{2j}-(uv)^{2g-2+d-3j}}{1-(uv)^2}\Big((1-u^2)^g(1-v^2)^g-(1+u)^g(1+v)^g\Big)+$$

$$+\frac{(1-(uv)^{g-1+(d-3j)/2})^2-(1-(uv)^j)^2}{(1-uv)^2}(1+u)^g(1+v)^g\cdot$$

$$\cdot\Big((1+u)^g(1+v)^g-1\Big)\Bigg]\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}=$$

$$=\frac{1}{2}\Bigg[\frac{(uv)^{2j}-(uv)^{2g-2+d-3j}}{1-(uv)^2}\Big((1-u^2)^g(1-v^2)^g-(1+u)^g(1+v)^g\Big)+$$

$$+\frac{(uv)^{2g-2+d-3j}-2(uv)^{g-1+(d-3j)/2}-(uv)^{2j}+2(uv)^j}{(1-uv)^2}(1+u)^g(1+v)^g\cdot$$

$$\cdot\Big((1+u)^g(1+v)^g-1\Big)\Bigg]\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)x^j};$$

$$q_6^{j\equiv_2 d}-p_6^{j\equiv_2 d}=$$

$$=\frac{(1-(uv)^{g-2+(d-3j)/2})(1-(uv)^{g-1+(d-3j)/2})-(1-(uv)^{j-1})(1-(uv)^j)}{(1-uv)(1-(uv)^2)}\cdot$$

$$\cdot(1+u)^g(1+v)^g\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}=$$

$$=\frac{(1+u)^g(1+v)^g}{(1-uv)(1-(uv)^2)}\cdot\{(uv)^j+(uv)^{j-1}-(uv)^{2j-1}-(uv)^{g-1+(d-3j)/2}+$$

$$-(uv)^{g-2+(d-3j)/2}+(uv)^{2g-3+d-3j}\}\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)}.$$

All these terms coincide with analogous terms in the proof of [M, proposition 6.4], except for the extra multiplicative factor in that case given by $-(1+u)^g(1+v)^g$. In particular,

- $q_1^{j\not\equiv_2 d}-p_1^{j\not\equiv_2 d}$ is associated to $e(X_2^+)-e(X_2^-)$ in the proof of proposition 6.4;

- $q_2^{j\equiv_2 d} - p_2^{j\equiv_2 d}$ is associated to $e(X_1^+) - e(X_1^-)$ in the proof of proposition 6.4;
- $q_i^{j\equiv_2 d} - p_i^{j\equiv_2 d}$ is associated to $e(X_i^+) - e(X_i^-)$ in the proof of proposition 6.4 for all $i = 3, \cdots, 6$.

All these identifications are obtained by setting $d_1 := d$, $d_2 := 0$, $n := d - j$, so that the invariants N_1 and N_2 (see the notation before proposition 6.3 in [M]) are given by:

$$N_1 = d_1 - d_2 - n = d - (d - j) = j,$$
$$N_2 = g - 1 - d_1 + 3\frac{n}{2} = g - 1 - d + 3\frac{d-j}{2} = g - 1 + \frac{d-3j}{2}.$$

Then we can simply use exactly the same computations of [M], proof of proposition 6.4 in order to compute the next quantities. We do anyway all the computations following that paper, since some of the intermediate passages are missing; moreover, we will need to do twice almost all the same computation (see the next section), so we will do it once here and then simply state the differences with the second computation. The only difference with [M] is that all signs are changed and that in [M] there is an additional multiplicative term $(1+u)^g(1+v)^g$ (the sign is given by the fact that we are crossing critical values right-to-left, while in [M] the crossing is done left-to-right; the extra multiplicative term is because on holomorphic triples we have an extra contribution from a Jacobian). For simplicity, we will still use the notation N_1 for j and N_2 for $g - 1 + \frac{d-3j}{2}$. For every $0 \leq j < d/3$ such that $d - j$ is even we define:

$$\mathcal{HD}(j, even) := \mathcal{HD}(G(\alpha(j)^-; 3, d, 1)) - \mathcal{HD}(G(\alpha(j)^+; 3, d, 1)) =$$
$$= \sum_{i=1,\cdots,6} (q_i^{j\equiv_2 d} - p_i^{j\equiv_2 d}) = \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)} \cdot$$
$$\cdot \left\{ \frac{(uv)^{2N_1} - (uv)^{2N_2}}{2(1-uv)^2(1-(uv)^2)} \cdot \left[2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g + \right.\right.$$
$$\left. -(1+u)^{2g}(1+v)^{2g}(1+2(uv)^{g+1} - (uv)^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2 \right] +$$
$$+ \frac{(uv)^{N_1} - (uv)^{N_2}}{(1-uv)^2}(uv)^{g-1}(1+u)^{2g}(1+v)^{2g} \left[\frac{(uv)^{N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_2+2}x}{1-(uv)^2x} - 1 \right] +$$
$$+ \frac{(uv)^{N_2} - (uv)^{2N_2} - (uv)^{N_1} + (uv)^{2N_1}}{(1-uv)^2} \cdot$$
$$\cdot \left[(1+u)^{2g}(1+v)^{2g} - (1+u)^g(1+v)^g \right] [1 - (uv)^{g-1}] +$$
$$+ \frac{(uv)^{N_2-1} - (uv)^{2N_2-1} - (uv)^{N_1-1} + (uv)^{2N_1-1}}{(1-uv)^2} (1+u)^g(1+v)^g[1-(uv)^g] +$$
$$+ \frac{1}{2} \left[\frac{(uv)^{2N_1} - (uv)^{2N_2}}{1-(uv)^2} \left((1-u^2)^g(1-v^2)^g - (1+u)^g(1+v)^g \right) + \right.$$
$$\left. + \frac{(uv)^{2N_2} - 2(uv)^{N_2} - (uv)^{2N_1} + 2(uv)^{N_1}}{(1-uv)^2} \left((1+u)^{2g}(1+v)^{2g} - (1+u)^g(1+v)^g \right) \right]$$

$$+(1+u)^g(1+v)^g\frac{(uv)^{N_1}+(uv)^{N_1-1}-(uv)^{2N_1-1}-(uv)^{N_2}-(uv)^{N_2-1}+(uv)^{2N_2-1}}{(1-uv)(1-(uv)^2)}\Bigg\} =$$

$$= \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)} \cdot \big\{[(uv)^{N_1}-(uv)^{N_2}]\mathcal{HD}^1(N_1,N_2)+$$
$$+[(uv)^{2N_1}-(uv)^{2N_2}]\mathcal{HD}^2(N_1,N_2)\big\}$$

where

$$\mathcal{HD}^1(N_1,N_2) := \frac{(uv)^{g-1}(1+u)^{2g}(1+v)^{2g}}{(1-uv)^2}\cdot\left[\frac{(uv)^{N_1}}{1-(uv)^{-1}x}-\frac{(uv)^{N_2+2}x}{1-(uv)^2x}-1\right]+$$
$$-\left[(1+u)^{2g}(1+v)^{2g}-(1+u)^g(1+v)^g\right]\frac{1-(uv)^{g-1}}{(1-uv)^2}+$$
$$-(uv)^{-1}(1+u)^g(1+v)^g\frac{1-(uv)^g}{(1-uv)^2}+\frac{(1+u)^{2g}(1+v)^{2g}-(1+u)^g(1+v)^g}{(1-uv)^2}+$$
$$+\frac{(1+(uv)^{-1})(1+u)^g(1+v)^g}{(1-uv)(1-(uv)^2)} =$$
$$= \frac{(uv)^{g-1}(1+u)^{2g}(1+v)^{2g}}{(1-uv)^2}\cdot\left[\frac{(uv)^{N_1}}{1-(uv)^{-1}x}-\frac{(uv)^{N_2+2}x}{1-(uv)^2x}-1\right]+$$
$$+\frac{(1+u)^{2g}(1+v)^{2g}(uv)^{g-1}}{(1-uv)^2}+$$
$$+\frac{(1+u)^g(1+v)^g}{(1-uv)^2}\left(-(uv)^{g-1}-(uv)^{-1}(1-(uv)^g)+\frac{1+(uv)^{-1}}{1+uv}\right)=$$
$$= \frac{(uv)^{g-1}(1+u)^{2g}(1+v)^{2g}}{(1-uv)^2}\cdot\left[\frac{(uv)^{N_1}}{1-(uv)^{-1}x}-\frac{(uv)^{N_2+2}x}{1-(uv)^2x}\right]$$

and

$$\mathcal{HD}^2(N_1,N_2)=\mathcal{HD}^2 := \frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g}{(1-uv)^2(1-(uv)^2)}+$$
$$-\frac{(1+u)^{2g}(1+v)^{2g}(1+2(uv)^{g+1}-(uv)^2)}{2(1-uv)^2(1-(uv)^2)}-\frac{(1-u^2)^g(1-v^2)^g}{2(1-(uv)^2)}+$$
$$+\frac{[(1+u)^{2g}(1+v)^{2g}-(1+u)^g(1+v)^g](1-(uv)^{g-1})}{(1-uv)^2}+$$
$$+\frac{(uv)^{-1}(1+u)^g(1+v)^g(1-(uv)^g)}{(1-uv)^2}+\frac{(1-u^2)^g(1-v^2)^g}{2(1-(uv)^2)}+$$
$$-\frac{(1+u)^g(1+v)^g}{2(1-(uv)^2)}-\frac{(1+u)^{2g}(1+v)^{2g}-(1+u)^g(1+v)^g}{2(1-uv)^2}+$$
$$-\frac{(uv)^{-1}(1+u)^g(1+v)^g)}{(1-uv)(1-(uv)^2)}=$$

$$= \frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g}{(1-uv)^2(1-(uv)^2)} +$$

$$+ \frac{(1+u)^{2g}(1+v)^{2g}}{(1-uv)^2} \cdot \left(-\frac{1+2(uv)^{g+1}-(uv)^2}{2(1-(uv)^2)} + (1-(uv)^{g-1}) - \frac{1}{2} \right) +$$

$$+(1+u)^g(1+v)^g \cdot \left(\frac{-(1-(uv)^{g-1}) + (uv)^{-1}(1-(uv)^g)}{(1-uv)^2} + \right.$$

$$\left. - \frac{1}{2(1-(uv)^2)} + \frac{1}{2(1-uv)^2} - \frac{(uv)^{-1}}{(1-uv)(1-(uv)^2)} \right) =$$

$$= \frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g}{(1-uv)^2(1-(uv)^2)} + \frac{(1+u)^{2g}(1+v)^{2g}}{(1-uv)^2} \cdot$$

$$\cdot \frac{-1-2(uv)^{g+1}+(uv)^2+1-(uv)^2-2(uv)^{g-1}+2(uv)^{g+1}}{2(1-(uv)^2)} +$$

$$+ \frac{(1+u)^g(1+v)^g}{2(1-uv)^2(1+uv)} \cdot \left(2((uv)^{-1}-1)(1+uv) - (1-uv) + (1+uv) - 2(uv)^{-1} \right) =$$

$$= \frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (uv)^{g-1}(1+u)^{2g}(1+v)^{2g}}{(1-uv)^2(1-(uv)^2)} =$$

$$= \frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^{2g}(1+v)^{2g}}{(1-uv)^2(1-(uv)^2)} +$$

$$- \frac{(uv)^{g-1}(1-uv)(1+u)^{2g}(1+v)^{2g}}{(1-uv)^2(1-(uv)^2)}.$$

So we get that

$$\mathcal{HD}(j, \text{even}) = (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)} \cdot$$

$$\cdot \left\{ [(uv)^{N_1} - (uv)^{N_2}] \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2} \cdot \left[\frac{(uv)^{N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_2+2}x}{1-(uv)^2x} \right] + \right.$$

$$+[(uv)^{2N_1} - (uv)^{2N_2}] \cdot \left[\frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} + \right.$$

$$\left. \left. - \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)} \right] \right\} =$$

$$= (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)} \cdot$$

$$\cdot \left\{ \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2} \left(\frac{(uv)^{2N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_1+N_2+2}x}{1-(uv)^2x} - \frac{(uv)^{N_1+N_2}}{1-(uv)^{-1}x} + \right. \right.$$

$$\left. + \frac{(uv)^{2N_2+2}x}{1-(uv)^2x} - \frac{(uv)^{2N_1} - (uv)^{2N_2}}{1+uv} \right) +$$

$$\left. +[(uv)^{2N_1} - (uv)^{2N_2}] \cdot \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \right\} =$$

$$= (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j}}{(1-x)(1-uvx)} \cdot$$
$$\cdot \Bigg\{ [(uv)^{2N_1} - (uv)^{2N_2}] \cdot \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} +$$
$$+ \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)} \left(\frac{(uv)^{2N_1+1}(1+(uv)^{-2}x)}{1-(uv)^{-1}x} + \frac{(uv)^{2N_2}(1+(uv)^3x)}{1-(uv)^2x} + \right.$$
$$\left. - \frac{(uv)^{N_1+N_2}(1+uv)(1-uvx^2)}{(1-(uv)^{-1}x)(1-(uv)^2x)} \right) \Bigg\}. \tag{14.9}$$

14.3.3 The polynomials for $G^-(\alpha(k); 3, d, 1)$

In the following chapter we will also need the Hodge-Deligne polynomial of the moduli space $G^-(\alpha(k); 3, d, 1)$. This polynomial will have 2 different expressions according to $d-k$ odd or even.

Lemma 14.3.1. *If $d-k$ is odd we have*

$$\mathcal{HD}(G^-(\alpha(k); 3, d, 1)) = q_1^{k \not\equiv_2 d} =$$
$$= \mathcal{HD}(M^s(2, odd)) \frac{1-(uv)^{2g-2+d-3k}}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-k}}{(1-x)(1-uvx)} =$$
$$= (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} \cdot [x^{-k} - (uv)^{2g-2+d-3k} x^{-k}] \cdot$$
$$\cdot \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}.$$

If $d-k$ is even, then

$$\mathcal{HD}(G^-(\alpha(k); 3, d, 1)) = \sum_{i=1,\cdots,6} q_i^{k \equiv_2 d}.$$

Now we denote by $N_1 = k$ and $N_2 = g-1+(d-3k)/2$ and we write:

$$q_2^{k \equiv_2 d} = \frac{1-(uv)^{N_2}(uv)^{g-1}}{(1-uv)^2} (1+u)^{2g}(1+v)^{2g} \cdot$$
$$\operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-k}}{(1-x)(1-uvx)} \cdot \left[\frac{(uv)^{N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_2+2}x}{1-(uv)^2x} - 1 \right] =$$
$$= \frac{1-(uv)^{N_2}}{(1-uv)^2} (uv)^{g-1}(1+u)^{2g}(1+v)^{2g} \cdot$$
$$\operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-k}}{(1-x)(1-uvx)} \cdot \left[\frac{(uv)^{N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_2+2}x}{1-(uv)^2x} - 1 \right] +$$
$$+ \frac{1-(uv)^{g-1}}{(1-uv)^2} (1+u)^{2g}(1+v)^{2g} \cdot$$

$$\underset{x^0}{\mathrm{coeff}} \frac{(1+ux)^g(1+vx)^g x^{-k}}{(1-x)(1-uvx)} \cdot \left[\frac{(uv)^{N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_2+2}x}{1-(uv)^2x} - 1\right].$$

Therefore,

$$\mathcal{HD}(G^-(\alpha(k);3,d,1)) = \underset{x^0}{\mathrm{coeff}} \frac{(1+ux)^g(1+vx)^g x^{-k}}{(1-x)(1-uvx)} \cdot$$
$$\cdot \left\{ \frac{1-(uv)^{2N_2}}{2(1-uv)^2(1-(uv)^2)} \cdot \left[2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g + \right.\right.$$
$$\left. -(1+u)^{2g}(1+v)^{2g}(1+2(uv)^{g+1}-(uv)^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2\right] +$$
$$+\frac{1-(uv)^{N_2}}{(1-uv)^2}(uv)^{g-1}(1+u)^{2g}(1+v)^{2g}\left[\frac{(uv)^{N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_2+2}x}{1-(uv)^2x} - 1\right] +$$
$$+\frac{(uv)^{N_2}-(uv)^{2N_2}}{(1-uv)^2} \cdot \left[(1+u)^{2g}(1+v)^{2g} - (1+u)^g(1+v)^g\right] \cdot [1-(uv)^{g-1}] +$$
$$+\frac{(uv)^{N_2-1}-(uv)^{2N_2-1}}{(1-uv)^2}(1+u)^g(1+v)^g[1-(uv)^g] +$$
$$+\frac{1}{2}\left[\frac{1-(uv)^{2N_2}}{1-(uv)^2}\left((1-u^2)^g(1-v^2)^g - (1+u)^g(1+v)^g\right) + \right.$$
$$\left. +\frac{(uv)^{2N_2}-2(uv)^{N_2}+1}{(1-uv)^2}\left((1+u)^{2g}(1+v)^{2g} - (1+u)^g(1+v)^g\right)\right] +$$
$$+(1+u)^g(1+v)^g \frac{1-(uv)^{N_2}-(uv)^{N_2-1}+(uv)^{2N_2-1}}{(1-uv)(1-(uv)^2)} +$$
$$\left. +\frac{1-(uv)^{g-1}}{(1-uv)^2}(1+u)^{2g}(1+v)^{2g}\left[\frac{(uv)^{N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_2+2}x}{1-(uv)^2x} - 1\right]\right\}.$$

Now a careful comparison with the computation for $\mathcal{HD}(j, \text{even})$ with j replaced by k proves that this is equal to

$$\underset{x^0}{\mathrm{coeff}} \frac{(1+ux)^g(1+vx)^g x^{-k}}{(1-x)(1-uvx)} \cdot \left\{ [1-(uv)^{N_2}]\mathcal{HD}^1(N_1,N_2) + [1-(uv)^{2N_2}]\mathcal{HD}^2 + \right.$$
$$\left. +\frac{1-(uv)^{g-1}}{(1-uv)^2}(1+u)^{2g}(1+v)^{2g}\left[\frac{(uv)^{N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_2+2}x}{1-(uv)^2x} - 1\right]\right\}$$

where $\mathcal{HD}^1(N_1,N_2)$ and $\mathcal{HD}^2$ are computed as before (with j replaced by d). Therefore, if $d-k$ is even we get:

$$\mathcal{HD}(G^-(\alpha(k);3,d,1)) = (1+u)^g(1+v)^g \underset{x^0}{\mathrm{coeff}} \frac{(1+ux)^g(1+vx)^g x^{-k}}{(1-x)(1-uvx)} \cdot$$
$$\cdot \left\{ [1-(uv)^{N_2}] \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2} \cdot \left[\frac{(uv)^{N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_2+2}x}{1-(uv)^2x}\right] + \right.$$
$$+\frac{1-(uv)^{g-1}}{(1-uv)^2}(1+u)^g(1+v)^g\left[\frac{(uv)^{N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_2+2}x}{1-(uv)^2x} - 1\right] + [1-(uv)^{2N_2}] \cdot$$

$$\cdot \left[\frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} - \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)} \right] \Bigg\} =$$

$$= (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-k}}{(1-x)(1-uvx)} \cdot \Bigg\{ \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2} \cdot$$

$$\cdot \left(\frac{(uv)^{N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_2+2}x}{1-(uv)^2 x} - \frac{(uv)^{N_1+N_2}}{1-(uv)^{-1}x} + \frac{(uv)^{2N_2+2}x}{1-(uv)^2x} + \right.$$

$$\left. - \frac{(uv)^{N_1}}{1-(uv)^{-1}x} + \frac{(uv)^{N_2+2}x}{1-(uv)^2x} + 1 - \frac{1-(uv)^{2N_2}}{1+uv} \right) +$$

$$+ \frac{(1+u)^g(1+v)^g}{(1-uv)^2} \left[\frac{(uv)^{N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_2+2}x}{1-(uv)^2x} - 1 \right] +$$

$$+ [1-(uv)^{2N_2}] \cdot \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \Bigg\} =$$

$$= (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-k}}{(1-x)(1-uvx)} \cdot \Bigg\{ \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2} \cdot$$

$$\cdot \left(\frac{(uv)^{2N_2+2}x}{1-(uv)^2x} - \frac{(uv)^{N_1+N_2}}{1-(uv)^{-1}x} + \frac{uv+(uv)^{2N_2}}{1+uv} \right) +$$

$$+ \frac{(1+u)^g(1+v)^g}{(1-uv)^2} \left[\frac{(uv)^{N_1}}{1-(uv)^{-1}x} - \frac{(uv)^{N_2+2}x}{1-(uv)^2x} - 1 \right] +$$

$$+ [1-(uv)^{2N_2}] \cdot \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \Bigg\}.$$

Therefore, by substituting $N_1 = k$ and $N_2 = g - 1 + (d-3k)/2$, we get:

Lemma 14.3.2. *If $d-k$ is even, then*

$$\mathcal{HD}(G^-(\alpha(k); 3, d, 1)) =$$

$$= (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} \cdot \Bigg\{ \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2} \cdot$$

$$\cdot \left(\frac{(uv)^{2g+d-3k}x^{1-k}}{1-(uv)^2x} - \frac{(uv)^{g-1+(d-k)/2}x^{-k}}{1-(uv)^{-1}x} + \frac{uvx^{-k} + (uv)^{2g-2+d-3k}x^{-k}}{1+uv} \right) +$$

$$+ \frac{(1+u)^g(1+v)^g}{(1-uv)^2} \left[\frac{(uv)^k x^{-k}}{1-(uv)^{-1}x} - \frac{(uv)^{g+1+(d-3k)/2}x^{1-k}}{1-(uv)^2x} - x^{-k} \right] +$$

$$+ [1-(uv)^{2g-2+d-3k}] x^{-k} \cdot \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \Bigg\}.$$

14.3.4 The "last" moduli space

We observe that $G^+(\alpha(0);3,d,1)=\varnothing$; therefore, we get that if $d=d-0$ is odd, then:

$$\begin{aligned}\mathcal{HD}(G_L(3,d,1)) &= \mathcal{HD}(G(\alpha(0)^-;3,d,1)) = \mathcal{HD}(G^-(\alpha(0);3,d,1)) = \\ &= \mathcal{HD}(G^-(\alpha(0);3,d,1)) - \mathcal{HD}(G^+(\alpha(0);3,d,1)) = \mathcal{HD}(0,\text{odd}).\end{aligned}$$

Now the function

$$f(x) := \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}$$

is holomorphic around $x=0$, so we get that $\operatorname{coeff}_{x^0} f(x) = f(0) = 1$. Therefore we get:

Proposition 14.3.3. *If d is odd, then:*

$$\begin{aligned}\mathcal{HD}(G(\alpha(0)^-;3,d,1)) &= \Big(1-(uv)^{2g-2+d}\Big)\cdot \\ &\cdot\frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^{2g}(1+v)^{2g}}{(1-uv)^2(1-(uv)^2)}.\end{aligned}$$

Since this moduli space is smooth, by setting $u=v=:t$, we get the Poincaré polynomial of $G(\alpha(0)^-;3,d,1)$:

$$\begin{aligned}P_{G(\alpha(0)^-;3,d,1)}(t) &= \Big(1-t^{2(2g-2+d)}\Big)\,\frac{(1+t)^{2g}(1+t^3)^{2g}-t^{2g}(1+t)^{4g}}{(1-t^2)^2(1-t^4)} = \\ &= \frac{(1+t)^{2g}[(1+t^3)^{2g}-t^{2g}(1+t)^{2g}](1-t^{2(2g-2+d)})}{(1-t^2)^2(1-t^4)}\end{aligned}$$

which coincides with the formula given in [BGMMN, corollary 8.7]. Note that in that paper this is the best that one can say about the moduli spaces $G(\alpha;3,d,1)$ for α non-critical. Indeed, if $d\geq 3$, then the value denoted by α_T in that paper is given by

$$\alpha_T = \frac{d-3}{2} = \alpha(1).$$

So there are no critical values that we have to cross in order to get to α_T starting form $\alpha(0)$. Therefore, the results of this section agree with those of [BGMMN] and improve them at least in the case $(n,d,k)=(3,d,1)$, $d\geq 3$ and odd.

If $d=d-0$ is even then

$$\begin{aligned}\mathcal{HD}(G(\alpha(0)^-;3,d,1)) &= \mathcal{HD}(0,\text{even}) = (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot \\ &\cdot\Bigg\{\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\cdot[1-(uv)^{2g-2+d}]+ \\ &+\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot\Bigg(\frac{uv(1+(uv)^{-2}x)}{1-(uv)^{-1}x}+\end{aligned}$$

$$+\frac{(uv)^{2g-2+d}(1+(uv)^3x)}{1-(uv)^2x}-\frac{(uv)^{g-1+d/2}(1+uv)(1-uvx^2)}{(1-(uv)^{-1}x)(1-(uv)^2x)}\Bigg)\Bigg\}.$$

Now if we write

$$f(x,u,v):=\frac{uv(1+(uv)^{-2}x)}{1-(uv)^{-1}x}+\frac{(uv)^{2g-2+d}(1+(uv)^3x)}{1-(uv)^2x}+$$
$$-\frac{(uv)^{g-1+d/2}(1+uv)(1-uvx^2)}{(1-(uv)^{-1}x)(1-(uv)^2x)},$$

then it turns out that f is holomorphic in x around $x=0$ and that

$$\operatorname*{coeff}_{x^0} f(x,u,v)=f(0,u,v)=uv+(uv)^{2g-2+d}-(uv)^{g-1+d/2}(1+uv)=$$
$$=uv(1+(uv)^{2g-3+d}-(uv)^{g-2+d/2}-(uv)^{g+d/2-1}).$$

So we get that:

Proposition 14.3.4. *If d is even, then*

$$\mathcal{HD}(G(\alpha(0)^-;3,d,1))=(1+u)^g(1+v)^g\Big\{[1-(uv)^{2g-2+d}]\cdot$$
$$\cdot\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}+$$
$$+\frac{(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot\Big(1+(uv)^{2g-3+d}-(uv)^{g-2+d/2}-(uv)^{g+d/2-1}\Big)\Big\}.$$

As a corollary, we get that the formula for the Poincaré polynomials (we recall that this moduli space are smooth for all d):

$$P_{G(\alpha(0)^-;3,d,1)}(t)=(1+t)^{2g}\Bigg\{[1-t^{2(2g-2+d)}]\cdot\frac{(1+t^3)^{2g}-t^{2g}(1+t)^{2g}}{(1-t^2)^2(1-t^2)}+$$
$$+\frac{t^{2g}(1+t)^{2g}}{(1-t^2)^2(1+t^2)}\cdot\Big(1+t^{2(2g-3+d)}-t^{2g-4+d}-t^{2g+d-2}\Big)\Bigg\}.$$

As a check of the correctness of proposition 14.3.4 we can compare our formula with corollary 8.0.11 in the case when $g=2$ and $d=2$. In this case the previous formula for the Hodge-Deligne polynomials becomes:

$$\mathcal{HD}(G(\alpha(0)^-;3,2,1))=(1+u)^2(1+v)^2\cdot$$
$$\cdot\Bigg\{[1-(uv)^4]\cdot\frac{(1+u^2v)^2(1+uv^2)^2-(uv)^2(1+u)^2(1+v)^2}{(1-uv)^2(1-(uv)^2)}+$$
$$+\frac{(uv)^2(1+u)^2(1+v)^2}{(1-uv)^2(1+uv)}\cdot\big(1+(uv)^3-uv-(uv)^2\big)\Bigg\}=$$
$$=(1+u)^2(1+v)^2\cdot\Bigg\{\frac{1+(uv)^2}{(1-uv)^2}\big((1+2u^2v+u^4v^2)(1+2uv^2+u^2v^4)+$$

$$-(uv)^2(1+2u+u^2)(1+2v+v^2)\big) + \frac{(uv)^2(1+u)^2(1+v)^2}{(1-uv)^2(1+uv)} \cdot (1-uv)^2(1+uv)\Big\} =$$
$$= (1+u)^2(1+v)^2 \cdot \big\{(1+u^2v^2)(1-uv)^{-2}\big(1+2uv^2+u^2v^4+2u^2v+4u^3v^3+$$
$$+2u^4v^5+u^4v^2+2u^5v^4+u^6v^6-u^2v^2-2u^2v^3-u^2v^4-2u^3v^2-4u^3v^3$$
$$-2u^3v^4-u^4v^2-2u^4v^3-u^4v^4\big)+(uv)^2(1+2u+u^2)(1+2v+v^2)\big\}.$$

Now a direct check proves that

$$1+2uv^2+u^2v^4+2u^2v+4u^3v^3+2u^4v^5+u^4v^2+2u^5v^4+u^6v^6-u^2v^2-2u^2v^3+$$
$$-u^2v^4-2u^3v^2-4u^3v^3-2u^3v^4-u^4v^2-2u^4v^3-u^4v^4=$$
$$=(1-uv)^2[1+2uv+2u^2v^2+2u^3v^3+2u^2v+2uv^2+u^4v^4+2u^3v^2+2u^2v^3].$$

Therefore by substituting we get:

$$\mathcal{HD}(G(\alpha(0)^-;3,2,1)) = (1+u)^2(1+v)^2 \cdot \big\{(1+u^2v^2)\cdot$$
$$\cdot[1+2uv+2u^2v^2+2u^3v^3+2u^2v+2uv^2+u^4v^4+2u^3v^2+2u^2v^3]+$$
$$+u^2v^2(1+2u+u^2)(1+2v+v^2)\big\} =$$
$$= (1+2u+u^2)(1+2v+v^2)\cdot\big\{1+2uv+2u^2v^2+2u^3v^3+2u^2v+2uv^2+u^4v^4+$$
$$+2u^3v^2+2u^2v^3+u^2v^2+2u^3v^3+2u^4v^4+2u^5v^5+2u^4v^3+2u^3v^4+u^6v^6+2u^5v^4+2u^4v^5+$$
$$+u^2v^2+2u^2v^3+u^2v^4+2u^3v^2+4u^3v^3+2u^3v^4+u^4v^2+2u^4v^3+u^4v^4\big\} =$$
$$= (1+2v+v^2+2u+4uv+2uv^2+u^2+2u^2v+u^2v^2)\cdot\big\{1+2uv+2u^2v+2uv^2+$$
$$+4u^2v^2+4u^3v^2+4u^2v^3+u^4v^2+8u^3v^3+u^2v^4+4u^4v^3+4u^3v^4+4u^4v^4+$$
$$+2u^5v^4+2u^4v^5+2u^5v^5+u^6v^6\big\} =$$
$$= 1+2uv+2u^2v+2uv^2+4u^2v^2+4u^3v^2+4u^2v^3+u^4v^2+8u^3v^3+$$
$$+u^2v^4+4u^4v^3+4u^3v^4+4u^4v^4+2u^5v^4+2u^4v^5+2u^5v^5+u^6v^6+$$
$$+2v+4uv^2+4u^2v^2+4uv^3+8u^2v^3+8u^3v^3+8u^2v^4+2u^4v^3+16u^3v^4+$$
$$+2u^2v^5+8u^4v^4+8u^3v^5+8u^4v^5+4u^5v^5+4u^4v^6+4u^5v^6+2u^6v^7+$$
$$+v^2+2uv^3+2u^2v^3+2uv^4+4u^2v^4+4u^3v^4+4u^2v^5+u^4v^4+8u^3v^5+$$
$$+u^2v^6+4u^4v^5+4u^3v^6+4u^4v^6+2u^5v^6+2u^4v^7+2u^5v^7+u^6v^8+$$
$$+2u+4u^2v+4u^3v+4u^2v^2+8u^3v^2+8u^4v^2+8u^3v^3+2u^5v^2+16u^4v^3+$$
$$+2u^3v^4+8u^5v^3+8u^4v^4+8u^5v^4+4u^6v^4+4u^5v^5+4u^6v^5+2u^7v^6+$$
$$+4uv+8u^2v^2+8u^3v^2+8u^2v^3+16u^3v^3+16u^4v^3+16u^3v^4+4u^5v^3+$$
$$+32u^4v^4+4u^3v^5+16u^5v^4+16u^4v^5+16u^5v^5+8u^6v^5+8u^5v^6+8u^6v^6+4u^7v^7+$$

$$+2uv^2+4u^2v^3+4u^3v^3+4u^2v^4+8u^3v^4+8u^4v^4+8u^3v^5+2u^5v^4+$$
$$+16u^4v^5+2u^3v^6+8u^5v^5+8u^4v^6+8u^5v^6+4u^6v^6+4u^5v^7+4u^6v^7+2u^7v^8+$$
$$+u^2+2u^3v+2u^4v+2u^3v^2+4u^4v^2+4u^5v^2+4u^4v^3+u^6v^2+8u^5v^3+u^4v^4+$$
$$+4u^6v^3+4u^5v^4+4u^6v^4+2u^7v^4+2u^6v^5+2u^7v^5+u^8v^6+$$
$$+2u^2v+4u^3v^2+4u^4v^2+4u^3v^3+8u^4v^3+8u^5v^3+8u^4v^4+2u^6v^3+$$
$$+16u^5v^4+2u^4v^5+8u^6v^4+8u^5v^5+8u^6v^5+4u^7v^5+4u^6v^6+4u^7v^6+2u^8v^7+$$
$$+u^2v^2+2u^3v^3+2u^4v^3+2u^3v^4+4u^4v^4+4u^5v^4+4u^4v^5+u^6v^4+8u^5v^5+$$
$$+u^4v^6+4u^6v^5+4u^5v^6+4u^6v^6+2u^7v^6+2u^6v^7+2u^7v^7+u^8v^8=$$
$$=1+2u+2v+u^2+v^2+6uv+8u^2v+8uv^2+6u^3v+6uv^3+21u^2v^2+2u^4v+2uv^4+$$
$$+26u^3v^2+26u^2v^3+50u^3v^3+17u^4v^2+17u^2v^4+52u^4v^3+52u^3v^4+6u^5v^2+6u^2v^5+$$
$$+u^6v^2+28u^5v^3+74u^4v^4+28u^3v^5+u^2v^6+6u^6v^3+6u^3v^6+52u^5v^4+52u^4v^5+$$
$$+17u^6v^4+17u^4v^6+50u^5v^5+26u^6v^5+26u^5v^6+2u^7v^4+2u^4v^7+21u^6v^6+6u^7v^5+$$
$$+6u^5v^7+8u^7v^6+8u^6v^7+6u^7v^7+u^8v^6+u^6v^8+2u^8v^7+2u^7v^8+u^8v^8.$$

This coincides exactly with formula (8.13. We remark that it should also be possible to compare the Hodge-Deligne polynomial obtained before for every g with the one described in theorem 8.0.10, but this would require some more work.

14.3.5 The polynomials of the other moduli spaces

In order to compute the Hodge-Deligne polynomials of all the other moduli spaces for $(3, d, 1)$ (for α non-critical) we proceed as in chapter 13. So if we use together (14.8) and (14.9), we get that for every $0 \leq k < d/3$

$$\mathcal{HD}(G(\alpha(k)^-;3,d,1))=\mathcal{HD}(G(\alpha(k)^-;3,d,1))-\mathcal{HD}(G(\alpha(0)^+;3,d,1))=$$
$$=\sum_{0\leq j\leq k, d-j \text{ odd}}\mathcal{HD}(j,\text{odd})+\sum_{0\leq j\leq k, d-j \text{ even}}\mathcal{HD}(j,\text{even})=$$
$$=(1+u)^g(1+v)^g\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\left\{\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\cdot\right.$$
$$\cdot\left[\sum_{0\leq j\leq k}x^{-j}\left((uv)^{2j}-(uv)^{2g-2+d-3j}\right)\right]+\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot$$
$$\cdot\left[\sum_{0\leq j\leq k, d-j \text{ even}}x^{-j}\left(\frac{(uv)^{2j+1}(1+(uv)^{-2}x)}{1-(uv)^{-1}x}+\frac{(uv)^{2g-2+d-3j}(1+(uv)^3x)}{1-(uv)^2x}+\right.\right.$$
$$\left.\left.\left.-\frac{(uv)^{g-1+(d-j)/2}(1+uv)(1-uvx^2)}{(1-(uv)^{-1}x)(1-(uv)^2x)}\right)\right]\right\}.$$

Now if we extend the two summations for $j < 0$, this does not change the result since in both cases we are adding a function $f(x,u,v)$ that is holomorphic in x around $x = 0$ and such that $\operatorname{coeff}_{x^0} f(x,u,v) = f(0,u,v) = 0$. So this does not affect the results after taking the coefficient of x^0. So we can write

$$\begin{aligned}\mathcal{HD}(G(\alpha(k)^-;3,d,1)) &= (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot \\ &\cdot\left\{\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\cdot A_k+\right. \\ &+\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot\left(B_k\frac{1+(uv)^{-2}x}{1-(uv)^{-1}x}+\right. \\ &\left.\left.+C_k\frac{1+(uv)^3x}{1-(uv)^2x}-D_k\frac{(1+uv)(1-uvx^2)}{(1-(uv)^{-1}x)(1-(uv)^2x)}\right)\right\},\end{aligned}$$

where we use the following notation:

$$h := -j, \quad l := \frac{d-j}{2}, \quad l_0 := \left\lceil\frac{d-k}{2}\right\rceil,$$

and

$$\begin{aligned}A_k &:= \sum_{-\infty<j\le k} x^{-j}(uv)^{2j} - x^{-j}(uv)^{2g-2+d-3j} = \\ &= \sum_{-\infty<j\le 0}(uv)^{2k}x^{-k}(uv)^{2j}x^{-j}-(uv)^{2g-2+d-3k}x^{-k}x^{-j}(uv)^{-3j} = \\ &= \sum_{0\le h<+\infty}(uv)^{2k}x^{-k}((uv)^{-2}x)^h-(uv)^{2g-2+d-3k}x^{-k}((uv)^3x)^h = \\ &= \frac{(uv)^{2k}x^{-k}}{1-(uv)^{-2}x}-\frac{(uv)^{2g-2+d-3k}x^{-k}}{1-(uv)^3x};\end{aligned}$$

$$\begin{aligned}B_k &:= \sum_{-\infty<j\le k, d-j \text{ even}} x^{-j}(uv)^{2j+1} = \sum_{(d-k)/2\le l<+\infty} x^{2l-d}(uv)^{2d-4l+1} = \\ &= \sum_{l_0\le l<+\infty} x^{2l-d}(uv)^{2d-4l+1} = x^{2l_0-d}(uv)^{2d-4l_0+1}\sum_{0\le l<+\infty}((uv)^{-4}x^2)^l = \\ &= \frac{(uv)^{2d-4l_0+1}x^{2l_0-d}}{1-(uv)^{-4}x^2};\end{aligned}$$

$$\begin{aligned}C_k &:= \sum_{-\infty<j\le k, d-j \text{ even}} x^{-j}(uv)^{2g-2+d-3j} = \sum_{(d-k)/2\le l<+\infty} x^{2l-d}(uv)^{2g-2-2d+6l} = \\ &= \sum_{l_0\le l<+\infty} x^{2l-d}(uv)^{2g-2-2d+6l} = x^{2l_0-d}(uv)^{2g-2-2d+6l_0}\sum_{0\le l<+\infty}((uv)^6x^2)^l =\end{aligned}$$

$$= \frac{(uv)^{2g-2-2d+6l_0} x^{2l_0-d}}{1-(uv)^6 x^2};$$

$$D_k := \sum_{-\infty<j\leq k, d-j \text{ even}} x^{-j}(uv)^{g-1+(d-j)/2} = \sum_{(d-k)/2\leq l<+\infty} x^{2l-d}(uv)^{g-1+l} =$$

$$= \sum_{l_0\leq l<+\infty} x^{2l-d}(uv)^{g-1+l} = x^{2l_0-d}(uv)^{g-1+l_0} \sum_{0\leq l<+\infty} (uvx^2)^l =$$

$$= \frac{(uv)^{g-1+l_0} x^{2l_0-d}}{1-uvx^2}.$$

So we get:

$$\mathcal{HD}(G(\alpha(k)^-);3,d,1) = (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot$$
$$\cdot\Bigg\{ \frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \cdot \Bigg(\frac{(uv)^{2k}x^{-k}}{1-(uv)^{-2}x} +$$
$$- \frac{(uv)^{2g-2+d-3k}x^{-k}}{1-(uv)^3x} \Bigg) + \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)} \cdot \Bigg(\frac{(uv)^{2d-4l_0+1}x^{2l_0-d}}{1-(uv)^{-4}x^2} \cdot$$
$$\cdot \frac{1+(uv)^{-2}x}{1-(uv)^{-1}x} + \frac{(uv)^{2g-2-2d+6l_0}x^{2l_0-d}}{1-(uv)^6x^2} \cdot \frac{1+(uv)^3x}{1-(uv)^2x} +$$
$$- \frac{(uv)^{g-1+l_0}x^{2l_0-d}}{1-uvx^2} \cdot \frac{(1+uv)(1-uvx^2)}{(1-(uv)^{-1}x)(1-(uv)^2x)} \Bigg) \Bigg\}.$$

Therefore, by rearranging and simplifying we get:

Theorem 14.3.5. *For every smooth projective irreducible curve C of genus $g \geq 2$ and for every $d > 0$ and for every critical value*

$$\alpha(k) = (d-3k)/2, \quad 0 \leq k < d/3$$

the following formula holds for the Hodge-Deligne polynomial of the moduli space $G(\alpha(k)^-;3,d,1)$.

$$\mathcal{HD}(G(\alpha(k)^-;3,d,1)) = (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot$$
$$\cdot\Bigg\{ \frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)} \cdot \Bigg(\frac{(uv)^{2k}x^{-k}}{1-(uv)^{-2}x} +$$
$$- \frac{(uv)^{2g-2+d-3k}x^{-k}}{1-(uv)^3x} \Bigg) + \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)} \cdot \Bigg(\frac{(uv)^{2d-4l_0+1}x^{2l_0-d}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)} +$$
$$+ \frac{(uv)^{2g-2-2d+6l_0}x^{2l_0-d}}{(1-(uv)^3x)(1-(uv)^2x)} - \frac{(1+uv)(uv)^{g-1+l_0}x^{2l_0-d}}{(1-(uv)^{-1}x)(1-(uv)^2x)} \Bigg) \Bigg\}, \tag{14.10}$$

where $l_0 := \lceil (d-k)/2 \rceil$.

Remark 14.3.1. As a check for the correctness of the formula above, let us denote by p the previous polynomial and let us verify Poincaré duality for it. As in the previous chapter, we simply substitute u and v with their inverses and x with uvx.

$$p(u^{-1},v^{-1}) = (uv)^{-g}(1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+vx)^g(1+ux)^g}{(1-uvx)(1-x)}\cdot$$

$$\cdot\left\{\frac{(uv)^{-3g}(1+u^2v)^g(1+uv^2)^g-(uv)^{-2g}(1+u)^g(1+v)^g}{(uv)^{-4}(1-uv)^2((uv)^2-1)}\cdot\left(\frac{(uv)^{-3k}x^{-k}}{1-(uv)^3x}+\right.\right.$$

$$\left.-\frac{(uv)^{3k+2-2g-d}(uv)^{-k}x^{-k}}{1-(uv)^{-2}x}\right)+\frac{(uv)^{1-2g}(1+u)^g(1+v)^g}{(uv)^{-3}(1-uv)^2(1+uv)}\cdot$$

$$\cdot\left(\frac{(uv)^{4l_0-2d-1}(uv)^{2l_0-d}x^{2l_0-d}}{(1-(uv)^3x)(1-(uv)^2x)}+\frac{(uv)^{2+2d-2g-6l_0}(uv)^{2l_0-d}x^{2l_0-d}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)}+\right.$$

$$\left.\left.-\frac{(uv)^{-1}(1+uv)(uv)^{1-g-l_0}(uv)^{2l_0-d}x^{2l_0-d}}{(1-(uv)^2x)(1-(uv)^{-1}x)}\right)\right\}=$$

$$=(uv)^{-g}(1+u)^g(1+v)^g \operatorname*{coeff}_{x^0}\frac{(1+vx)^g(1+ux)^g}{(1-x)(1-uvx)}\cdot$$

$$\cdot\left\{(uv)^{4-3g}\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}(uv)^{2-2g-d}\cdot\right.$$

$$\cdot\left(\frac{(uv)^{2k}x^{-k}}{1-(uv)^{-2}x}-\frac{(uv)^{2g-2+d-3k}x^{-k}}{1-(uv)^3x}\right)+(uv)^{5-3g}\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot$$

$$\cdot(uv)^{1-2g-d}\left(\frac{(uv)^{2d-4l_0+1}x^{2l_0-d}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)}+\frac{(uv)^{2g-2-2d+6l_0}x^{2l_0-d}}{(1-(uv)^3x)(1-(uv)^2x)}+\right.$$

$$\left.\left.-\frac{(1+uv)(uv)^{g-1+l_0}x^{2l_0-d}}{(1-(uv)^{-1}x)(1-(uv)^2x)}\right)\right\}=(uv)^{-(6g+d-6)}p(u,v).$$

Now according to [BGMN], the moduli spaces $G(\alpha;3,d,1)$ are smooth for α non-critical and their dimension coincides with the expected dimension $\beta(3,d,1)=6g+d-6$, so the previous polynomials satisfy Poincaré duality.

Remark 14.3.2. Up to a multiplicative term $(1+u)^g(1+v)^g$ (see remark 13.3.1) the previous polynomials coincide with the ones described in [M, theorem 6.5] for the moduli spaces of stable triples, after setting $d_1:=d$, $d_2:=0$, $n_0:=d-k$, so that

$$\overline{n_0}=2\lfloor(n_0+1)/2\rfloor=2\lceil n_0/2\rceil=2\lceil(d-k)/2\rceil=2l_0.$$

As a corollary, we can compute the Hodge-Deligne polynomials for the moduli spaces of stable objects at any critical value:

$$G^{\mathrm{s}}(\alpha(k);3,d,1)\simeq G(\alpha(k)^+;3,d,1)\smallsetminus G^+(\alpha(k);3,d,1)\simeq$$

$$\simeq G(\alpha(k)^-;3,d,1)\smallsetminus G^-(\alpha(k);3,d,1).$$

This result will be used in the computation for the case $n=4,k=1$. The polynomials will have 2 different forms according to $d-k$ being odd or even. In both cases, we simply consider the difference between the formula of the previous theorem and the formula for the Hodge-Deligne polynomial of $G^{-}(\alpha(k);3,d,1)$ already written.

If $d-k$ is odd, then $l_0=\lceil (d-k)/2\rceil=(d-k+1)/2$, so by combining theorem 14.3.5 with lemma 14.3.1 we get:

$$\mathcal{HD}(G^{\mathrm{s}}(\alpha(k);3,d,1))=(1+u)^g(1+v)^g\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot$$
$$\cdot\left\{\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\cdot\left(\frac{(uv)^{2k}x^{-k}}{1-(uv)^{-2}x}+\right.\right.$$
$$\left.-\frac{(uv)^{2g-2+d-3k}x^{-k}}{1-(uv)^3x}\right)+\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot\left(\frac{(uv)^{2k-1}x^{1-k}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)}+\right.$$
$$\left.\left.+\frac{(uv)^{2g+d+1-3k}x^{1-k}}{(1-(uv)^3x)(1-(uv)^2x)}-\frac{(1+uv)(uv)^{g-1+(d-k+1)/2}x^{1-k}}{(1-(uv)^{-1}x)(1-(uv)^2x)}\right)\right\}+$$
$$-(1+u)^g(1+v)^g\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot[x^{-k}-(uv)^{2g-2+d-3k}x^{-k}]\cdot$$
$$\cdot\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}=$$
$$=(1+u)^g(1+v)^g\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot$$
$$\cdot\left\{\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\cdot\left(\frac{(uv)^{2k}x^{-k}}{1-(uv)^{-2}x}+\right.\right.$$
$$\left.-\frac{(uv)^{2g-2+d-3k}x^{-k}}{1-(uv)^3x}-x^{-k}+(uv)^{2g-2+d-3k}x^{-k}\right)+\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot$$
$$\cdot\left(\frac{(uv)^{2k-1}x^{1-k}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)}+\frac{(uv)^{2g+d+1-3k}x^{1-k}}{(1-(uv)^3x)(1-(uv)^2x)}+\right.$$
$$\left.\left.-\frac{(1+uv)(uv)^{g-1+(d-k+1)/2}x^{1-k}}{(1-(uv)^{-1}x)(1-(uv)^2x)}\right)\right\}.$$

Therefore, we get that:

Corollary 14.3.6. *For every curve C as before and for every critical value*

$$\alpha(k)=(d-3k)/2,\quad 0\leq k<d/3$$

such that $d-k$ is odd, the following formula holds:

$$\mathcal{HD}(G^{s}(\alpha(k);3,d,1))=(1+u)^g(1+v)^g\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot$$
$$\cdot\left\{\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\cdot\left(\frac{(uv)^{2k}x^{-k}}{1-(uv)^{-2}x}+\right.\right.$$

$$-\frac{(uv)^{2g+1+d-3k}x^{1-k}}{1-(uv)^3x} - x^{-k}\Bigg) + \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot$$
$$\cdot\left(\frac{(uv)^{2k-1}x^{1-k}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)} + \frac{(uv)^{2g+d+1-3k}x^{1-k}}{(1-(uv)^3x)(1-(uv)^2x)} + \right.$$
$$\left. - \frac{(1+uv)(uv)^{g-1+(d-k+1)/2}x^{1-k}}{(1-(uv)^{-1}x)(1-(uv)^2x)}\right)\Bigg\}.$$

As a check for the correctness of this formula, we can consider the case $k=0$, where we obtain the zero polynomial. This agrees with the fact that the space $G^{\mathrm{s}}(d/2;3,d,1)$ is empty since $d/2$ is the last critical value for $(3,d,1)$.

If $d-k$ is even, then $l_0 = \lceil (d-k)/2 \rceil = (d-k)/2$, so by combining theorem 14.3.5 with lemma 14.3.2 we get:

$$\mathcal{HD}(G^{\mathrm{s}}(\alpha(k);3,d,1)) = (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot$$
$$\cdot\left\{\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\cdot\left(\frac{(uv)^{2k}x^{-k}}{1-(uv)^{-2}x} + \right.\right.$$
$$\left. -\frac{(uv)^{2g-2+d-3k}x^{-k}}{1-(uv)^3x}\right) + \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot\left(\frac{(uv)^{2k+1}x^{-k}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)} + \right.$$
$$\left.\left. + \frac{(uv)^{2g+d-2-3k}x^{-k}}{(1-(uv)^3x)(1-(uv)^2x)} - \frac{(1+uv)(uv)^{g-1+(d-k)/2}x^{-k}}{(1-(uv)^{-1}x)(1-(uv)^2x)}\right)\right\} +$$
$$-(1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot\left\{\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2}\cdot\right.$$
$$\cdot\left(\frac{(uv)^{2g+d-3k}x^{1-k}}{1-(uv)^2x} - \frac{(uv)^{g-1+(d-k)/2}x^{-k}}{1-(uv)^{-1}x} + \frac{uvx^{-k}+(uv)^{2g-2+d-3k}x^{-k}}{1+uv}\right) +$$
$$+\frac{(1+u)^g(1+v)^g}{(1-uv)^2}\left[\frac{(uv)^kx^{-k}}{1-(uv)^{-1}x} - \frac{(uv)^{g+1+(d-3k)/2}x^{1-k}}{1-(uv)^2x} - x^{-k}\right] +$$
$$\left. +[1-(uv)^{2g-2+d-3k}]x^{-k}\cdot\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\right\} =$$
$$= (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot\left\{\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\cdot\right.$$
$$\cdot\left(\frac{(uv)^{2k}x^{-k}}{1-(uv)^{-2}x} - \frac{(uv)^{2g-2+d-3k}x^{-k}}{1-(uv)^3x} - x^{-k} + (uv)^{2g-2+d-3k}x^{-k}\right) + \frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot$$
$$\cdot\left(\frac{(uv)^{2k+1}x^{-k}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)} + \frac{(uv)^{2g+d-2-3k}x^{-k}}{(1-(uv)^3x)(1-(uv)^2x)} - \frac{(1+uv)(uv)^{g-1+(d-k)/2}x^{-k}}{(1-(uv)^{-1}x)(1-(uv)^2x)} + \right.$$
$$\left. -\frac{(uv)^{2g+d-3k}(1+uv)x^{1-k}}{1-(uv)^2x} + \frac{(uv)^{g-1+(d-k)/2}(1+uv)x^{-k}}{1-(uv)^{-1}x} - uvx^{-k} - (uv)^{2g-2+d-3k}x^{-k}\right) +$$

$$-\frac{(1+u)^g(1+v)^g}{(1-uv)^2}\left[\frac{(uv)^k x^{-k}}{1-(uv)^{-1}x}-\frac{(uv)^{g+1+(d-3k)/2}x^{1-k}}{1-(uv)^2 x}-x^{-k}\right]\Bigg\}.$$

Therefore, by rearranging we get that:

Corollary 14.3.7. *For every critical value*

$$\alpha(k)=(d-3k)/2,\quad 0\le k<d/3$$

such that $d-k$ is even, the following formula holds:

$$\mathcal{HD}(G^s(\alpha(k);3,d,1))=$$

$$=(1+u)^g(1+v)^g \operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot\Bigg\{\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\cdot$$

$$\cdot\left(\frac{(uv)^{2k}x^{-k}}{1-(uv)^{-2}x}-\frac{(uv)^{2g+1+d-3k}x^{1-k}}{1-(uv)^3 x}-x^{-k}\right)+\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot$$

$$\cdot\Bigg(\frac{(uv)^{2k+1}x^{-k}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)}+\frac{(uv)^{2g+d+4-3k}x^{2-k}}{(1-(uv)^3x)(1-(uv)^2x)}-\frac{(1+uv)(uv)^{g+1+(d-k)/2}x^{1-k}}{(1-(uv)^{-1}x)(1-(uv)^2x)}+$$

$$-uvx^{-k}\Bigg)-\frac{(1+u)^g(1+v)^g}{(1-uv)^2}\left[\frac{(uv)^k x^{-k}}{1-(uv)^{-1}x}-\frac{(uv)^{g+1+(d-3k)/2}x^{1-k}}{1-(uv)^2 x}-x^{-k}\right]\Bigg\}.$$

Also in this case, a check for the correctness of this formula is given by considering the case $k=0$, where we obtain the zero polynomial, that agrees with the fact that the space $G^{\mathrm{s}}(d/2;3,d,1)$ is empty.

Chapter 15

Case n=4, k=1

First of all, let us compute the critical values for the triple $(4, d, 1)$. By [BGMN, §2 and proposition 4.2], the non-zero virtual critical values are all in the set

$$\left\{ \frac{nd' - n'd}{n'k - nk'} \text{ s.t. } 0 \leq k' \leq k, \quad 0 < n' < n, \quad n'k \neq nk', \quad d' \in \mathbb{Z} \right\} \cap \left] 0, \frac{d}{n-k} \right[.$$

In our case, this gives

$$\left\{ \frac{4d' - n'd}{n' - 4k'} \text{ s.t. } k = 0, 1, \quad n' = 1, 2, 3, \quad d' \in \mathbb{Z} \right\} \cap \left] 0, \frac{d}{3} \right[.$$

So we have the following 6 types of non-zero virtual critical values:

(1) if $k' = 1, n' = 1$, $\{\frac{d}{3} - \frac{4}{3}d' \text{ s.t. } 0 < d' < \frac{d}{4}\}$;

(2) if $k' = 1, n' = 2$, $\{d - 2d' \text{ s.t. } \frac{d}{3} < d' < \frac{d}{2}\}$;

(3) if $k' = 1, n' = 3$, $\{3d - 4d' \text{ s.t. } \frac{2}{3}d < d' < \frac{3}{4}d\}$;

(4) if $k' = 0, n' = 3$, $\{\frac{4}{3}d' - d \text{ s.t. } \frac{3}{4}d < d' < d\}$;

(5) if $k' = 0, n' = 2$, $\{2d' - d \text{ s.t. } \frac{d}{2} < d' < \frac{2}{3}d\}$;

(6) if $k' = 0, n' = 1$, $\{4d' - d \text{ s.t. } \frac{d}{4} < d' < \frac{d}{3}\}$.

Now the first 3 sets coincide with the last 3 ones (the bijection is given by replacing d' in the first 3 sets by $d - d'$ in the last 3 ones), so we need to consider only the first 3 sets. The best way of parametrizing all such objects at the same time is to consider the set

$$\left\{ \alpha(j) := \frac{d - 2j}{3} \text{ s.t. } 0 < j < \frac{d}{2} \right\} . \tag{15.1}$$

Since we will also need to cross the value $d/3$, we will consider also $\alpha(0) = d/3$ as a critical value. Such a set contains (1), (2) and (3) by setting respectively $j = 2d'$, $j = 3d' - d$ and $j = 6d' - 4d$. According to this parametrization and to the identification of (1), (2), (3) with (4), (5), (6) we will get that:

- the cases (1) and (4) are possible only for those values of $0 \leq j < d/2$ such that $j \equiv 0 \bmod 2$;
- the cases (2) and (5) are possible only for those values of $0 \leq j < d/2$ such that $j \equiv -d \equiv 2d \bmod 3$;
- the cases (3) and (6) are possible for those values of $0 \leq j < d/2$ such that $j \equiv -4d \equiv 2d \bmod 6$.

In particular, if we rewrite the previous relations mod 6, we get that the class of j mod 6 can only belong to the set $\{0,2,4,2d,2d+3\}_{\text{mod } 6} = \{0,2,4,2d+3\}_{\text{mod } 6}$. In particular, this proves that at most 4 of the 6 classes of j modulo 6 are obtained, so the set (15.1) is overabundant. In particular,

- if $d \equiv 0 \bmod 3$ (i.e. $d \equiv 0,3 \bmod 6$) then all these values coincide with $\{0,2,3,4\}$;
- if $d \equiv 1 \bmod 3$ (i.e. $d \equiv 1,4 \bmod 6$) then all these values coincide with $\{0,2,4,5\}$;
- if $d \equiv 2 \bmod 3$ (i.e. $d \equiv 2,5 \bmod 6$) then all these values coincide with $\{0,1,2,4\}$.

So in all the various cases not all the possible values of j correspond to actual critical values. In particular, we will be interested in crossing the critical values in the interval $[\alpha_T, d/3]$. By [BGMMN, lemma 2.4], α_T is given by $(d-4)/3$ whenever $d \geq 4$. Therefore the critical values we are interested in are given by

$$\alpha_T = \frac{d-4}{3} = \alpha(2) < \frac{d-2}{3} = \alpha(1) < \frac{d}{3} = \alpha(0).$$

In other terms, we will be interested in particular in $j \in \{0,1,2\}$. The values $0,2$ are always obtained in all the 3 cases, while the value $j = 1$ is obtained only in the case $d \equiv 2 \bmod 3$.

15.1 The moduli spaces $G^+(\alpha(j); 4, d, 1)$

Let us fix any critical value $\alpha(j) = (d-2j)/3$ with $0 \leq j < d/2$ and let us consider any object $(E,V) \in G^+(\alpha(j); 4, d, 1)$ (if $j = 0$, we will obtain the empty set and the zero polynomial, so this will not give any problem for our computation). Since $n = 4$, then all the objects of $G^+(\alpha(j); 4, d, 1)$ have length r of the $\alpha(j)$-Jordan-Hölder filtration equal to $2,3$ or 4. So let us consider the 3 different cases separately.

15.1.1 Case $r = 2$

By applying lemma 1.0.6, any (E,V) in $G^+(\alpha(j); 4, d, 1)$ with length of the α_c-JHF equal to 2 sits in a non-split exact sequence:

$$0 \to (Q_1, W_1) \to (E,V) \to (Q_2, W_2) \to 0 \tag{15.2}$$

with conditions (a)-(b). Then condition (a) implies that $k_1 = 0$, but a priori n_1 can be either equal to $1, 2$ or 3.

(1) If $n_1 = 3$, then $n_2 = 1$; since $\mu_{\alpha(j)}(E,V) = d/3 - j/6$, then condition (b) implies that $d_1 = d - j/2$. Therefore, *this case is possible only if* $j \equiv 0 \bmod 2$. If we assume that condition, then both $d_1 = d - j/2$ and $d_2 = j/2$ are non-negative integers. Since $r = 2$, we must impose that both $(Q_1, W_1) = (Q_1, 0)$ and (Q_2, W_2) are $\alpha(j)$-stable. Since there are no critical values for $(3, d_1, 0)$ and $(1, d_2, 1)$, this simply means that we are considering all pairs $(Q_1, 0), (Q_2, W_2)$ such that:

$$(Q_1, 0) \in M^{\mathrm{s}}(3, d - j/2) = G_1, \quad (Q_2, W_2) \in G(1, j/2, 1) = G_2.$$

Since $\mathbb{H}^0_{21} = \mathbb{H}^2_{21} = 0$, we get

$$\begin{aligned} \dim \mathrm{Ext}^1((Q_2, W_2), (Q_1, 0)) &= C_{21} = \\ &= n_1 n_2 (g-1) - d_1 n_2 + d_2 n_1 + k_2 d_1 - k_2 n_1 (g-1) - k_1 k_2 = \\ &= 3(g-1) - (d - j/2) + 3j/2 + (d - j/2) - 3(g-1) = 3j/2. \end{aligned}$$

Then we can apply proposition 5.0.5 for $r = 2$. So for every critical value $\alpha(j)$ such that $j \equiv 0 \bmod 2$ we get a contribution to $G^+(\alpha(j);4,d,1)$ by a projective bundle over $G_1 \times G_2$ with fibers isomorphic to $\mathbb{P}^{3j/2-1}$. So we get the polynomial:

$$p_1^{j \equiv_2 0} := \mathcal{HD}(M^s(3, d - j/2)) \frac{1 - (uv)^{3j/2}}{1 - uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g (1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}.$$

If $j = 0$, this is the zero polynomial, as it should be.

Now we recall that we are assuming that j is even, so:

- $d - j/2 \equiv 0 \bmod 3$ if and only if $j \equiv 2d \bmod 6$;
- $d - j/2 \not\equiv 0 \bmod 3$ if and only if $j \equiv 2d + 2 \bmod 6$ or $j \equiv 2d + 4 \bmod 6$.

In the first case, we don't know an explicit formula for the Hodge-Deligne polynomial of $M^s(3, d - j/2)$; in the second case we have an explicit formula, as described in chapter 8 (and such a formula does not depend on j or d). We will denote the corresponding 2 polynomials by $\mathcal{HD}(M(3, j \equiv_6 2d))$ and $\mathcal{HD}(M^{\mathrm{s}}(3, j \equiv_6 2d+2)) = \mathcal{HD}(M^{\mathrm{s}}(3, j \equiv_6 2d+4))$ respectively. According to that notation, we denote by $p_1^{j \equiv_6 2d}$ and $p_1^{j \equiv_6 2d+2} = p_1^{j \equiv_6 2d+4}$ the corresponding polynomials.

(2) If $n_1 = 2$, then $n_2 = 2$. Moreover, condition (b) implies that $d_1 = (2d - j)/3$, so *this case is possible only if* $j \equiv 2d \bmod 3$, that is $j \in \{2d, 2d+3\}_{\bmod 6}$. If we assume that condition, then both d_1 and $d_2 = d - d_1 = (d + j)/3$ are non-negative integers.

Now we recall that both $(Q_1, 0)$ and (Q_2, W_2) must be strictly $\alpha(j)$-stable (otherwise, the length of the Jordan-Hölder filtration would be bigger than 2). So we need to consider 2 cases:

(a) if $j \equiv 2d \bmod 6$, then $d_1 = (2d-j)/3$ is even, so we are considering

$$(Q_1, 0) \in M^{\mathrm{s}}\left(2, \frac{2d-j}{3}\right) = M^{\mathrm{s}}(2, \mathrm{even}) =: G_1;$$

(b) if $j \equiv 2d+3 \bmod 6$, then $d_1 = (2d-j)/3$ is odd, so we are considering

$$(Q_1, 0) \in M^{\mathrm{s}}\left(2, \frac{2d-j}{3}\right) = M^{\mathrm{s}}(2, \mathrm{odd}) = M^{\mathrm{ss}}(2, \mathrm{odd}) =: G_1'.$$

Analogously, (Q_2, W_2) must be an object of the moduli space $G^{\mathrm{s}}(\alpha(j); 2, (d+j)/3, 1)$. Such a scheme is not empty if and only if $0 < \alpha(j) < (d+j)/3$, but this condition is automatically satisfied by definition of $\alpha(j)$ for all $j > 0$ (for $j = 0$ the moduli space of semistable objects is non-empty, while the stable locus is empty). Then we have to verify if $\alpha(j)$ is critical for the triple $(2, (d+j)/3, 1)$. According to the computations of chapter 13, $\alpha(j)$ is critical for such a triple if and only if $\alpha(j) = (d+j)/3 - 2k$ for some $0 \leq k < (d+j)/6$. So this gives:

$$\frac{d-2j}{3} = \alpha(j) = \frac{d+j}{3} - 2k = \frac{d+j-6k}{3} \quad \Leftrightarrow \quad j = 2k.$$

So $\alpha(j)$ is critical for $(2, (d+j)/3, 1)$ if and only if $j = 2k$ for some $0 \leq k < (d+j)/6$. If we set $j = 2k$, this is equivalent to imposing $0 \leq j < (d+j)/3$, that is equivalent to $0 \leq j < d/2$. These are exactly the conditions we already put on j, so $\alpha(j)$ is critical for $(2, (d+j)/3, 1)$ if and only if j is any admissible value (i.e. $0 \leq j < d/2$) such that $j \equiv 0 \bmod 2$. Now we have to distinguish 2 cases as follows.

(i) If $j \equiv 0 \bmod 2$, then $(d-2j)/3$ is a critical value for $(2, (d+j)/3, 1)$. In particular, if we set $k := j/2 \in \mathbb{N}_0$, then we can write $(d-2j)/3 = (d+j)/3 - 2k$ and we need to consider

$$(Q_2, W_2) \in G^{\mathrm{s}}\left(\frac{d+j}{3} - 2k; 2, \frac{d+j}{3}, 1\right) =: G_2.$$

According to corollary 13.3.2 with d replaced by $(d+j)/3$ and k replaced by $j/2$, we have that

$$\mathcal{HD}(G_2) = \frac{(1+u)^g(1+v)^g}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} \cdot$$
$$\cdot \left[\frac{(uv)^{j/2}x^{-j/2}}{1-x(uv)^{-1}} - \frac{(uv)^{g+1+(d-2j)/3}x^{1-j/2}}{1-x(uv)^2} - x^{-j/2}\right].$$

By remark 13.3.4, this is the zero polynomial if $k = 0$, i.e. if $j = 0$.

(ii) If $j \equiv 1 \bmod 2$, then we can define $k := (j-1)/2 \in \mathbb{N}_0$, so that

$$\frac{d-2j}{3} = \frac{d+j}{3} - 2k - 1.$$

Then if we recall that the critical values of $(2, (d+j)/3, 1)$ are of the form $(d+j)/3-2k$, we need to consider

$$(Q_2, W_2) \in G^{\mathrm{s}}\left(\frac{d+j}{3} - 2k - 1; 2, \frac{d+j}{3}, 1\right) =$$
$$= G^{\mathrm{s}}\left(\frac{d+j}{3} - 2k - \varepsilon; 2, \frac{d+j}{3}, 1\right) =: G_2'.$$

According to theorem 13.3.1 with d replaced by $(d+j)/3$ and k replaced by $(j-1)/2$, we have that

$$\mathcal{HD}(G_2') = \frac{(1+u)^g(1+v)^g}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} \cdot$$
$$\cdot \left[\frac{(uv)^{(j-1)/2}x^{(1-j)/2}}{1-x(uv)^{-1}} - \frac{(uv)^{g+(d-2j)/3}x^{(1-j)/2}}{1-x(uv)^2}\right].$$

We recall that we are under the hypothesis $j \in \{2d, 2d+3\}_{\mathrm{mod}\ 6}$. Therefore, under that condition we have $j \equiv 0 \bmod 2$ if and only if $j \equiv 2d \bmod 6$ and $j \equiv 1 \bmod 2$ if and only if $j \equiv 2d+3 \bmod 6$. So cases (a) and (b) match with cases (i) and (ii) respectively.

Since $\mathbb{H}^0_{21} = \mathbb{H}^2_{21} = 0$ for all values of j, we get:

$$\dim \operatorname{Ext}^1((Q_2, W_2), (Q_1, 0)) = C_{21} =$$
$$= n_1 n_2 (g-1) - d_1 n_2 + d_2 n_1 + k_2 d_1 - k_2 n_1 (g-1) - k_1 k_2 =$$
$$= 4(g-1) - 2\tfrac{2d-j}{3} + 2\tfrac{d+j}{3} + \tfrac{2d-j}{3} - 2(g-1) = 2g-2+j.$$

So for every critical value $\alpha(j)$ such that $j \equiv 2d \bmod 3$, we get a contribution to $G^+(\alpha(j); 4, d, 1)$ by:

- a projective bundle over $G_1 \times G_2$ with fibers isomorphic to $\mathbb{P}^{2g-3+j}$ if $j \equiv 2d \bmod 6$;
- a projective bundle over $G_1' \times G_2'$ with the same fibers if $j \equiv 2d+3 \bmod 6$;

So we get the polynomials

$$p_2^{j\equiv_6 2d} = \frac{1}{2(1-uv)(1-(uv)^2)}\Big(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g +$$
$$-(1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1}-u^2v^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2\Big) \cdot$$
$$\cdot \frac{(1+u)^g(1+v)^g}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} \cdot$$
$$\cdot \left[\frac{(uv)^{j/2}x^{-j/2}}{1-x(uv)^{-1}} - \frac{(uv)^{g+1+(d-2j)/3}x^{1-j/2}}{1-x(uv)^2} - x^{-j/2}\right] \cdot \frac{1-(uv)^{2g-2+j}}{1-uv}$$

and

$$p_2^{j\equiv_6 2d+3} = \frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^{2g}(1+v)^{2g}}{(1-uv)(1-(uv)^2)}\cdot$$

$$\cdot\frac{(1+u)^g(1+v)^g}{1-uv}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot$$

$$\cdot\left[\frac{(uv)^{(j-1)/2}x^{(1-j)/2}}{1-x(uv)^{-1}}-\frac{(uv)^{g+(d-2j)/3}x^{(1-j)/2}}{1-x(uv)^2}\right]\cdot\frac{1-(uv)^{2g-2+j}}{1-uv}$$

according to the 2 possible values of j modulo 6.

If $j=0$ (and $d\equiv 0 \bmod 3$), then we are in the first case and the associated polynomial p_2 is the zero polynomial, as it should be.

(3) If $n_1=1$, then $n_2=3$. Moreover, condition (b) implies that $d_1=(2d-j)/6$, so *this case is possible only if* $j\equiv 2d \bmod 6$. If we assume that condition, then both d_1 and $d_2=d-d_1=(4d+j)/6$ are non-negative integers.

Now both $(Q_1,0)$ and (Q_2,W_2) must be strictly $\alpha(j)$-stable. For $(Q_1,0)$, this simply amounts to considering all possible $(Q_1,0)\in J^{(2d-j)/6}=:G_1$. On the other hand, (Q_2,W_2) must be an object of the moduli space $G^{\mathrm{s}}(\alpha(j);3,(4d+j)/6,1)$. Such a scheme is non-empty if and only if $\alpha(j)<(4d+j)/12$, but this condition is automatically satisfied by definition of $\alpha(j)$ for all $j>0$ (if $j=0$, the semistable locus is non-empty, while the stable locus is empty). Then we have to verify if $\alpha(j)$ is critical for the triple $(3,(4d+j)/6,1)$. According to the computations of chapter 14, $\alpha(j)$ is critical for such a triple if and only if $\alpha(j)=d_2/2-3k/2=(4d+j)/12-3k/2$ for some $0\le k<d_2/3=(4d+j)/18$. So this gives:

$$\frac{d-2j}{3}=\alpha(j)=\frac{4d+j}{12}-\frac{3k}{2}=\frac{4d+j-18k}{12}\quad\Leftrightarrow\quad j=2k.$$

So $\alpha(j)$ is critical for $(3,d_2,1)$ if and only if $j=2k$ for some $0\le k<(4d+j)/18$. If we set $j:=2k$, this is equivalent to imposing $0\le j<(4d+j)/9$. These conditions are equivalent to $0\le j<d/2$, that are exactly the conditions we already put on j. Therefore, $\alpha(j)$ is critical for $(3,d_2,1)$ if and only if j is any admissible value (i.e. $0\le j<d/2$) such that $j\equiv 0 \bmod 2$. But we recall that the case we are considering is possible only when $j\equiv 2d \bmod 6$, that implies $j\equiv 0 \bmod 2$. Therefore, when this case is possible, $\alpha(j)$ is always critical for $(3,d_2,1)$. Then if we define $k:=j/2\in\mathbb{N}_0$, we need to consider

$$(Q_2,W_2)\in G^{\mathrm{s}}\left(\frac{1}{2}\left(\frac{4d+j}{6}-3k\right);3,\frac{4d+j}{6},1\right)=:G_2.$$

Now we recall that according to chapter 14 we have 2 different formulae for the Hodge-Deligne polynomial of $G^{\mathrm{s}}((d'-3k')/2;3,d',1)$ depending on $d'-k'$ being odd or even. In our case $d'=(4d+j)/6$ and $k'=j/2$, so

$$d'-k'=\frac{4d+j}{6}-\frac{j}{2}=\frac{4d-2j}{6}=\frac{2d-j}{3}.$$

Since in this case we are assuming $j\equiv 2d \bmod 6$, then $d'-k'$ is even, so we can apply corollary 14.3.7 with d replaced by $(4d+j)/6$ and k replaced by $j/2$ and we get

$$\begin{gathered}
\mathcal{HD}(G_2)=(1+u)^g(1+v)^g\cdot\\
\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot\left\{\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\cdot\right.\\
\cdot\left(\frac{(uv)^jx^{-j/2}}{1-(uv)^{-2}x}-\frac{(uv)^{2g+1+(2d-4j)/3}x^{1-j/2}}{1-(uv)^3x}-x^{-j/2}\right)+\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot\\
\cdot\left(\frac{(uv)^{j+1}x^{-j/2}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)}+\frac{(uv)^{2g+4+(2d-4j)/3}x^{2-j/2}}{(1-(uv)^3x)(1-(uv)^2x)}-\frac{(1+uv)(uv)^{g+1+(2d-j)/6}x^{1-j/2}}{(1-(uv)^{-1}x)(1-(uv)^2x)}+\right.\\
\left.\left.-uvx^{-j/2}\right)-\frac{(1+u)^g(1+v)^g}{(1-uv)^2}\left[\frac{(uv)^{j/2}x^{-j/2}}{1-(uv)^{-1}x}-\frac{(uv)^{g+1+(d-3j)/3}x^{1-j/2}}{1-(uv)^2x}-x^{-j/2}\right]\right\}.
\end{gathered}$$

Now for all values of j we have that $\mathbb{H}^0_{21}=\mathbb{H}^2_{21}=0$, so:

$$\begin{aligned}
\dim \operatorname{Ext}^1((Q_2,W_2),(Q_1,0))=C_{21}=\\
=n_1n_2(g-1)-d_1n_2+d_2n_1+k_2d_1-k_2n_1(g-1)-k_1k_2=\\
=3(g-1)-3\tfrac{2d-j}{6}+\tfrac{4d+j}{6}+\tfrac{2d-j}{6}-(g-1)=2g-2+j/2.
\end{aligned}$$

Then we get a projective bundle over $G_1\times G_2$ with fibers isomorphic to $\mathbb{P}^{2g-3+j/2}$. So we get the polynomial

$$\begin{gathered}
p_3^{j\equiv_6 2d}:=\mathcal{HD}(G_1)\mathcal{HD}(G_2)\mathcal{HD}(\mathbb{P}^{2g-3+j/2})=\\
=(1+u)^g(1+v)^g\frac{1-(uv)^{2g-2+j/2}}{1-uv}\mathcal{HD}(G_2)=\\
=(1+u)^{2g}(1+v)^{2g}\frac{1-(uv)^{2g-2+j/2}}{1-uv}\cdot\\
\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot\left\{\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\cdot\right.\\
\cdot\left(\frac{(uv)^jx^{-j/2}}{1-(uv)^{-2}x}-\frac{(uv)^{2g+1+(2d-4j)/3}x^{1-j/2}}{1-(uv)^3x}-x^{-j/2}\right)+\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot\\
\cdot\left(\frac{(uv)^{j+1}x^{-j/2}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)}+\frac{(uv)^{2g+4+(2d-4j)/3}x^{2-j/2}}{(1-(uv)^3x)(1-(uv)^2x)}-\frac{(1+uv)(uv)^{g+1+(2d-j)/6}x^{1-j/2}}{(1-(uv)^{-1}x)(1-(uv)^2x)}+\right.\\
\left.\left.-uvx^{-j/2}\right)-\frac{(1+u)^g(1+v)^g}{(1-uv)^2}\left[\frac{(uv)^{j/2}x^{-j/2}}{1-(uv)^{-1}x}-\frac{(uv)^{g+1+(d-3j)/3}x^{1-j/2}}{1-(uv)^2x}-x^{-j/2}\right]\right\}.
\end{gathered}$$

15.1.2 Case $r = 3$

In this case the graded of (E, V) is necessarily made of 3 objects of the form $(Q_1, 0)$, $(Q_2, 0)$, (Q_3, W_3) (a priori not necessarily in this order) where 2 of the Q_i's are line bundles and one is a vector bundle of rank 2. So we need to consider 3 possibilities.

(a) If Q_3 is the vector bundle of rank 2, then necessarily Q_1 and Q_2 are line bundles of the same degree $d_1 = d_2 = (2d - j)/6$ and Q_3 has degree $d_3 = (d + j)/3$.

(b) If Q_3 is a line bundle, then we get that $d_3 = j/2$; if Q_1 is a line bundle, then it has degree $d_1 = (2d - j)/6$ and Q_2 is a vector bundle of rank 2 and degree $d_2 = (2d - j)/3$.

(c) If Q_3 is a line bundle, then we get that $d_3 = j/2$; if Q_2 is a line bundle, then it has degree equal to $d_2 = (2d - j)/6$ and Q_1 is a vector bundle of rank 2 and degree $d_1 = (2d - j)/3$.

Therefore, *all the* 3 *cases are possible only when* $j \equiv 2d \bmod 6$. For each case we have to consider 2 different subcases according to the various $\alpha(j)$-canonical filtrations.

(1) Unique $\alpha(j)$-Jordan-Hölder filtration. If the filtration is unique, we need to fix the order of the 3 objects of the graded. The object (Q_3, W_3) must be necessarily the last object of the graded, otherwise it destabilizes (E, V) for $\alpha(j)^+$. Therefore we have the following possibilities.

(1a) Let us suppose that the graded is given by $(Q_1, 0) \oplus (Q_2, 0) \oplus (Q_3, W_3)$ with:

- Q_1 and Q_2 both line bundles of degree $d_1 = d_2 = (2d - j)/6$;
- (Q_3, W_3) with Q_3 vector bundle of rank 2 and degree $d_3 = (d + j)/3$.

In this case $\mathrm{Hom}\,((Q_3, W_3), (Q_2, 0)) = 0$ by lemma 1.0.4: indeed both objects are α_c-stable with the same slope and they are not isomorphic since their types are different. Then we have to consider two subcases as follows

(1a-i) If we suppose that $(Q_1, 0) \not\simeq (Q_2, 0)$, then we can apply proposition 6.1.2 in order to parametrize all the corresponding (E, V)'s. In this case $\mathrm{Hom}((Q_2, 0), (Q_1, 0)) = 0$, so the invariant a of that proposition can only assume the value

$$a = \dim \mathrm{Ext}^1((Q_2, 0), (Q_1, 0)) = C_{21} = n_1 n_2 (g - 1) - d_1 n_2 + d_2 n_1 = g - 1$$

on the set $U_a = G_1 \times G_2 \smallsetminus \Delta_{12}$. So we have a projective bundle R_a over U_a with fibers isomorphic to $\mathbb{P}^{a-1} = \mathbb{P}^{g-2}$. If we write $E_2 = (E_2, 0)$ for any non-split extension of Q_2 by Q_1, we get that E_2 is a vector bundle of rank $N_2 = 2$ and degree $D_2 = 2d_1 = (2d - j)/3$. Moreover, $\mathrm{Ext}^2((Q_3, W_3), (E_2, 0)) = 0$ because $k_3 = 1$ and also $\mathrm{Hom}(-, -) = 0$; therefore we get that the invariant b can assume only the value:

$$b = \dim \mathrm{Ext}^1((Q_3, W_3), (E_2, 0)) =$$

$$= N_2n_3(g-1) - D_2n_3 + d_3N_2 + k_3D_2 - k_3N_2(g-1) =$$
$$= 4(g-1) - 2D_2 + 2d_3 + D_2 - 2(g-1) = 2(g-1) - D_2 + 2d_3 =$$
$$= 2g-2-(2d-j)/3 + 2(d+j)/3 = 2g-2+j.$$

Moreover, the invariant c can only assume the value:

$$c = \dim \operatorname{Ext}^1((Q_3,W_3),(Q_1,0)) = C_{31} =$$
$$= n_1n_3(g-1) - d_1n_3 + d_3n_1 + k_3d_1 - k_3n_1(g-1) =$$
$$= 2(g-1) - 2d_1 + d_3 + d_1 - (g-1) = g-1-d_1+d_3 =$$
$$= g-1-(2d-j)/6 + (d+j)/3 = g-1+j/2.$$

Therefore, we can assume that $U_{a,b,c} = R_a \times G_3$ and we have a bundle $R_{a,b,c}$ over $U_{a,b,c}$ with fibers isomorphic to $\mathbb{P}^{b-1} \setminus \mathbb{P}^{c-1} = \mathbb{P}^{2g-3+j} \setminus \mathbb{P}^{g-2+j/2}$, that parametrizes all the (E,V)'s under consideration. We recall that $G_1 = G_2 = J^{(d-2j)/6}C$. Since $j \equiv 2d \bmod 6$, then we can define $k := j/2 \in \mathbb{N}_0$; since $0 \leq j < d/2$, then we get that $0 \leq k < (d+j)/6$. So we are considering all the (Q_3,W_3)'s in the scheme

$$G_3 = G^{\mathrm{s}}\left(\frac{d-2j}{3};2,\frac{d+j}{3},1\right) = G^{\mathrm{s}}\left(\frac{d+j}{3}-2k;2,\frac{d+j}{3},1\right).$$

Then we can use corollary 13.3.2 with d replaced by $(d+j)/3$ and k replaced by $j/2$ and we get that

$$\mathcal{HD}(G_3) = \frac{(1+u)^g(1+v)^g}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot$$
$$\cdot\left[\frac{(uv)^{j/2}x^{-j/2}}{1-x(uv)^{-1}} - \frac{(uv)^{g+(d-2j)/3+1}x^{1-j/2}}{1-x(uv)^2} - x^{-j/2}\right]. \tag{15.3}$$

Then we get the Hodge-Deligne polynomial

$$p_4^{j\equiv_6 2d} := \mathcal{HD}(R_{a,b,c}) = \left(\mathcal{HD}(\mathbb{P}^{2g-3+j}) - \mathcal{HD}(\mathbb{P}^{g-2+j/2})\right)\mathcal{HD}(G_3)\mathcal{HD}(R_a) =$$
$$= \left(\mathcal{HD}(\mathbb{P}^{2g-3+j}) - \mathcal{HD}(\mathbb{P}^{g-2+j/2})\right)\mathcal{HD}(\mathbb{P}^{g-2})\cdot$$
$$\cdot\mathcal{HD}(G_3)\cdot\mathcal{HD}(G_1)\left(\mathcal{HD}(G_1)-1\right) =$$
$$= \frac{(uv)^{g-1+j/2}-(uv)^{2g-2+j}}{1-uv}\cdot\frac{1-(uv)^{g-1}}{1-uv}\cdot\frac{(1+u)^g(1+v)^g}{1-uv}\cdot$$
$$\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot\left[\frac{(uv)^{j/2}x^{-j/2}}{1-x(uv)^{-1}} - \frac{(uv)^{g+(d-2j)/3+1}x^{1-j/2}}{1-x(uv)^2} - x^{-j/2}\right]\cdot$$
$$\cdot(1+u)^g(1+v)^g((1+u)^g(1+v)^g-1).$$

By using remark 13.3.4 we get that if $j=0$ (that is, if $k=0$), then p_4 is the zero polynomial.

(1a-ii) Let us suppose that $(Q_1,0) \simeq (Q_2,0)$. Since $k_3 = 1$, then we get that $\mathrm{Ext}^2((Q_3,W_3),(Q_1,0)) = 0$. So we can apply proposition 6.1.4 in order to parametrize all the corresponding (E,V)'s. In this case we need to compute the invariants:

$$a = \dim\ \mathrm{Ext}^1((Q_2,0),(Q_1,0)) = C_{21} + 1 = g$$

and

$$b = \dim\ \mathrm{Ext}^1((Q_3,W_3),(Q_1,0)) = C_{31} = g - 1 + j/2.$$

Therefore, we get a projective bundle R_a over $G_1 = G_2$ with fibers isomorphic to $\mathbb{P}^{g-1}$; the (E,V)'s we are interested in are parametrized by a bundle $R_{a,b}$ over $R_a \times G_3$ with fibers isomorphic to $\mathbb{C}^{g-2+j/2} \times \mathbb{P}^{g-2+j/2}$. In this case the schemes G_1 and G_3 are as in case (1a-i), so we get the polynomial:

$$\begin{aligned}
p_5^{j\equiv_6 2d} &:= \mathcal{HD}(G_1)\mathcal{HD}(G_3)\mathcal{HD}(\mathbb{C}^{g-2+j/2})\mathcal{HD}(\mathbb{P}^{g-2+j/2})\mathcal{HD}(\mathbb{P}^{g-1}) = \\
&= (1+u)^g(1+v)^g\frac{(1+u)^g(1+v)^g}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} \cdot \\
&\cdot \left[\frac{(uv)^{j/2}x^{-j/2}}{1-x(uv)^{-1}} - \frac{(uv)^{g+(d-2j)/3+1}x^{1-j/2}}{1-x(uv)^2} - x^{-j/2}\right] \cdot \\
&\cdot (uv)^{g-2+j/2}\frac{1-(uv)^{g-1+j/2}}{1-uv} \cdot \frac{1-(uv)^g}{1-uv}.
\end{aligned}$$

As in the proof for p_4, also p_5 is zero if $j = 0$.

(1b) Let us suppose that the graded is given by $(Q_1,0) \oplus (Q_2,0) \oplus (Q_3,W_3)$, where:

- Q_1 is a line bundle of degree $d_1 = (2d-j)/6$;
- Q_2 is a vector bundle of rank 2 and degree $d_2 = (2d-j)/3$;
- Q_3 is a line bundle of degree $d_3 = j/2$.

Since Q_1 is a line bundle and Q_2 is a vector bundle of rank 2, then these 2 coherent systems cannot be isomorphic; since they are both α_c-stable, then lemma 1.0.4 implies that $\mathrm{Hom}((Q_2,0),(Q_1,0)) = 0$. Moreover, we have also that $\mathrm{Hom}\,((Q_3,W_3),(Q_2,0)) = 0$ because both objects are α_c-stable with the same slope and they are not isomorphic. Then we can apply proposition 6.1.2 in order to parametrize all the corresponding (E,V)'s. In this case the invariant a can only assume the value

$$\begin{aligned}
a = \dim\ \mathrm{Ext}^1((Q_2,0),(Q_1,0)) &= C_{21} = n_1n_2(g-1) - d_1n_2 + d_2n_1 = \\
&= 2(g-1) - \frac{2d-j}{3} + \frac{2d-j}{3} = 2g-2
\end{aligned}$$

on the set $U_a = G_1 \times G_2$. So we get a projective bundle R_a over U_a with fibers isomorphic to $\mathbb{P}^{2g-3}$. If we write $E_2 = (E_2,0)$ for any extension of Q_2 by Q_1, we get that $N_2 = 3$ and

$D_2 = d_1 + d_2 = d - j/2$. Moreover, $\mathrm{Ext}^2((Q_3,W_3),(E_2,0)) = 0$ because $k_3 = 1$ and also $\mathrm{Hom}(-,-) = 0$; therefore we get that the invariant b can assume only the value:

$$\begin{aligned} b &= \dim\,\mathrm{Ext}^1((Q_3,W_3),(E_2,0)) = \\ &= N_2 n_3(g-1) - D_2 n_3 + d_3 N_2 + k_3 D_2 - k_3 N_2(g-1) = \\ &= 3(g-1) - D_2 + 3d_3 + D_2 - 3(g-1) = 3d_3 = 3j/2. \end{aligned}$$

Moreover, the invariant c can only assume the value:

$$\begin{aligned} c &= \dim\,\mathrm{Ext}^1((Q_3,W_3),(Q_1,0)) = C_{31} = \\ &= n_1 n_3(g-1) - d_1 n_3 + d_3 n_1 + k_3 d_1 - k_3 n_1(g-1) = \\ &= (g-1) - d_1 + d_3 + d_1 - (g-1) = d_3 = j/2. \end{aligned}$$

Therefore, we get that $U_{a,b,c} = R_a \times G_3$ and we get a bundle $R_{a,b,c}$ over $U_{a,b,c}$ with fibers isomorphic to $\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1} = \mathbb{P}^{3j/2-1} \smallsetminus \mathbb{P}^{j/2-1}$. Now the objects (Q_i,W_i)'s vary in the following sets:

$$(Q_1,0) \in G_1 = J^{(2d-j)/6}C, \quad (Q_2,0) \in G_2 = M^{\mathrm{s}}(2,(2d-j)/3),$$
$$(Q_3,W_3) \in G_3 = G(1,j/2,1).$$

Since we are assuming that $j \equiv 2d \bmod 6$, then $(2d-j)/3$ is even, so for the scheme G_2 we need to use formula (8.11). Then we get the Hodge-Deligne polynomial

$$\begin{aligned} p_6^{j\equiv_6 2d} &:= \mathcal{HD}(R_{a,b,c}) = \Big(\mathcal{HD}(\mathbb{P}^{3j/2-1}) - \mathcal{HD}(\mathbb{P}^{j/2-1})\Big)\mathcal{HD}(G_3)\mathcal{HD}(R_a) = \\ &= \frac{(uv)^{j/2} - (uv)^{3j/2}}{1-uv} \cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)} \cdot \\ &\quad \cdot \mathcal{HD}(\mathbb{P}^{2g-3}) \cdot \mathcal{HD}(J^{(2d-j)/6}C) \cdot \mathcal{HD}(M^{\mathrm{s}}(2,\mathrm{even})) = \\ &= \frac{(uv)^{j/2} - (uv)^{3j/2}}{1-uv} \cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)} \cdot \frac{1-(uv)^{2g-2}}{1-uv} \cdot \\ &\quad \cdot (1+u)^g(1+v)^g \cdot \frac{1}{2(1-uv)(1-(uv)^2)}\Big(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g + \\ &\quad -(1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1} - u^2v^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2\Big). \end{aligned}$$

Also in this case, if $j = 0$, then we get the zero polynomial.

(1c) Let us suppose that the graded is given by $(Q_1,0) \oplus (Q_2,0) \oplus (Q_3,W_3)$ where:

- Q_1 is a vector bundle of rank 2 and degree $d_1 = (2d-j)/3$;
- Q_2 is a line bundle of degree $d_2 = (2d-j)/6$;
- Q_3 is a line bundle of degree $d_3 = j/2$.

Also in this case $\mathrm{Hom}\left((Q_3, W_3), (Q_2, 0)\right) = 0$. Since Q_1 is a vector bundle of rank 2 and Q_2 is a line bundle, then these 2 coherent systems cannot be isomorphic; therefore we can apply again proposition 6.1.2 in order to parametrize all the corresponding (E, V)'s. Since $\mathrm{Hom}((Q_2, W_2), (Q_1, W_1)) = 0$, then the invariant a can only assume the value

$$a = \dim \mathrm{Ext}^1((Q_2, 0), (Q_1, 0)) = C_{21} = n_1 n_2 (g-1) - d_1 n_2 + d_2 n_1 =$$
$$= 2(g-1) - (2d-j)/3 + (2d-j)/3 = 2g-2$$

on the set $U_a = G_1 \times G_2$. So we will get a projective bundle R_a over U_a with fibers isomorphic to $\mathbb{P}^{2g-3}$. If we write $E_2 = (E_2, 0)$ for any non-split extension of Q_2 by Q_1, we get that $N_2 = 3$ and $D_2 = d_1 + d_2 = d - j/2$. Moreover, $\mathrm{Ext}^2((Q_3, W_3), (E_2, 0)) = 0$ because $k_3 = 1$ and also $\mathrm{Hom}(-,-) = 0$; therefore we get that the invariant b can assume only the value:

$$b = \dim \mathrm{Ext}^1((Q_3, W_3), (E_2, 0)) =$$
$$= N_2 n_3 (g-1) - D_2 n_3 + d_3 N_2 + k_3 D_2 - k_3 N_2 (g-1) =$$
$$= 3(g-1) - D_2 + 3d_3 + D_2 - 3(g-1) = 3d_3 = 3j/2.$$

Moreover, the invariant c can only assume the value:

$$c = \dim \mathrm{Ext}^1((Q_3, W_3), (Q_1, 0)) = C_{31} =$$
$$= n_1 n_3 (g-1) - d_1 n_3 + d_3 n_1 + k_3 d_1 - k_3 n_1 (g-1) =$$
$$= 2(g-1) - d_1 + 2d_3 + d_1 - 2(g-1) = 2d_3 = j.$$

Therefore, we get that $U_{a,b,c} = R_a \times G_3$ and we get a bundle $R_{a,b,c}$ over $U_{a,b,c}$ with fibers isomorphic to $\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1} = \mathbb{P}^{3j/2-1} \smallsetminus \mathbb{P}^{j-1}$. Now the objects (Q_i, W_i)'s vary in the following sets:

$$(Q_1, 0) \in G_1 = M^{\mathrm{s}}(2, (2d-j)/3), \quad (Q_2, 0) \in G_2 = J^{(2d-j)/6} C,$$
$$(Q_3, W_3) \in G_3 = G(1, j/2, 1).$$

Since we are assuming that $j \equiv 2d \bmod 6$, then $(2d-j)/3$ is even, so for the scheme G_1 we need to use formula (8.11). Then we get the Hodge-Deligne polynomial

$$p_7^{j \equiv_6 2d} := \mathcal{HD}(R_{a,b,c}) = \Big(\mathcal{HD}(\mathbb{P}^{3j/2-1}) - \mathcal{HD}(\mathbb{P}^{j-1})\Big) \mathcal{HD}(G_3) \mathcal{HD}(R_a) =$$
$$= \frac{(uv)^j - (uv)^{3j/2}}{1-uv} \cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g (1+vx)^g x^{-j/2}}{(1-x)(1-uvx)} \cdot$$
$$\cdot \mathcal{HD}(M^{\mathrm{s}}(2, \mathrm{even})) \cdot \mathcal{HD}(\mathbb{P}^{2g-3}) \cdot \mathcal{HD}(J^{(2d-j)/6} C) =$$
$$= \frac{(uv)^j - (uv)^{3j/2}}{1-uv} \cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g (1+vx)^g x^{-j/2}}{(1-x)(1-uvx)} \cdot \frac{1-(uv)^{2g-2}}{1-uv} \cdot$$
$$\cdot (1+u)^g (1+v)^g \cdot \frac{1}{2(1-uv)(1-(uv)^2)} \Big(2(1+u)^g (1+v)^g (1+u^2 v)^g (1+uv^2)^g +$$

$$-(1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1}-u^2v^2)-(1-u^2)^g(1-v^2)^g(1-uv)^2\Big).$$

Also in this case, if $j=0$, then we get the zero polynomial.

This concludes the computations for the case when (E,V) has a unique $\alpha(j)$-Jordan-Hölder filtration of length 3.

(2) Not unique $\alpha(j)$-Jordan-Hölder filtration. In this case the $\alpha(j)$-canonical filtration has length $s=2$ (it cannot be equal to 1 because this would imply that the corresponding (E,V)'s are semistable only at the critical value and they are not stable either on the left or on the right of any such value). If the length of the canonical filtration is $s=2$, by using the same argument used before we get that the $\alpha(j)$-canonical filtration of (E,V) is of type (2,1) and it is given by

$$0\subset(Q_1,0)\oplus(Q_2,0)\subset(E,V)$$

with $(E,V)/((Q_1,0)\oplus(Q_2,0))=(Q_3,W_3)$. In this situation the 2 cases denoted by (b) and (c) before coincide since the order of $(Q_1,0)$ and $(Q_2,0))$ is not important. Therefore, we have only to consider cases (a) and (b).

(2a) Let us suppose that the (Q_i,W_i)'s are described as in (a). Then $(Q_1,0)$ and $(Q_2,0)$ are of the same type, so we need to consider 2 subcases.

(2a-i) Let us suppose that $(Q_1,0)\not\simeq(Q_2,0)$; since these objects are of the same type, then we can use proposition 7.2.2 in order to have a global parametrization. In this case the invariant a can assume only the value

$$a=\dim\,\mathrm{Ext}^1\left((Q_3,W_3),(Q_1,0)\right)=C_{31}=$$

$$=n_1n_3(g-1)-d_1n_3+d_3n_1+k_3d_1-k_3n_1(g-1)=$$

$$=2(g-1)-\frac{2d-j}{3}+\frac{d+j}{3}+\frac{2d-j}{6}-(g-1)=g-1+j/2$$

and analogously also b can only assume the value $b=g-1+j/2$. Therefore the only schemes we are interested in are those described in (c) and (d) in that proposition. From the point of view of Hodge-Deligne polynomials, we can assume that there is a unique index $i=j$, so that there is only one scheme of type (d). Using the last part of proposition 7.2.2, we get that the (E,V)'s we are interested in are parametrized by a scheme $M/\mathbb{Z}_2$ and from the point of view of Hodge-Deligne polynomials we can assume that M is the scheme

$$(G_1\times G_2\smallsetminus\Delta_{12})\times G_3\times\mathbb{P}^{g-2+j/2}\times\mathbb{P}^{g-2+j/2},$$

where $\mathbb{Z}_2$ acts by:

$$\Big(Q_1,Q_2,(Q_3,W_3),\mu_1,\mu_2\Big)\mapsto\Big(Q_2,Q_1,(Q_3,W_3),\mu_2,\mu_1\Big).$$

Let us write $M' := G_1 \times \mathbb{P}^{g-2+j/2} = J^{(2d-j)/6}C \times \mathbb{P}^{g-2+j/2}$. Then

$$\mathcal{HD}(M')(u,v) = \frac{1-(uv)^{g-1+j/2}}{1-uv}(1+u)^g(1+v)^g.$$

Therefore we can compute:

$$A := \mathcal{HD}\Big((M' \times M')/\mathbb{Z}_2\Big)(u,v) =$$

$$= \frac{1}{2}\Big((\mathcal{HD}(M')(u,v))^2 + \mathcal{HD}(M')(-u^2,-v^2)\Big) =$$

$$= \frac{1}{2}\left(\frac{(1-(uv)^{g-1+j/2})^2}{(1-uv)^2}(1+u)^{2g}(1+v)^{2g} + \frac{1-(uv)^{2g-2+j}}{1-(uv)^2}(1-u^2)^g(1-v^2)^g\right)$$

and

$$B := \mathcal{HD}\Big((\Delta_{12} \times \mathbb{P}^{g-2+j/2} \times \mathbb{P}^{g-2+j/2})/\mathbb{Z}_2\Big) =$$

$$= \mathcal{HD}(\Delta_{12}) \cdot \mathcal{HD}\Big((\mathbb{P}^{g-2+j/2} \times \mathbb{P}^{g-2+j/2})/\mathbb{Z}_2\Big) =$$

$$= \frac{1}{2}(1+u)^g(1+v)^g\left(\frac{(1-(uv)^{g-1+j/2})^2}{(1-uv)^2} + \frac{1-(uv)^{2g-2+j}}{1-(uv)^2}\right).$$

Since $j \equiv 2d \bmod 6$, then it makes sense to define $k := j/2 \in \mathbb{N}_0$ and we get that (Q_3, W_3) varies in the scheme

$$G_3 = G^{\mathrm{s}}\left(\frac{d-2j}{3}; 2, \frac{d+j}{3}, 1\right) = G^{\mathrm{s}}\left(\frac{d+j}{3} - 2k; 2, \frac{d+j}{3}, 1\right).$$

So we can use formula (15.3) in order to compute $\mathcal{HD}(G_3)$ Finally, we can compute:

$$p_8^{j\equiv_6 2d} := \mathcal{HD}(M/\mathbb{Z}_2) = \mathcal{HD}(G_3)\cdot(A-B) =$$

$$= \frac{(1+u)^g(1+v)^g}{2(1-uv)} \underset{x^0}{\operatorname{coeff}} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot$$

$$\cdot\left[\frac{(uv)^{j/2}x^{-j/2}}{1-x(uv)^{-1}} - \frac{(uv)^{g+(d-2j)/3+1}x^{1-j/2}}{1-x(uv)^2} - x^{-j/2}\right]\cdot$$

$$\left\{\frac{(1-(uv)^{g-1+j/2})^2}{(1-uv)^2}(1+u)^{2g}(1+v)^{2g} + \frac{1-(uv)^{2g-2+j}}{1-(uv)^2}(1-u^2)^g(1-v^2)^g + \right.$$

$$\left. -(1+u)^g(1+v)^g\left(\frac{(1-(uv)^{g-1+j/2})^2}{(1-uv)^2} + \frac{1-(uv)^{2g-2+j}}{1-(uv)^2}\right)\right\}.$$

Using remark 13.3.4 if $j=0$, we get $p_8 = 0$.

(2a-ii) If we are in case (a) and $(Q_1,0) \simeq (Q_2,0)$, then the corresponding (E,V)'s are parametrized using proposition 7.2.3. From the point of view of Hodge-Deligne polynomials, we can assume that there is a single index i. Also in this case, there is only one value for the invariant a, namely $a = g-1+j/2$ as in (2a-i). So we can assume that the (E,V)'s we are interested in are parametrized by a grassmannian $\mathrm{Grass}(2,R_a)$ where R_a is a vector bundle over $U_a = G_1 \times G_3$ with fibers isomorphic to $\mathbb{C}^{g-1+j/2}$. Here G_1 and G_3 are as in case (2a-i), so we get the polynomial:

$$\begin{aligned}
p_9^{j\equiv_6 2d} &:= \mathcal{HD}(\mathrm{Grass}(2,R_a)) = \\
&= \mathcal{HD}\Big(\mathrm{Grass}(2,g-1+j/2)\Big)\cdot\mathcal{HD}(G_1)\cdot\mathcal{HD}(G_3) = \\
&= \frac{(1-(uv)^{g-2+j/2})(1-(uv)^{g-1+j/2})}{(1-uv)^2(1-(uv)^2)}(1+u)^{2g}(1+v)^{2g})\cdot \\
&\cdot \operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot\left[\frac{(uv)^{j/2}x^{-j/2}}{1-x(uv)^{-1}} - \frac{(uv)^{g+(d-2j)/3+1}x^{1-j/2}}{1-x(uv)^2} - x^{-j/2}\right].
\end{aligned}$$

Using again remark 13.3.4 we get that also in this case $p_9 = 0$ for $j = 0$.

(2b) We have also to consider case (b) (that coincides with case (c)). In that case Q_1 and Q_2 have different ranks, so in particular their types are different, so we can simply apply proposition 7.2.1 in order to parametrize the corresponding (E,V)'s. In this case the invariants a can only assume the value:

$$a = \dim \mathrm{Ext}^1((Q_3,W_3),(Q_1,0)) = j/2$$

(this is computed as the invariant c is computed in (1b)), and b can only assume the value:

$$\begin{aligned}
b &= \dim \mathrm{Ext}^1((Q_3,W_3),(Q_2,0)) = \\
&= n_2n_3(g-1) - d_2n_3 + d_3n_2 + k_3d_2 - k_3n_2(g-1) = \\
&= 2(g-1) - d_2 + j + d_2 - 2(g-1) = j.
\end{aligned}$$

From the point of view of Hodge-Deligne polynomials we can assume that there is a single index i. So we can assume that the (E,V)'s we are interested in are parametrized by a scheme R that comes with a sequence of two projective fibrations

$$R \longrightarrow A \longrightarrow U = G_1 \times G_2 \times G_3;$$

the first fibration has fibers isomorphic to $\mathbb{P}^{b-1} = \mathbb{P}^{j-1}$, while the second fibration has fibers isomorphic to $\mathbb{P}^{a-1} = \mathbb{P}^{j/2-1}$. Here the schemes G_i for $i = 1,2,3$ coincide with those described in case (1b). So we get the polynomial:

$$\begin{aligned}
p_{10}^{j\equiv_6 2d} &= \mathcal{HD}(\mathbb{P}^{j-1})\mathcal{HD}(\mathbb{P}^{j/2-1})\mathcal{HD}(G_1)\mathcal{HD}(G_2)\mathcal{HD}(G_3) = \\
&= \frac{1-(uv)^j}{1-uv}\cdot\frac{1-(uv)^{j/2}}{1-uv}(1+u)^g(1+v)^g\cdot \\
&\cdot\frac{1}{2(1-uv)(1-(uv)^2)}\Big(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g +
\end{aligned}$$

$$-(1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1}-u^2v^2)-(1-u^2)^g(1-v^2)^g(1-uv)^2\Big)\cdot$$

$$\cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}.$$

So also in this case we get the zero polynomial for $j=0$.

15.1.3 Case $r=4$

In this case the graded is necessarily made of 4 objects of the form $(Q_i,0)_{i=1,2,3},(Q_4,W_4)$ where (Q_4,W_4) must be necessarily the last object of the graded in order not to destabilize (E,V) for $\alpha(j)^+$. Moreover, every Q_i for $i=1,2,3$ must be a line bundle and the stability conditions prove that the Q_i's for $i=1,2,3$ must all have the same degree $d_1=d_2=d_3=(2d-j)/6$. Therefore, *this case is possible only when* $j\equiv 2d$ mod 6. Moreover, Q_4 is a line bundle of degree $d_4=d-3d_1=d-(2d-j)/2=j/2$.

Then we need to consider several different subcases according to the various possible $\alpha(j)$-canonical filtrations. The cases we will consider are those when such a filtration is of one of following types: $(1,1,1,1)$ (unique $\alpha(j)$-Jordan-Hölder filtration), $(3,1)$, $(1,2,1)$ and $(2,1,1)$. A priori we should also consider the cases $(1,1,2)$, $(1,3)$ and (4); none of these 3 cases is actually possible since in each case we will have a quotient $(E,V)\twoheadrightarrow(Q_3,0)$ and this would prove that (E,V) is not α_c^+-stable, so these 3 cases do not occur in the description of $G^+(\alpha_c;4,d,1)$.

(1) Canonical filtration of type (1,1,1,1) (unique $\alpha(j)$-Jordan-Hölder filtration). Since the $(Q_i,0)$'s for $i=1,2,3$ are all of the same type, we need to consider 4 subcases according to the various relations between them:

(a) $(Q_1,0)\simeq(Q_2,0)\not\simeq(Q_3,0)$;

(b) $(Q_1,0)\not\simeq(Q_2,0)\not\simeq(Q_3,0)$;

(c) $(Q_1,0)\simeq(Q_2,0)\simeq(Q_3,0)$;

(d) $(Q_1,0)\not\simeq(Q_2,0)\simeq(Q_3,0)$.

(1a) Let us suppose that $(Q_1,0)\simeq(Q_2,0)\not\simeq(Q_3,0)$. Then we can apply proposition 6.2.2. In this case we need to compute the invariants a,b,c,d,e,f. In order to do that, let $E_2=(E_2,0)$ be any non-split extension of Q_2 by Q_1 and let (E'',V'') be any non-split extension of (Q_4,W_4) by $(Q_3,0)$. Then E_2 is a vector bundle of rank $N_2=2$ and degree $D_2=2d_1=(2d-j)/3$; E'' is a vector bundle of rank $N''=2$ and degree $D''=d_3+d_4=(2d-j)/6+j/2=(d+j)/3$. Moreover, the dimension of V'' is $K''=1$. Since $(Q_1,0)\simeq(Q_2,0)$, we have:

$$a=\dim\,\mathrm{Ext}^1((Q_2,0),(Q_1,0))=C_{21}+1=n_1n_2(g-1)-d_1n_2+d_2n_1+1=g.$$

Since $(Q_3,0)\not\simeq(Q_4,W_4)$, we have

$$b = \dim\ \mathrm{Ext}^1((Q_4,W_4),(Q_3,0)) = C_{43} =$$
$$= n_3n_4(g-1) - d_3n_4 + d_4n_3 + k_4d_3 - k_4n_3(g-1) =$$
$$= (g-1) - d_3 + d_4 + d_3 - (g-1) = d_4 = j/2.$$

Moreover, since $(Q_1,0)$ is of the same type of $(Q_3,0)$ and since $(Q_1,0) \not\simeq (Q_4,W_4)$, we have:

$$f = \dim\ \mathrm{Ext}^1((Q_4,W_4),(Q_1,0)) = b = j/2.$$

Now $(E_2,0)$ is a non-split extension of $(Q_1,0)$ by itself and (E'',V'') is a non-split extension of (Q_4,W_4) by $(Q_3,0)$. So as in (6.44) and (6.62) we get:

$$\mathrm{Hom}((E'',V''),(E_2,0)) = 0 = \mathrm{Hom}((E'',V''),(Q_1,0)).$$

Moreover, we have also that $\mathrm{Hom}((Q_4,W_4),(E_2,0)) = 0$ because the graded of $(E_2,0)$ does not contain any object isomorphic to (Q_4,W_4). Then we can compute also the following invariants.

$$c = \dim\ \mathrm{Ext}^1((E'',V''),(E_2,0)) =$$
$$= N_2N''(g-1) - D_2N'' + D''N_2 + K''D_2 - K''N_2(g-1) =$$
$$= 4(g-1) - 2(2d-j)/3 + 2(d+j)/3 + (2d-j)/3 - 2(g-1) = 2g-2+j.$$

$$d = \dim\ \mathrm{Ext}^1((Q_4,W_4),(E_2,0)) =$$
$$= N_2n_4(g-1) - D_2n_4 + d_4N_2 + k_4D_2 - k_4N_2(g-1) =$$
$$= 2(g-1) + 2d_4 - 2(g-1) = 2d_4 = j.$$

$$e = \dim\ \mathrm{Ext}^1((E'',V''),(Q_1,0)) =$$
$$= n_1N''(g-1) - d_1N'' + D''n_1 + K''d_1 - K''n_1(g-1) =$$
$$= 2(g-1) - (2d-j)/3 + (d+j)/3 + (2d-j)/6 - (g-1) = g-1+j/2.$$

So each invariant can only assume one value. From the point of view of Hodge-Deligne polynomials, we can ignore the indices i,j,l of proposition 6.2.2, so without loss of generality we can assume that we have the following description:

- $U_a = G_1 = G_2$ and there is a projective bundle R_a over it with fibers isomorphic to $\mathbb{P}^{a-1} = \mathbb{P}^{g-1}$;
- $U^b = G_3 \times G_4$ and there is a projective bundle R^b over it with fibers isomorphic to $\mathbb{P}^{b-1} = \mathbb{P}^{j/2-1}$;

- from the point of view of Hodge-Deligne polynomials we can assume that $R_a \times R^b = G_1 \times G_3 \times G_4 \times \mathbb{P}^{g-1} \times \mathbb{P}^{j/2-1}$. So we can assume that

$$U_{a,b,c,d,e,f} = (G_1 \times G_3 \smallsetminus \Delta_{13}) \times G_4 \times \mathbb{P}^{g-1} \times \mathbb{P}^{j/2-1};$$

there is a bundle $R_{a,b,c,d,e,f}$ over it with fibers isomorphic to $\mathbb{C}^{e-1} \times (\mathbb{P}^{c-e-1} \smallsetminus \mathbb{P}^{d-f-1}) = \mathbb{C}^{g-2+j/2} \times (\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-1})$.

The (E, V)'s we are interested in are parametrized by the scheme $R_{a,b,c,d,e,f}$. In this case

$$G_1 = G_2 = G_3 = J^{(2d-j)/6}C, \quad G_4 = G(1, j/2, 1).$$

So we get the polynomial

$$p_{11}^{j \equiv_6 2d} = \mathcal{HD}(R_{a,b,c,d,e,f}) = (\mathcal{HD}(G_1)^2 - \mathcal{HD}(G_1))\mathcal{HD}(G_4)\mathcal{HD}(\mathbb{P}^{g-1})\mathcal{HD}(\mathbb{P}^{j/2-1})\cdot$$

$$\cdot\mathcal{HD}(\mathbb{C}^{g-2+j/2} \times (\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-1})) =$$

$$= (1+u)^g(1+v)^g((1+u)^g(1+v)^g - 1) \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}\cdot$$

$$\cdot\frac{1-(uv)^g}{1-uv} \cdot \frac{1-(uv)^{j/2}}{1-uv} \cdot (uv)^{g-2+j/2} \cdot \frac{(uv)^{j/2} - (uv)^{g-1+j/2}}{1-uv}.$$

Also in this case, if $j = 0$, then we get that the polynomial is zero.

(1b) Let us suppose that $(Q_1, 0) \not\simeq (Q_2, 0) \not\simeq (Q_3, 0)$; we don't need to fix any additional condition on the relations between $(Q_1, 0)$ and $(Q_3, 0)$. Then we can apply proposition 6.2.4. Also in this case we need to compute invariants a, b, c, d, e, f. In order to do that, let $E_2 = (E_2, 0)$ and (E'', V'') be as in case (1a). Then we get the same invariants computed before, except for the invariant a that now has value $g-1$ (instead of g). In particular, each invariant can only assume one value. From the point of view of Hodge-Deligne polynomials, we can ignore the indices i, j, l of that proposition, so we can assume that:

- $U_a = G_1 \times G_2 \smallsetminus \Delta_{12}$ and there is a projective bundle R_a over it with fibers isomorphic to $\mathbb{P}^{a-1} = \mathbb{P}^{g-2}$;
- $U^b = G_3 \times G_4$ and there is a bundle R^b over it with fibers isomorphic to $\mathbb{P}^{b-1} = \mathbb{P}^{j/2-1}$;
- from the point of view of Hodge-Deligne polynomials we can assume that $R_a \times R^b = (G_1 \times G_2 \smallsetminus \Delta_{12}) \times G_3 \times G_4 \times \mathbb{P}^{g-2} \times \mathbb{P}^{j/2-1}$. So we can assume that

$$U_{a,b,c,d,e,f} = ((G_1 \times G_2 \smallsetminus \Delta_{12}) \times G_3 \smallsetminus G_1 \times \Delta_{23}) \times G_4 \times \mathbb{P}^{g-2} \times \mathbb{P}^{j/2-1};$$

there is a bundle $R_{a,b,c,d,e,f}$ over it with fibers isomorphic to $\mathbb{P}^{c-1} \smallsetminus \mathbb{P}^{d+e-f-1} = \mathbb{P}^{2g-3+j} \smallsetminus \mathbb{P}^{g-2+j}$.

The (E,V)'s we are interested in are parametrized by the scheme $R_{a,b,c,d,e,f}$. Also in this case

$$G_1 = G_2 = G_3 = J^{(2d-j)/6}C, \quad G_4 = G(1, j/2, 1).$$

Now

$$\begin{aligned}\mathcal{HD}((G_1 \times G_2 \smallsetminus \Delta_{12}) \times G_3 \smallsetminus G_1 \times \Delta_{23}) &= \mathcal{HD}(G_1 \times G_2 \times G_3) - \mathcal{HD}(\Delta_{12} \times G_3 \cup G_1 \times \Delta_{23}) = \\ = \mathcal{HD}(G_1)^3 - \mathcal{HD}(\Delta_{12} \times G_3) &- \mathcal{HD}(G_1 \times \Delta_{23}) + \mathcal{HD}(\Delta_{12} \times G_3 \cap G_1 \times \Delta_{23}) = \\ = \mathcal{HD}(G_1)^3 - 2\mathcal{HD}(G_1)^2 &+ \mathcal{HD}(G_1) = \mathcal{HD}(G_1)(\mathcal{HD}(G_1) - 1)^2. \end{aligned} \tag{15.4}$$

So we get the polynomial

$$\begin{aligned} p_{12}^{j\equiv_6 2d} &= \mathcal{HD}(R_{a,b,c,d,e,f}) = \mathcal{HD}(G_1)(\mathcal{HD}(G_1)-1)^2\mathcal{HD}(G_4)\cdot \\ &\cdot \mathcal{HD}(\mathbb{P}^{g-2})\mathcal{HD}(\mathbb{P}^{j/2-1})\mathcal{HD}(\mathbb{P}^{2g-3+j} \smallsetminus \mathbb{P}^{g-2+j}) = \\ &= (1+u)^g(1+v)^g((1+u)^g(1+v)^g-1)^2 \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)} \cdot \\ &\cdot \frac{1-(uv)^{g-1}}{1-uv} \cdot \frac{1-(uv)^{j/2}}{1-uv} \cdot \frac{(uv)^{g-1+j}-(uv)^{2g-2+j}}{1-uv}. \end{aligned}$$

Also in this case, if $j = 0$, then we get that the polynomial is zero.

(1c)-(1d) As we stated in remark 6.2.3 we are still not able to give a geometric description of these 2 cases. We simply denote the corresponding polynomials by $p_{13}^{j\equiv_6 2d}$ and $p_{14}^{j\equiv_6 2d}$ respectively.

(2) Canonical filtration of type (3,1). Since the $(Q_i, 0)$'s for $i = 1, 2, 3$ are all of the same type, we need to consider 3 subcases as follows:

(a) there are no pairs of isomorphic objects among the $(Q_i, 0)$'s for $i = 1, 2, 3$;

(b) exactly 2 objects among the $(Q_i, 0)$'s are isomorphic; without loss of generality we can assume that they are $(Q_1, 0)$ and $(Q_2, 0)$;

(c) $(Q_1, 0) \simeq (Q_2, 0) \simeq (Q_3, 0)$.

(2a) Let us suppose that there are no pairs of isomorphic objects among the $(Q_i, 0)$'s for $i = 1, 2, 3$. Then we can apply proposition 7.4.4. In this case we need to compute the invariants a, b, c; the same computation that gives the invariants b and f in case (1a) proves that we have:

$$a = \dim \operatorname{Ext}^1((Q_4, W_4), (Q_1, 0)) = j/2.$$

Analogously, since $(Q_1, 0)$, $(Q_2, 0)$ and $(Q_3, 0)$ are all of the same type, we get that

$$b = \dim \operatorname{Ext}^1((Q_4, W_4), (Q_2, 0)) = j/2 = \dim \operatorname{Ext}^1((Q_4, W_4), (Q_3, 0)) = c.$$

Now $G_1 = J^{(2d-j)/6}C$ and $G_4 = G(\alpha(j); 1; j/2, 1)$, so both spaces are irreducible. Therefore the index i appearing in proposition 7.4.4 assumes only one value. Moreover, since $G_1 = G_2 = G_3$, also the indices j and k can assume only one value. Therefore, we get that

$$U^1_{a;i} = U^2_{b;j} = U^3_{c;k} = G_1 \times G_4.$$

Then we get that the only scheme $R_{a,b,c;i,j,k}$ that we will be interested in is $R_{a,a,a;i,i,i}$, that comes with a locally trivial fibration to

$$U_{a,a,a;i,i,i} = U^1_{a;i} \times_{G_4} U^2_{a;i} \times_{G_4} U^3_{a;i} = G_1 \times G_1 \times G_1 \times G_4$$

with fibers isomorphic to $\mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1}$. Since this is the only case, then the only object we need to consider is given by case (j) of that proposition, namely

$$R := (R_{a,a,a;i,i,i}|_{(G_1 \times G_1 \times G_1 \smallsetminus \Delta) \times G_4})/S_3,$$

where Δ is the big diagonal of $G_1 \times G_1 \times G_1$, i.e. the set of all triples of objects such that at least 2 of them are isomorphic. Every $\sigma \in S_3$ acts as follows on $U_{a,a,a;i,i,i}$ and $R_{a,a,a;i,i,i}$:

- $(Q_i, W_i)_{i=1,2,3,4} \mapsto ((Q_{\sigma(i)}, W_{\sigma(i)})_{i=1,2,3}, (Q_4, W_4))$;
- $(\mu_i)_{i=1,2,3} \mapsto (\mu_{\sigma(i)})_{i=1,2,3}$ for every point (μ_1, μ_2, μ_3) in the fiber over a quadruple $(Q_i, W_i)_{i=1,\cdots,4}$.

Moreover, there exists a finite disjoint covering of the base space $(G_1 \times G_1 \times G_1 \smallsetminus \Delta) \times G_4$ by locally closed subschemes T_l that are invariant under the action of S_3 on $G_1 \times G_1 \times G_1 \times G_4$; in addition, there exist trivializations of the fibrations from $R_{a,a,a;i,i,i}$ to $U_{a,a,i;i,i,i}$

$$R|_{T_l} \xrightarrow{\sim} T_l \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1}$$

that are compatible with the natural action of S_3 on $T_l \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1}$. From the point of view of Hodge-Deligne polynomials, we can therefore assume that R coincides with a scheme of the form M/S_3, where M is the scheme

$$(G_1 \times G_1 \times G_1 \smallsetminus \Delta) \times G_4 \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1}$$

and every $\sigma \in S_3$ acts on M as follows:

$$(Q_1, Q_2, Q_3, (Q_4, W_4), \mu_1, \mu_2, \mu_3) \mapsto$$
$$\mapsto \left(Q_{\sigma(1)}, Q_{\sigma(2)}, Q_{\sigma(3)}, (Q_4, W_4), \mu_{\sigma(1)}, \mu_{\sigma(2)}, \mu_{\sigma(3)}\right).$$

Let us consider the following schemes:

$$M' := G_1 \times \mathbb{P}^{j/2-1} = J^{(2d-j)/6}C \times \mathbb{P}^{j/2-1},$$

$$
\begin{aligned}
&\Delta_0 := \{(Q_1, Q_2, Q_3) \in G_1 \times G_1 \times G_1 \text{ s.t. } Q_1 \simeq Q_2 \simeq Q_3\},\\
&\Delta_1 := \{(Q_1, Q_2, Q_3) \in G_1 \times G_1 \times G_1 \text{ s.t. } Q_1 \simeq Q_2 \not\simeq Q_3\},\\
&\Delta_2 := \{(Q_1, Q_2, Q_3) \in G_1 \times G_1 \times G_1 \text{ s.t. } Q_1 \simeq Q_3 \not\simeq Q_2\},\\
&\Delta_3 := \{(Q_1, Q_2, Q_3) \in G_1 \times G_1 \times G_1 \text{ s.t. } Q_2 \simeq Q_3 \not\simeq Q_1\}. \qquad (15.5)
\end{aligned}
$$

Then $\Delta_0 \simeq G_1$ and $\Delta_i \simeq G_1 \times G_1 \smallsetminus \Delta'$ for $i = 1, 2, 3$, where Δ' is the diagonal of $G_1 \times G_1$. Moreover, we can write

$$\Delta = \Delta_0 \amalg \Delta_1 \amalg \Delta_2 \amalg \Delta_3.$$

Now we have that

$$
\begin{aligned}
M/S_3 \simeq \Big((M' \times M' \times M')/S_3 \smallsetminus \big((\Delta_0 \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1})/S_3 \amalg\\
\amalg((\Delta_1 \amalg \Delta_2 \amalg \Delta_3) \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1})/S_3)\big)\Big) \times G_4. \qquad (15.6)
\end{aligned}
$$

Now

$$\mathcal{HD}(M')(u, v) = (1 + u)^g (1 + v)^g \frac{1 - (uv)^{j/2}}{1 - uv}.$$

So we can use lemma 8.0.6 in order to compute

$$
\begin{aligned}
A := \mathcal{HD}((M' \times M' \times M')/S_3)(u, v) = \frac{1}{6}(\mathcal{HD}(M')(u, v))^3 +\\
+\frac{1}{2}\mathcal{HD}(M')(-u^2, -v^2) \cdot \mathcal{HD}(M')(u, v) + \frac{1}{3}\mathcal{HD}(M')(u^3, v^3) =\\
= (1 + u)^{3g}(1 + v)^{3g} \frac{(1 - (uv)^{j/2})^3}{6(1 - uv)^3} + (1 - u^2)^g (1 - v^2)^g (1 + u)^g (1 + v)^g \cdot\\
\cdot \frac{(1 - (uv)^j)(1 - (uv)^{j/2})}{2(1 - (uv)^2)(1 - uv)} + (1 + u^3)^g (1 + v^3)^g \frac{1 - (uv)^{3j/2}}{3(1 - (uv)^3)}.
\end{aligned}
$$

Now the action of S_3 on Δ_0 is trivial; moreover, we have

$$(\Delta_1 \amalg \Delta_2 \amalg \Delta_3)/S_3 \simeq \Delta_1/\mathbb{Z}_2 \simeq (G_1 \times G_1 \smallsetminus \Delta')/\mathbb{Z}_2 \simeq (G_1 \times G_1)/\mathbb{Z}_2 \smallsetminus G_1.$$

So we have:

$$(\Delta_0 \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1})/S_3 \simeq G_1 \times (\mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1})/S_3$$

and

$$
\begin{aligned}
((\Delta_1 \amalg \Delta_2 \amalg \Delta_3) \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1})/S_3 \simeq\\
\simeq \Big((G_1 \times G_1 \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1})/\mathbb{Z}_2 \smallsetminus G_1 \times (\mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1})/\mathbb{Z}_2\Big) \times \mathbb{P}^{j/2-1}.
\end{aligned}
$$

So we compute:

$$B_1 := \mathcal{HD}(G_1 \times (\mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1})/S_3)) =$$
$$= (1+u)^g(1+v)^g \cdot \left(\frac{(1-(uv)^{j/2})^3}{6(1-uv)^3} + \frac{(1-(uv)^j)(1-(uv)^{j/2})}{2(1-(uv)^2)(1-uv)} + \frac{1-(uv)^{3j/2}}{3(1-(uv)^3)} \right);$$
$$B_2 := \mathcal{HD}\Big((G_1 \times G_1 \times \mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1})/\mathbb{Z}_2 \times \mathbb{P}^{j/2-1}\Big) =$$
$$= \frac{1}{2} \cdot \left((1+u)^{2g}(1+v)^{2g} \frac{(1-(uv)^{j/2})^2}{(1-uv)^2} + (1-u^2)^g(1-v^2)^g \frac{1-(uv)^j}{1-(uv)^2} \right) \cdot \frac{1-(uv)^{j/2}}{1-uv};$$
$$B_3 = \mathcal{HD}(G_1 \times (\mathbb{P}^{j/2-1} \times \mathbb{P}^{j/2-1})/\mathbb{Z}_2) \times \mathbb{P}^{j/2-1}) =$$
$$= \frac{1}{2}(1+u)^g(1+v)^g \left(\frac{(1-(uv)^{j/2})^2}{(1-uv)^2} + \frac{1-(uv)^j}{1-(uv)^2} \right) \frac{1-(uv)^{j/2}}{1-uv}.$$

Then by considering everything together, we have:

$$p_{15}^{j \equiv_6 2d} := (A - B_1 - B_2 + B_3) \cdot \mathcal{HD}(G_4) =$$
$$= \Big\{ (1+u)^{3g}(1+v)^{3g} \frac{(1-(uv)^{j/2})^3}{6(1-uv)^3} + (1-u^2)^g(1-v^2)^g(1+u)^g(1+v)^g \cdot$$
$$\cdot \frac{(1-(uv)^j)(1-(uv)^{j/2})}{2(1-(uv)^2)(1-uv)} + (1+u^3)^g(1+v^3)^g \frac{1-(uv)^{3j/2}}{3(1-(uv)^3)} +$$
$$-(1+u)^g(1+v)^g \cdot \left(\frac{(1-(uv)^{j/2})^3}{6(1-uv)^3} + \frac{(1-(uv)^j)(1-(uv)^{j/2})}{2(1-(uv)^2)(1-uv)} + \frac{1-(uv)^{3j/2}}{3(1-(uv)^3)} \right) +$$
$$-\frac{1}{2} \cdot \left((1+u)^{2g}(1+v)^{2g} \frac{(1-(uv)^{j/2})^2}{(1-uv)^2} + (1-u^2)^g(1-v^2)^g \frac{1-(uv)^j}{1-(uv)^2} \right) \cdot \frac{1-(uv)^{j/2}}{1-uv} +$$
$$+\frac{1}{2}(1+u)^g(1+v)^g \left(\frac{(1-(uv)^{j/2})^2}{(1-uv)^2} + \frac{1-(uv)^j}{1-(uv)^2} \right) \frac{1-(uv)^{j/2}}{1-uv} \Big\} \cdot$$
$$\cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}.$$

(2b) Let us suppose that $(Q_1, 0) \simeq (Q_2, 0) \not\simeq (Q_3, 0)$. Then we can apply proposition 7.4.5. In this case we need to compute the invariants a and b. The same analysis of case (2a) proves that $a = b = j/2$ and that the indices i and j can only assume one value. Therefore, we get that

$$U^1_{a;i} = U^3_{b;j} = G_1 \times G_4.$$

and

$$V_{a,b;i,j} = (U^1_{a;i} \times_{G_4} U^3_{b;j}) \cap ((G_1 \times G_1 \smallsetminus \Delta) \times G_4) = (G_1 \times G_1 \smallsetminus \Delta) \times G_4.$$

The scheme we are looking at is $R_{a,b;i,j}$, that comes with a morphism

$$R_{a,b;i,j} \overset{\phi_2\circ\phi_1}{\longrightarrow} V_{a,b;i,j},$$

where ϕ_1 is a fibration with fibers isomorphic to $\mathbb{P}^{b-1}=\mathbb{P}^{j/2-1}$ and ϕ_2 is the grassmannian fibration of 2-planes associated to a vector bundle $Q_{a,b;i,j}$ over $V_{a,b;i,j}$ with rank $a=j/2$. So we get:

$$p_{16}^{j\equiv_6 2d} := \mathcal{HD}(R_{a,b;i,j}) = \mathcal{HD}(G_1\times G_1\smallsetminus\Delta)\mathcal{HD}(G_4)\mathcal{HD}(\mathbb{P}^{j/2-1})\mathcal{HD}(\mathrm{Grass}(2,j/2)) =$$
$$= \left((1+u)^{2g}(1+v)^{2g}-(1+v)^g(1+v)^g\right)\cdot \underset{x^0}{\mathrm{coeff}}\,\frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}\cdot$$
$$\cdot\frac{1-(uv)^{j/2}}{1-uv}\cdot\frac{(1-(uv)^{j/2-1})(1-(uv)^{j/2})}{(1-uv)(1-(uv)^2)}.$$

(2c) Let us suppose that $(Q_1,0)\simeq(Q_2,0)\simeq(Q_3,0)$. Then we can apply proposition 7.4.6. In this case the only invariant that we need is a. As before, $a=j/2$ and the index i can only assume one value. Therefore, we get that $U_{a;i}=G_1\times G_4$. The scheme we are looking at is the grassmannian of 3-planes associated to a locally free sheaf $R_{a;i}$ of rank $a=j/2$ over $U_{a;i}$. So we get:

$$p_{17}^{j\equiv_6 2d} := \mathcal{HD}(G_1)\mathcal{HD}(G_4)\mathcal{HD}(\mathrm{Grass}(3,j/2)) =$$
$$= (1+u)^g(1+v)^g\cdot \underset{x^0}{\mathrm{coeff}}\,\frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}\cdot$$
$$\cdot\frac{(1-(uv)^{j/2-2})(1-(uv)^{j/2-1})(1-(uv)^{j/2})}{(1-uv)(1-(uv)^2)(1-(uv)^3)}.$$

(3) Canonical filtration of type (2,1,1). Since the $(Q_i,0)$'s for $i=1,2,3$ are all of the same type and since the order of $(Q_1,0)$ and $(Q_2,0)$ is not important, we need to consider 4 cases as follows:

(a) $(Q_1,0)\not\simeq(Q_2,0)\simeq(Q_3,0)$;

(b) $(Q_i,0)\not\simeq(Q_j,0)$ for all $i\neq j\in\{1,2,3\}$;

(c) $(Q_1,0)\simeq(Q_2,0)\not\simeq(Q_3,0)$;

(d) $(Q_1,0)\simeq(Q_2,0)\simeq(Q_3,0)$.

(3a) Let us suppose that $(Q_1,0)\not\simeq(Q_2,0)\simeq(Q_3,0)$. Then we can apply proposition 7.5.1. In this case we need to compute the invariants a,b,c,d. In order to do that, let (E'',V'') be any non-split extension of (Q_4,W_4) by $(Q_3,0)$; then E'' is a vector bundle of rank $N''=2$ and degree $D''=d_3+d_4=(2d-j)/6+j/2=(d+j)/3$; moreover, the dimension of V'' is $K''=1$. Since $(Q_3,0)\not\simeq(Q_4,W_4)$, we have:

$$a=\dim\,\mathrm{Ext}^1((Q_4,W_4),(Q_3,0))=C_{43}=$$

$$= n_3n_4(g-1) - d_3n_4 + d_4n_3 + k_4d_3 - k_4n_3(g-1) =$$

$$= (g-1) - d_3 + d_4 + d_3 - (g-1) = d_4 = j/2.$$

By the same computation we get that

$$c = \dim \mathrm{Ext}^1((Q_4, W_4), (Q_1, 0)) = j/2.$$

Since (E'', V'') is a non-split extension of (Q_4, W_4) by $(Q_3, 0)$, then

$$\mathrm{Hom}((E'', V''), (Q_i, 0)) = 0 \text{ for } i = 1, 2, 3.$$

Therefore, we have that

$$b = \dim \mathrm{Ext}^1((E'', V''), (Q_1, 0)) =$$

$$= n_1N''(g-1) - d_1N'' + D''n_1 + K''d_1 - K''n_1(g-1) =$$

$$= 2(g-1) - (2d-j)/3 + (d+j)/3 + (2d-j)/6 - (g-1) = g - 1 + j/2.$$

By the same computation we get that

$$d = \dim \mathrm{Ext}^1((E'', V''), (Q_3, 0)) = g - 1 + j/2.$$

So each invariant can only assume one value. From the point of view of Hodge-Deligne polynomials, we can ignore the indices i and j of proposition 7.5.1, so we can assume that we have the following description.

- $U_a = G_3 \times G_4$ and there is a projective bundle R_a over it with fibers isomorphic to $\mathbb{P}^{a-1} = \mathbb{P}^{j/2-1}$; from the point of view of Hodge-Deligne polynomials we can assume that $U_a = G_3 \times G_4 \times \mathbb{P}^{j/2-1}$;

- from the point of view of Hodge-Deligne polynomials we can assume that $U_{a,b,c,d} = (G_1 \times G_3 \smallsetminus \Delta_{13}) \times G_4 \times \mathbb{P}^{j/2-1}$; there sequences of fibrations for $l = 1, 2, 3$:

$$R^l_{a,b,c,d} \xrightarrow{\phi^l} A^l_{a,b,c,d} \xrightarrow{\theta^l} U_{a,b,c,d}$$

  where:

  - ϕ^1 has fibers isomorphic to $\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-1}$, θ^1 has fibers isomorphic to $\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-2}$;
  - ϕ^2 has fibers isomorphic to $\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-1}$, θ^2 has fibers isomorphic to $\mathbb{P}^{j/2-2}$;
  - ϕ^3 has fibers isomorphic to $\mathbb{P}^{j/2-1}$, θ^3 has fibers isomorphic to $\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-2}$.

The (E,V)'s we are interested in are parametrized by the scheme $R^1_{a,b,c,d} \amalg R^2_{a,b,c,d} \amalg R^3_{a,b,c,d}$. In this case

$$G_1 = G_3 = J^{(2d-j)/6}C, \quad G_4 = G(1, j/2, 1).$$

So we get the polynomial

$$p_{18}^{j \equiv_6 2d} = \mathcal{HD}(R^1_{a,b,c,d}) + \mathcal{HD}(R^2_{a,b,c,d}) + \mathcal{HD}(R^3_{a,b,c,d}) =$$
$$= (\mathcal{HD}(G_1)^2 - \mathcal{HD}(G_1))\mathcal{HD}(G_4)\mathcal{HD}(\mathbb{P}^{j/2-1}) \cdot \{\mathcal{HD}(\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-1})\mathcal{HD}(\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-2}) +$$
$$+ \mathcal{HD}(\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-1})\mathcal{HD}(\mathbb{P}^{j/2-2}) + \mathcal{HD}(\mathbb{P}^{j/2-1})\mathcal{HD}(\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-2})\} =$$
$$= (1+u)^g(1+v)^g((1+u)^g(1+v)^g - 1) \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)} \cdot$$
$$\frac{1-(uv)^{j/2}}{1-uv} \cdot \{\mathcal{HD}(\mathbb{P}^{g-2+j/2})^2 - \mathcal{HD}(\mathbb{P}^{j/2-1})\mathcal{HD}(\mathbb{P}^{j/2-2})\} =$$
$$= (1+u)^g(1+v)^g((1+u)^g(1+v)^g - 1) \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)} \cdot$$
$$\frac{1-(uv)^{j/2}}{1-uv} \cdot \cdot \left\{ \frac{(1-(uv)^{g-1+j/2})^2}{(1-uv)^2} - \frac{(1-(uv)^{j/2})(1-(uv)^{j/2-1})}{(1-uv)^2} \right\}.$$

Also in this case, if $j=0$, then we get that the polynomial is zero.

(3b) Let us suppose that $(Q_i,0) \not\simeq (Q_j,0)$ for all $i \neq j \in \{1,2,3\}$. Then we can apply proposition 7.5.2. In this case we need to compute the invariants a,b,c,d,e. Also in this case, we denote by (E'',V'') any non-split extension of (Q_4,W_4) by $(Q_3,0)$; again we have $(N'',D'',K'') = (2,(d+j)/3,1)$. Since $(Q_i,0) \not\simeq (Q_4,W_4)$ for $i=1,2,3$, we have as before

$$a = \dim \operatorname{Ext}^1((Q_4,W_4),(Q_3,0)) = j/2,$$
$$c = \dim \operatorname{Ext}^1((Q_4,W_4),(Q_1,0)) = j/2,$$
$$e = \dim \operatorname{Ext}^1((Q_4,W_4),(Q_2,0)) = j/2.$$

Moreover, as before we get also:

$$b = \dim \operatorname{Ext}^1((E'',V''),(Q_1,0)) = g-1+j/2,$$
$$d = \dim \operatorname{Ext}^1((E'',V''),(Q_2,0)) = g-1+j/2.$$

So each invariant can only assume one value. From the point of view of Hodge-Deligne polynomials, we can ignore the indices i and j of that proposition, so we can assume that we have the following description.

- $U_a = G_3 \times G_4$ and there is a projective bundle $\varphi_a : R_a \to U_a$ with fibers isomorphic to $\mathbb{P}^{a-1} = \mathbb{P}^{j/2-1}$; from the point of view of Hodge-Deligne polynomials we can assume that $R_a = G_3 \times G_4 \times \mathbb{P}^{j/2-1}$.

- The scheme $U_{a,b,c,d,e}$ coincides with

$$\{((Q_1,W_1),(Q_2,W_2),(E'',V''))\in G_1\times G_2\times R_a \text{ s.t.}$$
$$(Q_l,W_l)\not\simeq\overline{\varphi}_a(E'',V'')\forall\, l=1,2,\ (Q_1,W_1)\not\simeq(Q_2,W_2)\}$$

where $\overline{\varphi}_a$ is the composition of φ_a with the projection to G_3. From the point of view of Hodge-Deligne polynomials we can assume that $U_{a,b,c,d,e}=(G_1\times G_2\times G_3\smallsetminus\Delta)\times G_4\times\mathbb{P}^{a-1}$, where Δ is the "big" diagonal of $G_1\times G_2\times G_3=G_1\times G_1\times G_1$. There are 2 schemes and 2 sequences of fibrations for $l=1,2$

$$R^l_{a,b,c,d,e}\xrightarrow{\phi^l}A^l_{a,b,c,d,e}\xrightarrow{\theta^l}U_{a,b,c,d,e}$$

where:

  - both ϕ^1 and θ^1 have fibers isomorphic to $\mathbb{P}^{g-2+j/2}\smallsetminus\mathbb{P}^{j/2-1}$;
  - ϕ^2 has fibers isomorphic to $\mathbb{P}^{g-2+j/2}\smallsetminus\mathbb{P}^{j/2-1}$, θ^2 has fibers isomorphic to $\mathbb{P}^{j/2-1}$;

The (E,V)'s we are interested in are parametrized by the schemes described in (b) and (c) in proposition 7.5.2 (case (a) doesn't occur since there are no choices of $(b,c)<(d,e)$). So the coherent systems (E,V)'s we are interested in are parametrized by the pair of schemes $R^1_{a,b,c,b,c}/\mathbb{Z}_2$ and $R^2_{a,b,c,b,c}$. Also in this case

$$G_1=G_2=G_3=J^{(2d-j)/6}C,\quad G_4=G(1,j/2,1).$$

Moreover, we have that $\Delta\simeq\Delta_0\amalg\Delta_1\amalg\Delta_2\amalg\Delta_3$, where the Δ_i's are as in (15.5). Now $\Delta_0\simeq G_1$ and $\Delta_i\simeq G_1\times G_1\smallsetminus\Delta'$ for $i=1,2,3$, where Δ' is the diagonal of $G_1\times G_1$, so

$$\mathcal{HD}(\Delta)=\mathcal{HD}(G_1)+3(\mathcal{HD}(G_1)^2-\mathcal{HD}(G_1))=$$
$$=\mathcal{HD}(G_1)(\mathcal{HD}(G_1)-2)=(1+u)^g(1+v)^g((1+u)^g(1+v)^g-2).$$

So we get that

$$\mathcal{HD}(R^2_{a,b,c,b,c})=\mathcal{HD}(G_1\times G_1\times G_1\smallsetminus\Delta)\mathcal{HD}(G_4)\mathcal{HD}(\mathbb{P}^{j/2-1})\cdot$$
$$\cdot\mathcal{HD}(\mathbb{P}^{g-2+j/2}\smallsetminus\mathbb{P}^{j/2-1})\mathcal{HD}(\mathbb{P}^{j/2-1})=$$
$$=(1+u)^g(1+v)^g((1+u)^g(1+v)^g-2)\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^gx^{-j/2}}{(1-x)(1-uvx)}\cdot$$
$$\cdot\frac{(1-(uv)^{j/2})^2}{(1-uv)^2}\cdot\frac{(uv)^{j/2}-(uv)^{g-1+j/2}}{1-uv}.$$

Now let us consider $R^1_{a,b,c,b,c}/\mathbb{Z}_2$. By proposition 7.5.2 from the point of view of Hodge-Deligne polynomials we can assume that $R^1_{a,b,c,b,c}/\mathbb{Z}_2$ is isomorphic to $(M/\mathbb{Z}_2)\times G_4\times\mathbb{P}^{j/2-1}$, where M is the scheme

$$M := (G_1 \times G_2 \times G_3 \smallsetminus \Delta) \times (\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-1})^2$$

and where $\mathbb{Z}_2$ acts on M by permutations on $G_1 \times G_2$ and on $(\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-1})^2$. Let us define the scheme

$$M' := G_1 \times (\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-1}).$$

By construction the action of $\mathbb{Z}_2$ is trivial on $\Delta_0 \amalg \Delta_1 \simeq G_1 \times G_1$ and it exchanges Δ_2 and Δ_3. Then

$$M/\mathbb{Z}_2 = \Big(M' \times M' \times G_1\Big)/\mathbb{Z}_2 \smallsetminus \Big((\Delta_0 \amalg \Delta_1 \times (\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-1})^2)/\mathbb{Z}_2 \amalg$$
$$\amalg (\Delta_2 \amalg \Delta_3 \times (\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-1})^2)/\mathbb{Z}_2\Big) =$$
$$= \Big(M' \times M'\Big)/\mathbb{Z}_2 \times G_1 \smallsetminus \Big(\Delta_0 \amalg \Delta_1 \times (\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-1})^2/\mathbb{Z}_2 \amalg$$
$$\amalg \Delta_2 \times (\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-1})^2\Big).$$

Now

$$\mathcal{HD}(M') = \mathcal{HD}(G_1) \times \mathcal{HD}(\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-1}) =$$
$$= (1+u)^g(1+v)^g \frac{(uv)^{j/2} - (uv)^{g-1+j/2}}{1-uv}.$$

Therefore, by [MOVG2, lemma 2.6] we get the following polynomial:

$$\mathcal{HD}((M' \times M')/\mathbb{Z}_2) = \frac{1}{2}\left((1+u)^{2g}(1+v)^{2g}\frac{((uv)^{j/2} - (uv)^{g-1+j/2})^2}{(1-uv)^2} + \right.$$
$$\left. + (1-u^2)^g(1-v^2)^g \frac{(uv)^j - (uv)^{2g-2+j}}{1-(uv)^2}\right).$$

Moreover, we can compute

$$\mathcal{HD}((\mathbb{P}^{g-2+j/2} \smallsetminus \mathbb{P}^{j/2-1})^2/\mathbb{Z}_2) =$$
$$= \frac{1}{2}\left(\frac{((uv)^{j/2} - (uv)^{g-1+j/2})^2}{(1-uv)^2} + \frac{(uv)^j - (uv)^{2g-2+j}}{1-(uv)^2}\right).$$

So we get that:

$$\mathcal{HD}(M/\mathbb{Z}_2) = \frac{1}{2}\left((1+u)^{2g}(1+v)^{2g}\frac{((uv)^{j/2} - (uv)^{g-1+j/2})^2}{(1-uv)^2} + \right.$$
$$\left. + (1-u^2)^g(1-v^2)^g \frac{(uv)^j - (uv)^{2g-2+j}}{1-(uv)^2}\right)(1+u)^g(1+v)^g +$$
$$-\frac{1}{2}(1+u)^{2g}(1+v)^{2g}\left(\frac{((uv)^{j/2} - (uv)^{g-1+j/2})^2}{(1-uv)^2} + \frac{(uv)^j - (uv)^{2g-2+j}}{1-(uv)^2}\right) +$$

$$-(1+u)^g(1+v)^g((1+u)^g(1+v)^g-1)\frac{((uv)^{j/2}-(uv)^{g-1+j/2})^2}{(1-uv)^2}.$$

So we conclude that

$$\mathcal{HD}(R^1_{a,b,c,b,c}/\mathbb{Z}_2)=\mathcal{HD}(M/\mathbb{Z}_2)\mathcal{HD}(G_4)\mathcal{HD}(\mathbb{P}^{j/2-1})=$$
$$=\left\{\frac{1}{2}\left((1+u)^{2g}(1+v)^{2g}\frac{((uv)^{j/2}-(uv)^{g-1+j/2})^2}{(1-uv)^2}+\right.\right.$$
$$\left.+(1-u^2)^g(1-v^2)^g\frac{(uv)^j-(uv)^{2g-2+j}}{1-(uv)^2}\right)(1+u)^g(1+v)^g+$$
$$-\frac{1}{2}(1+u)^{2g}(1+v)^{2g}\left(\frac{((uv)^{j/2}-(uv)^{g-1+j/2})^2}{(1-uv)^2}+\frac{(uv)^j-(uv)^{2g-2+j}}{1-(uv)^2}\right)+$$
$$\left.-(1+u)^g(1+v)^g((1+u)^g(1+v)^g-1)\frac{((uv)^{j/2}-(uv)^{g-1+j/2})^2}{(1-uv)^2}\right\}\cdot$$
$$\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^gx^{-j/2}}{(1-x)(1-uvx)}\cdot\frac{1-(uv)^{j/2}}{1-uv}.$$

So we get the polynomial:

$$p_{19}^{j\equiv_6 2d}=\mathcal{HD}(R^1_{a,b,c,b,c}/\mathbb{Z}_2)+\mathcal{HD}(R^2_{a,b,c,b,c})=$$
$$=\left\{\frac{1}{2}\left((1+u)^{2g}(1+v)^{2g}\frac{((uv)^{j/2}-(uv)^{g-1+j/2})^2}{(1-uv)^2}+\right.\right.$$
$$\left.+(1-u^2)^g(1-v^2)^g\frac{(uv)^j-(uv)^{2g-2+j}}{1-(uv)^2}\right)(1+u)^g(1+v)^g+$$
$$-\frac{1}{2}(1+u)^{2g}(1+v)^{2g}\left(\frac{((uv)^{j/2}-(uv)^{g-1+j/2})^2}{(1-uv)^2}+\frac{(uv)^j-(uv)^{2g-2+j}}{1-(uv)^2}\right)+$$
$$\left.-(1+u)^g(1+v)^g((1+u)^g(1+v)^g-1)\frac{((uv)^{j/2}-(uv)^{g-1+j/2})^2}{(1-uv)^2}\right\}\cdot$$
$$\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^gx^{-j/2}}{(1-x)(1-uvx)}\cdot\frac{1-(uv)^{j/2}}{1-uv}+$$
$$+(1+u)^g(1+v)^g((1+u)^g(1+v)^g-2)\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^gx^{-j/2}}{(1-x)(1-uvx)}\cdot$$
$$\cdot\frac{(1-(uv)^{j/2})^2}{(1-uv)^2}\cdot\frac{(uv)^{j/2}-(uv)^{g-1+j/2}}{1-uv}=$$
$$=(1+u)^g(1+v)^g\frac{1-(uv)^{j/2}}{(1-uv)^2}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^gx^{-j/2}}{(1-x)(1-uvx)}\cdot$$
$$\cdot\left\{\frac{1}{2}\left((1+u)^{2g}(1+v)^{2g}\frac{((uv)^{j/2}-(uv)^{g-1+j/2})^2}{1-uv}+(1-u^2)^g(1-v^2)^g\frac{(uv)^j-(uv)^{2g-2+j}}{1+uv}\right)+\right.$$

$$-\frac{1}{2}(1+u)^g(1+v)^g\left(\frac{((uv)^{j/2}-(uv)^{g-1+j/2})^2}{1-uv}+\frac{(uv)^j-(uv)^{2g-2+j}}{1+uv}\right)+$$

$$-((1+u)^g(1+v)^g-1)\frac{((uv)^{j/2}-(uv)^{g-1+j/2})^2}{1-uv}+$$

$$\left.+((1+u)^g(1+v)^g-2)\frac{(1-(uv)^{j/2})\cdot((uv)^{j/2}-(uv)^{g-1+j/2})}{1-uv}\right\}=$$

$$=(1+u)^g(1+v)^g\frac{1-(uv)^{j/2}}{(1-uv)^2}((uv)^{j/2}-(uv)^{g-1+j/2})\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}\cdot$$

$$\cdot\left\{\frac{1}{1-uv}\left[((uv)^{j/2}-(uv)^{g-1+j/2})\left(\frac{1}{2}(1+u)^{2g}(1+v)^{2g}-\frac{1}{2}(1+u)^g(1+v)^g+\right.\right.\right.$$

$$\left.\left.-(1+u)^g(1+v)^g+1\right)+((1+u)^g(1+v)^g-2)(1-(uv)^{j/2})\right]+$$

$$\left.+\frac{(uv)^{j/2}+(uv)^{g-1+j/2}}{1+uv}\left[\frac{1}{2}(1-u^2)^g(1-v^2)^g-\frac{1}{2}(1+u)^g(1+v)^g\right]\right\}=$$

$$=(1+u)^g(1+v)^g\frac{1-(uv)^{j/2}}{(1-uv)^2}((uv)^{j/2}-(uv)^{g-1+j/2})\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}\cdot$$

$$\cdot\left\{\frac{1}{1-uv}\left[\frac{1}{2}((uv)^{j/2}-(uv)^{g-1+j/2})((1+u)^g(1+v)^g-1)((1+u)^g(1+v)^g-2)+\right.\right.$$

$$\left.+((1+u)^g(1+v)^g-2)(1-(uv)^{j/2})\right]+$$

$$\left.+\frac{(uv)^{j/2}+(uv)^{g-1+j/2}}{2(1+uv)}(1+u)^g(1+v)^g((1-u)^g(1-v)^g-1)\right\}.$$

Also in this case, if $j=0$, then we get that the polynomial is zero.

(3c) Let us suppose that $(Q_1,0)\simeq(Q_2,0)\not\simeq(Q_3,0)$. Then we can apply proposition 7.5.3. In this case we need to compute the invariants a,b,c. Also in this case we get

$$a=\dim\operatorname{Ext}^1((Q_4,W_4),(Q_3,0))=j/2,$$

$$b=\dim\operatorname{Ext}^1((E'',V''),(Q_1,0))=g-1+j/2,$$

$$c=\dim\operatorname{Ext}^1((Q_4,W_4),(Q_1,0))=j/2,$$

so each invariant can only assume one value. From the point of view of Hodge-Deligne polynomials, we can ignore the indices i and j of that proposition, so we can assume that we have the following description.

- $U_a=G_3\times G_4$ and there is a projective bundle $\varphi_a:R_a\to U_a$ with fibers isomorphic to $\mathbb{P}^{a-1}=\mathbb{P}^{j/2-1}$;

- $U_{a,b,c}$ is the set

$$\{((Q_1,0),(E'',V''))\in G_1\times R_a \text{ s.t. } (Q_1,0)\not\simeq\overline{\varphi}_a(E'',V'')\},$$

where $\overline{\varphi}_a$ is the composition of φ_a with the projection to G_3. From the point of view of Hodge-Deligne polynomials we can assume that $U_{a,b,c}=(G_1\times G_3\smallsetminus\Delta)\times G_4\times\mathbb{P}^{j/2-1}$, where Δ is the diagonal of $G_1\times G_3=G_1\times G_1$. The (E,V)'s we are interested in are parametrized by a scheme $R_{a,b,c}$. Such a scheme comes with a fibration $R_{a,b,c}\to U_{a,b,c}$ with fibers isomorphic to $Grass(2,b)\smallsetminus Grass(2,c)=Grass(2,g-1+j/2)\smallsetminus Grass(2,j/2)$.

Then we get the polynomial

$$\begin{aligned}
p_{20}^{j\equiv_6 2d} = \mathcal{HD}(R_{a,b,c}) = (\mathcal{HD}(G_1)^2-\mathcal{HD}(G_1))\times\mathcal{HD}(G_4)\times\mathcal{HD}(\mathbb{P}^{j/2-1})\cdot\\
\cdot\{\mathcal{HD}(Grass(2,g-1+j/2)-\mathcal{HD}(Grass(2,j/2)\}=\\
=(1+u)^g(1+v)^g((1+u)^g(1+v)^g-1)\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^gx^{-j/2}}{(1-x)(1-uvx)}\cdot\frac{1-(uv)^{j/2}}{1-uv}\cdot\\
\cdot\left\{\frac{(1-(uv)^{g-2+j/2})(1-(uv)^{g-1+j/2})-(1-(uv)^{j/2-1})(1-(uv)^{j/2})}{(1-uv)(1-(uv)^2)}\right\}.
\end{aligned}$$

(3d) Let us suppose that $(Q_1,0)\simeq(Q_2,0)\simeq(Q_3,0)$. Then we can apply proposition 7.5.4. In this case the invariants a,b and the scheme R_a are as in (3c). Moreover, the scheme $U_{a,b}$ coincides with R_a and the (E,V)'s we are interested in are parametrized by a scheme $R_{a,b}$. Such a scheme comes with a fibration $R_{a,b}\to U_{a,b}$ with fibers isomorphic to $Grass(2,b)\smallsetminus Grass(2,a-1)=Grass(2,g-1+j/2)\smallsetminus Grass(2,j/2-1)$. Then we get the polynomial

$$\begin{aligned}
p_{21}^{j\equiv_6 2d}=\mathcal{HD}(R_{a,b})=\mathcal{HD}(G_1)\mathcal{HD}(G_4)\times\mathcal{HD}(\mathbb{P}^{j/2-1})\cdot\\
\cdot(\mathcal{HD}(Grass(2,g-1+j/2))-\mathcal{HD}(Grass(2,j/2-1))=\\
=(1+u)^g(1+v)^g\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^gx^{-j/2}}{(1-x)(1-uvx)}\cdot\frac{1-(uv)^{j/2}}{1-uv}\cdot\\
\cdot\left\{\frac{(1-(uv)^{g-2+j/2})(1-(uv)^{g-1+j/2})-(1-(uv)^{j/2-2})(1-(uv)^{j/2-1})}{(1-uv)(1-(uv)^2)}\right\}.
\end{aligned}$$

(4) Canonical filtration of type (1,2,1). In this situation we need to consider the following subcases:

(a) $(Q_1,0)\simeq(Q_2,0)\not\simeq(Q_3,0)$;

(b) $(Q_i,0)\not\simeq(Q_j,0)$ for all $j\neq j\in\{1,2,3\}$;

(c) $(Q_1,0)\not\simeq(Q_2,0)\simeq(Q_3,0)$;

(d) $(Q_1,0)\simeq(Q_2,0)\not\simeq(Q_3,0)$.

As we said in remark 12.2.1 we are able to describe completely only case (a).

(4a) If we suppose that $(Q_1,0)\simeq(Q_2,0)\not\simeq(Q_3,0)$, then we can apply proposition 7.6.1. In this case we need to compute the invariants a,b,c,d,e. In order to do that, first of all let us fix any pair of non-split extensions of the form

$$0\to(Q_i,0)\to(E_{4i},V_{4i})\to(Q_4,W_4)\to 0$$

for $i=2,3$ and let us denote by

$$0\to(Q_2,0)\oplus(Q_3,0)\to(E'',V'')\to(Q_4,W_4)\to 0$$

their sum. Then for $i=2,3$ the coherent system (E_{4i},V_{4i}) has rank $N_{4i}=2$, degree $D_{4i}=(d+j)/3$ and $K_{4i}=1$. Moreover, (E'',V'') is a coherent system of rank $N''=3$, degree $D''=d_2+d_3+d_4=(4d+j)/6$ and with $K''=1$. Since $(Q_2,0)$ and $(Q_3,0)$ are not isomorphic to (Q_4,W_4), as in case (3) we have that:

$$a=\dim\ \mathrm{Ext}^1((Q_4,W_4),(Q_2,0))=j/2,$$
$$b=\dim\ \mathrm{Ext}^1((Q_4,W_4),(Q_3,0))=j/2.$$

Moreover,

$$c=\dim\ \mathrm{Ext}^1((E'',V''),(Q_2,0))=$$
$$=n_2N''(g-1)-d_2N''+D''n_2+K''d_2-K''n_2(g-1)=$$
$$=3(g-1)-\frac{2d-j}{2}+\frac{4d+j}{6}+\frac{2d-j}{6}-(g-1)=2g-2+j/2.$$

In addition,

$$d=\dim\ \mathrm{Ext}^1((E_{42},V_{42}),(Q_2,0))=$$
$$=n_2N_{42}(g-1)-d_2N_{42}+D_{42}n_2+K_{42}d_2-K_{42}n_2(g-1)=$$
$$=2(g-1)-\frac{2d-j}{3}+\frac{d+j}{3}+\frac{2d-j}{6}-(g-1)=g-1+j/2$$

and analogously,

$$e=\dim\ \mathrm{Ext}^1((E_{43},V_{43}),(Q_2,0))=g-1+j/2.$$

So each invariant can assume only one value. By proposition 7.6.1 we can therefore suppose that we have the following description.

- $U_a^2=G_2\times G_4$, $U_b^3=G_3\times G_4$, $U_{a,b}=G_2\times G_3\times G_4$ and $V_{a,b}=(G_2\times G_3\setminus\Delta)\times G_4$; there is a fibration $R_{a,b}\to V_{a,b}$ with fibers isomorphic to $\mathbb{P}^{a-1}\times\mathbb{P}^{b-1}=(\mathbb{P}^{j/2-1})^2$;

- $U_{a,b,c,d,e} = R_{a,b}$ and there is a fibration $R_{a,b,c,d,e} \to U_{a,b,c,d,e}$ with fibers isomorphic to $\mathbb{P}^{c-1} \smallsetminus \mathbb{P}^{d+e-a-1} = \mathbb{P}^{2g-3+j/2} \smallsetminus \mathbb{P}^{2g-3+j/2} = \varnothing$.

Therefore we have proved that there are no (E,V)'s of type $(4,d,1)$ with $\alpha(j)$-canonical filtration of type $(1,2,1)$ and graded $(Q_2,0)\oplus(Q_2,0)\oplus(Q_3,0)\oplus(Q_4,W_4)$ with $(Q_2,0) \not\simeq (Q_3,0)$. So we get simply

$$p_{22}^{j\equiv_6 2d} := \mathcal{HD}(R_{a,b,c,d,e}) = \mathcal{HD}(\varnothing) = 0.$$

(4b)-(4c)-(4d) Currently we are not able to compute the polynomials for these 3 cases. We name such unknown polynomials by $p_{23}^{j\equiv_6 2d}$, $p_{24}^{j\equiv_6 2d}$ and $p_{25}^{j\equiv_6 2d}$ respectively.

Remark 15.1.1. The previous numerical computations together with lemma 12.2.4 actually suggest that also the scheme that should be considered in (4b) is the empty scheme and that consequently also $p_{23}^{j\equiv_6 2d}$ should be equal to zero. Currently, we cannot say anything about (4c) and (4d).

15.2 The moduli spaces $G^-(\alpha(j);4,d,1)$

Also in this case the length r of the filtration of any object (E,V) in $G^-(\alpha(j);4,d,1)$ can only be equal to $2,3$ or 4. So let us consider the 3 different cases.

15.2.1 Case $r=2$

By applying lemma 1.0.6, we get that any (E,V) that belongs to $G^+(\alpha(j);4,d,1)$ with length of the α_c-JHF equal to 2 sits in a non-split exact sequence:

$$0 \to (Q_1,W_1) \to (E,V) \to (Q_2,W_2) \to 0 \tag{15.7}$$

with conditions (a')-(b'). Then condition (a') implies that $k_2 = 0$, so $k_1 = 1$, but a priori n_2 can be either equal to $1,2$ or 3.

(1) If $n_2 = 3$, then $n_2 = 1$; since $\mu_{\alpha(j)}(E,V) = d/3 - j/6$, then condition (b') implies that $d_2 = d - j/2$. Therefore, *this case is possible only if* $j \equiv 0 \bmod 2$. If we assume that condition, then both $d_2 = d - j/2$ and $d_1 = j/2$ are non-negative integers. Since $r = 2$, we must impose that both (Q_1,W_1) and $(Q_2,W_2) = (Q_2,0)$ are $\alpha(j)$-stable. Since there are no critical values for $(1,d_1,1)$ and $(3,d_2,0)$, this simply means that we are considering all the pairs $(Q_1,W_1),(Q_2,0)$ such that:

$$(Q_1,W_1) \in G(1,j/2,1) = G_1, \quad (Q_2,0) \in M^{\mathrm{s}}(3,d-j/2) = G_2.$$

Since $\mathbb{H}^0_{21} = \mathbb{H}^2_{21} = 0$, we get

$$\begin{aligned}\dim \operatorname{Ext}^1((Q_2,0),(Q_1,W_1)) = C_{21} = n_1 n_2(g-1) - d_1 n_2 + d_2 n_1 = \\ = 3(g-1) - 3j/2 + d - j/2 = 3g - 3 + d - 2j.\end{aligned}$$

So for every critical value $\alpha(j)$ such that $j \equiv 0 \bmod 2$ we get a contribution to $G^{-}(\alpha(j);4,d,1)$ by a projective bundle over $G_1 \times G_2$ with fiber $\mathbb{P}^{3g-4+d-2j}$. So we get the polynomial:

$$q_1^{j\equiv_2 0} := \mathcal{HD}(M^s(3,d-j/2))\frac{1-(uv)^{3g-3+d-2j}}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}.$$

Now we recall that we are assuming that j is even, so:

- $d - j/2 \equiv 0 \bmod 3$ if and only if $j \equiv 2d \bmod 6$;
- $d - j/2 \not\equiv 0 \bmod 3$ if and only if $j \equiv 2d+2 \bmod 6$ or $j \equiv 2d+4 \bmod 6$.

In the first case, we don't know an explicit formula for the Hodge-Deligne polynomial of $M^s(3,d-j/2)$; in the second case we have an explicit formula, as described in chapter 8 (and such a formula does not depend on j or d). We will denote the corresponding 2 polynomials by $\mathcal{HD}(M(3,j \equiv_6 2d))$ and $\mathcal{HD}(M^{\mathrm{s}}(3,j\equiv_6 2d+2)) = \mathcal{HD}(M^{\mathrm{s}}(3,j\equiv_6 2d+4))$ respectively. According to that notation, we denote by $q_1^{j\equiv_6 2d}$ and $q_1^{j\equiv_6 2d+2} = q_1^{j\equiv_6 2d+4}$ the corresponding polynomials.

(2) If $n_2 = 2$, then $n_1 = 2$. Moreover, condition (b') implies that $d_2 = (2d-j)/3$, so *this case is possible only if* $j \equiv 2d \bmod 3$, that is $j \in \{2d, 2d+3\}_{\bmod 6}$. If we assume that condition, then both d_2 and $d_1 = d - d_1 = (d+j)/3$ are non-negative integers.

Now we recall that both (Q_1, W_1) and $(Q_2, 0)$ must be strictly $\alpha(j)$-stable (otherwise, the length of the Jordan-Hölder filtration would be bigger than 2). So we need to consider 2 cases:

(a) if $j \equiv 2d \bmod 6$, then $d_2 = (2d-j)/3$ is even, so we are considering

$$(Q_2, 0) \in M^{\mathrm{s}}\left(2, \frac{2d-j}{3}\right) = M^{\mathrm{s}}(2,\mathrm{even}) =: G_2;$$

(b) if $j \equiv 2d+3 \bmod 6$, then $d_2 = (2d-j)/3$ is odd, so we are considering

$$(Q_2, 0) \in M^{\mathrm{s}}\left(2, \frac{2d-j}{3}\right) = M^{\mathrm{s}}(2,\mathrm{odd}) = M^{\mathrm{ss}}(2,\mathrm{odd}) =: G_2'.$$

Analogously, (Q_1, W_1) must be an object of the moduli space $G^{\mathrm{s}}(\alpha(j);2,(d+j)/3,1)$. Such a scheme is not empty if and only if $0 < \alpha(j) < (d+j)/3$, but this condition is automatically satisfied by definition of $\alpha(j)$ for all $j > 0$ (for $j = 0$ the moduli space of semistable objects is non-empty, while the stable locus is empty). Then we have to verify if $\alpha(j)$ is critical for the triple $(2,(d+j)/3,1)$. According to the computations of chapter 13, $\alpha(j)$ is critical for such a triple if and only if $\alpha(j) = (d+j)/3 - 2k$ for some $0 \le k < (d+j)/6$. So this gives:

$$\frac{d-2j}{3} = \alpha(j) = \frac{d+j}{3} - 2k = \frac{d+j-6k}{3} \quad \Leftrightarrow \quad j = 2k.$$

So $\alpha(j)$ is critical for $(2,(d+j)/3,1)$ if and only if $j=2k$ for some $0\leq k<(d+j)/6$. If we set $j=2k$, this is equivalent to imposing $0\leq j<(d+j)/3$, that is equivalent to $0\leq j<d/2$. These are exactly the conditions we already put on j, so $\alpha(j)$ is critical for $(2,(d+j)/3,1)$ if and only if j is any admissible value (i.e. $0\leq j<d/2$) such that $j\equiv 0 \bmod 2$. Now we have to distinguish 2 cases as follows.

(i) If $j\equiv 0 \bmod 2$, then $(d-2j)/3$ is a critical value for $(2,(d+j)/3,1)$. In particular, if we write $k:=j/2\in\mathbb{N}_0$, then we can write $(d-2j)/3=(d+j)/3-2k$ and we need to consider

$$(Q_1,W_1)\in G^{\mathrm{s}}\left(\frac{d+j}{3}-2k;2,\frac{d+j}{3},1\right)=:G_1.$$

According to corollary 13.3.2 with d replaced by $(d+j)/3$ and k replaced by $j/2$, we have that

$$\mathcal{HD}(G_1)=\frac{(1+u)^g(1+v)^g}{1-uv}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot$$
$$\cdot\left[\frac{(uv)^{j/2}x^{-j/2}}{1-x(uv)^{-1}}-\frac{(uv)^{g+1+(d-2j)/3}x^{1-j/2}}{1-x(uv)^2}-x^{-j/2}\right].$$

(ii) If $j\equiv 1 \bmod 2$, then we can define $k:=(j-1)/2\in\mathbb{N}_0$, so that

$$\frac{d-2j}{3}=\frac{d+j}{3}-2k-1.$$

We recall that the critical values of $(2,(d+j)/3,1)$ are of the form $(d+j)/3-2k$, so we need to consider

$$(Q_1,W_1)\in G^{\mathrm{s}}\left(\frac{d+j}{3}-2k-1;2,\frac{d+j}{3},1\right)=$$
$$=G^{\mathrm{s}}\left(\frac{d+j}{3}-2k-\varepsilon;2,\frac{d+j}{3},1\right)=:G_1'.$$

According to theorem 13.3.1 with d replaced by $(d+j)/3$ and k replaced by $(j-1)/2$, we have that

$$\mathcal{HD}(G_1')=\frac{(1+u)^g(1+v)^g}{1-uv}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot$$
$$\cdot\left[\frac{(uv)^{(j-1)/2}x^{(1-j)/2}}{1-x(uv)^{-1}}-\frac{(uv)^{g+(d-2j)/3}x^{(1-j)/2}}{1-x(uv)^2}\right].$$

We recall that we are under the hypothesis $j \in \{2d, 2d+3\}_{\text{mod } 6}$. Therefore, under that condition we have $j \equiv 0 \bmod 2$ if and only if $j \equiv 2d \bmod 6$ and $j \equiv 1 \bmod 2$ if and only if $j \equiv 2d+3 \bmod 6$. So cases (a) and (b) match with the cases (i) and (ii) respectively.

Since $\mathbb{H}^0_{21} = \mathbb{H}^2_{21} = 0$ for all values of j, we get:

$$\begin{aligned}\dim \operatorname{Ext}^1((Q_2,0),(Q_1,W_1)) &= C_{21} = n_1 n_2(g-1) - d_1 n_2 + d_2 n_1 = \\ &= 4(g-1) - 2\tfrac{d+j}{3} + 2\tfrac{2d-j}{3} = 4g - 4 + \tfrac{2d-4j}{3}.\end{aligned}$$

So for every critical value $\alpha(j)$ such that $j \equiv 2d \bmod 3$, we get a contribution to $G^-(\alpha(j);4,d,1)$ by:

- a projective bundle over $G_1 \times G_2$ with fiber $\mathbb{P}^{4g-5+(2d-4j)/3}$ if $j \equiv 2d \bmod 6$;
- a projective bundle over $G_1' \times G_2'$ with the same fiber if $j \equiv 2d+3 \bmod 6$;

So we get the polynomials

$$\begin{aligned} q_2^{j\equiv_6 2d} = &\frac{1}{2(1-uv)(1-(uv)^2)}\Big(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g + \\ &-(1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1}-u^2v^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2\Big)\cdot \\ &\cdot\frac{(1+u)^g(1+v)^g}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot \\ &\cdot\left[\frac{(uv)^{j/2}x^{-j/2}}{1-x(uv)^{-1}} - \frac{(uv)^{g+1+(d-2j)/3}x^{1-j/2}}{1-x(uv)^2} - x^{-j/2}\right]\cdot\frac{1-(uv)^{4g-4+(2d-4j)/3}}{1-uv}\end{aligned}$$

and

$$\begin{aligned} q_2^{j\equiv_6 2d+3} = &\frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^{2g}(1+v)^{2g}}{(1-uv)(1-(uv)^2)}\cdot \\ &\cdot\frac{(1+u)^g(1+v)^g}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot \\ &\cdot\left[\frac{(uv)^{(j-1)/2}x^{(1-j)/2}}{1-x(uv)^{-1}} - \frac{(uv)^{g+(d-2j)/3}x^{(1-j)/2}}{1-x(uv)^2}\right]\cdot\frac{1-(uv)^{4g-4+(2d-4j)/3}}{1-uv}\end{aligned}$$

according to the 2 possible values of j modulo 6.

(3) If $n_2 = 1$, then $n_1 = 3$. Moreover, condition (b') implies that $d_2 = (2d-j)/6$, so *this case is possible only if* $j \equiv 2d \bmod 6$. If we assume that condition, then both d_2 and $d_1 = d - d_1 = (4d+j)/6$ are non-negative integers.

Now both (Q_1,W_1) and $(Q_2,0)$ must be strictly $\alpha(j)$-stable. For $(Q_2,0)$, this simply amounts to considering all possible objects of $J^{(2d-j)/6} =: G_2$. On the other hand, (Q_1,W_1)

must be an object of the moduli space $G^{\mathrm{s}}(\alpha(j);3,(4d+j)/6,1)$. Such a scheme is non-empty if and only if $\alpha(j) < (4d+j)/12$, but this condition is automatically satisfied by definition of $\alpha(j)$ for all $j>0$ (if $j=0$, the semistable locus is non-empty, while the stable locus is empty). Then we have to verify if $\alpha(j)$ is critical for the triple $(3,(4d+j)/6,1)$. According to the computations of chapter 14, $\alpha(j)$ is critical for such a triple if and only if $\alpha(j)=d_1/2-3k/2=(4d+j)/12-3k/2$ for some $0\leq k<d_2/3=(4d+j)/18$. So this gives:

$$\frac{d-2j}{3}=\alpha(j)=\frac{4d+j}{12}-\frac{3k}{2}=\frac{4d+j-18k}{12}\quad\Leftrightarrow\quad j=2k.$$

So $\alpha(j)$ is critical for $(3,d_1,1)$ if and only if $j=2k$ for some $0\leq k<(4d+j)/18$. If we set $j:=2k$, this is equivalent to imposing $0\leq j<(4d+j)/9$. These conditions are equivalent to $0\leq j<d/2$, that are exactly the conditions we already put on j. Therefore, $\alpha(j)$ is critical for $(3,d_1,1)$ if and only if j is any admissible value (i.e. $0\leq j<d/2$) such that $j\equiv 0 \bmod 2$. But we recall that the case we are considering is possible only when $j\equiv 2d \bmod 6$, that implies $j\equiv 0 \bmod 2$. Therefore, when this case is possible $\alpha(j)$ is always critical for $(3,d_1,1)$. Then if we define $k:=j/2\in\mathbb{N}_0$, we need to consider

$$(Q_1,W_1)\in G^{\mathrm{s}}\left(\frac{1}{2}\left(\frac{4d+j}{6}-3k\right);3,\frac{4d+j}{6},1\right)=:G_1.$$

Now we recall that according to chapter 14 we have 2 different formulae for the Hodge-Deligne polynomial of $G^{\mathrm{s}}((d'-3k')/2;3,d',1)$ depending on $d'-k'$ being odd or even. In our case $d'=(4d+j)/6$ and $k'=j/2$, so

$$d'-k'=\frac{4d+j}{6}-\frac{j}{2}=\frac{4d-2j}{6}=\frac{2d-j}{3}.$$

Since in this case we are assuming $j\equiv 2d \bmod 6$, then $d'-k'$ is even, so we can apply corollary 14.3.7 with d replaced by $(4d+j)/6$ and k replaced by $j/2$ and we get

$$\begin{aligned}
&\mathcal{HD}(G_1)=(1+u)^g(1+v)^g\cdot\\
&\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot\Bigg\{\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\cdot\\
&\cdot\left(\frac{(uv)^jx^{-j/2}}{1-(uv)^{-2}x}-\frac{(uv)^{2g+1+(2d-4j)/3}x^{1-j/2}}{1-(uv)^3x}-x^{-j/2}\right)+\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot\\
&\cdot\Bigg(\frac{(uv)^{j+1}x^{-j/2}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)}+\frac{(uv)^{2g+4+(2d-4j)/3}x^{2-j/2}}{(1-(uv)^3x)(1-(uv)^2x)}-\frac{(1+uv)(uv)^{g+1+(2d-j)/6}x^{1-j/2}}{(1-(uv)^{-1}x)(1-(uv)^2x)}+\\
&-uvx^{-j/2}\Bigg)-\frac{(1+u)^g(1+v)^g}{(1-uv)^2}\left[\frac{(uv)^{j/2}x^{-j/2}}{1-(uv)^{-1}x}-\frac{(uv)^{g+1+(d-3j)/3}x^{1-j/2}}{1-(uv)^2x}-x^{-j/2}\right]\Bigg\}.
\end{aligned}$$

Now for all values of j we have that $\mathbb{H}^0_{21}=\mathbb{H}^2_{21}=0$, so:

$$\begin{aligned}\dim \operatorname{Ext}^1((Q_2,0),(Q_1,W_1))&=C_{21}=n_1n_2(g-1)-d_1n_2+d_2n_1=\\&=3(g-1)-\tfrac{4d+j}{6}+\tfrac{2d-j}{2}=3g-3+\tfrac{d-2j}{3}.\end{aligned}$$

Then we will get a projective bundle over $G_1 \times G_2$ with fibers $\mathbb{P}^{3g-4+(d-j)/3}$. So we get the polynomial

$$\begin{aligned}
q_3^{j\equiv_6 2d} &:= \mathcal{HD}(G_1)\mathcal{HD}(G_2)\mathcal{HD}(\mathbb{P}^{3g-4+(d-2j)/3} = \\
&= (1+u)^g(1+v)^g\frac{1-(uv)^{3g-3+(d-2j)/3}}{1-uv}\mathcal{HD}(G_1) = \\
&= (1+u)^{2g}(1+v)^{2g}\frac{1-(uv)^{3g-3+(d-2j)/3}}{1-uv}\cdot
\end{aligned}$$

$$\cdot \operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot\left\{\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^2(1-(uv)^2)}\cdot\right.$$

$$\cdot\left(\frac{(uv)^jx^{-j/2}}{1-(uv)^{-2}x}-\frac{(uv)^{2g+1+(2d-4j)/3}x^{1-j/2}}{1-(uv)^3x}-x^{-j/2}\right)+\frac{(uv)^{g-1}(1+u)^g(1+v)^g}{(1-uv)^2(1+uv)}\cdot$$

$$\cdot\left(\frac{(uv)^{j+1}x^{-j/2}}{(1-(uv)^{-2}x)(1-(uv)^{-1}x)}+\frac{(uv)^{2g+4+(2d-4j)/3}x^{2-j/2}}{(1-(uv)^3x)(1-(uv)^2x)}-\frac{(1+uv)(uv)^{g+1+(2d-j)/6}x^{1-j/2}}{(1-(uv)^{-1}x)(1-(uv)^2x)}+\right.$$

$$\left.\left.-uvx^{-j/2}\right)-\frac{(1+u)^g(1+v)^g}{(1-uv)^2}\left[\frac{(uv)^{j/2}x^{-j/2}}{1-(uv)^{-1}x}-\frac{(uv)^{g+1+(d-3j)/3}x^{1-j/2}}{1-(uv)^2x}-x^{-j/2}\right]\right\}.$$

15.2.2 Case $r=3$

In this case the graded of (E,V) is necessarily made of 3 objects of the form (Q_1,W_1), $(Q_2,0)$, $(Q_3,0)$ (a priori not necessarily in this order) where 2 of the Q_i's are line bundles and one is a vector bundle of rank 2. So we need to consider 3 possibilities.

(a) If Q_1 is the vector bundle of rank 2, then necessarily Q_2 and Q_3 are line bundles of the same degree $d_2=d_3=(2d-j)/6$ and Q_1 has degree $d_1=(d+j)/3$.

(b) If Q_1 is a line bundle, then we get that $d_1=j/2$; if Q_2 is a line bundle, then it has degree $d_2=(2d-j)/6$ and Q_3 is a vector bundle of rank 2 and degree $d_3=(2d-j)/3$.

(c) If Q_1 is a line bundle, then we get that $d_1=j/2$; if Q_3 is a line bundle, then it has degree $d_3=(2d-j)/6$ and Q_2 is a vector bundle of rank 2 and degree $d_2=(2d-j)/3$.

Therefore, *all the* 3 *cases are possible only when* $j\equiv 2d \bmod 6$. For each case we have to consider 2 different subcases according to the various $\alpha(j)$-canonical filtrations.

(1) Unique $\alpha(j)$-Jordan-Hölder filtration. If the filtration is unique, we need to fix the order of the 3 objects of the graded. The object (Q_1,W_1) must be necessarily the first object of the graded, otherwise it destabilizes (E,V) for $\alpha(j)^-$. Therefore we have the following possibilities.

(1a) Let us suppose that the graded is given by $(Q_1,W_1)\oplus(Q_2,0)\oplus(Q_3,0)$ with:

- Q_2 and Q_3 both line bundles of degree $d_2 = d_3 = (2d - j)/6$;
- (Q_1, W_1) with Q_1 vector bundle of rank 2 and degree $d_1 = (d + j)/3$.

In this case $\mathrm{Hom}\left((Q_1, W_1), (Q_2, 0)\right) = 0$ because both objects are α_c-stable with the same slope and they are not isomorphic. Then we have to consider two subcases as follows

(1a-i) If we suppose that $(Q_2, 0) \not\simeq (Q_3, 0)$, then we can apply proposition 6.1.6 in order to parametrize all the corresponding (E, V)'s. In this case $\mathrm{Hom}((Q_3, 0), (Q_2, 0)) = 0$, so the invariant a of that proposition can only assume the value

$$a = \dim\ \mathrm{Ext}^1((Q_3, 0), (Q_2, 0)) = C_{32} = n_2 n_3 (g-1) - d_2 n_3 + d_3 n_2 = g - 1$$

on the set $U_a = G_2 \times G_3 \smallsetminus \Delta_{23}$. So we have a projective bundle R_a over U_a with fibers isomorphic to $\mathbb{P}^{a-1} = \mathbb{P}^{g-2}$. If we write $E'' = (E'', 0)$ for any non-split extension of Q_3 by Q_2, we get that it is a vector bundle of rank $N'' = 2$ and degree $D'' = 2d_2 = (2d-j)/3$. Moreover, $\mathrm{Ext}^2((E'', 0), (Q_1, W_1)) = 0$ because $k_1 = 1$ and also $\mathrm{Hom}(-,-) = 0$; therefore we get that the invariant b can assume only the value:

$$\begin{aligned} b &= \dim\ \mathrm{Ext}^1((E'', V''), (Q_1, W_1)) = n_1 N''(g-1) - d_1 N'' + D'' n_1 = \\ &= 4(g-1) - 2\frac{d+j}{3} + 2\frac{2d-j}{3} = 4g - 4 + \frac{2d-4j}{3}. \end{aligned}$$

Moreover, the invariant c can only assume the value:

$$\begin{aligned} c &= \dim\ \mathrm{Ext}^1((Q_3, 0), (Q_1, W_1)) = C_{31} = n_1 n_3 (g-1) - d_1 n_3 + d_3 n_1 = \\ &= 2(g-1) - \frac{d+j}{3} + \frac{2d-j}{3} = 2g - 2 + (d-2j)/3. \end{aligned}$$

Therefore, we get that $U_{a,b,c} = G_1 \times R_a$ and we have a bundle $R_{a,b,c}$ over $U_{a,b,c}$ with fibers isomorphic to $\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1} = \mathbb{P}^{4g-5+(2d-4j)/3} \smallsetminus \mathbb{P}^{2g-3+(d-2j)/3}$, that parametrizes all the (E, V)'s under consideration. We recall that $G_2 = G_3 = J^{(d-2j)/6} C$. Since $j \equiv 2d \bmod 6$, then we can define $k := j/2 \in \mathbb{N}_0$; since $0 \leq j < d/2$, then we get that $0 \leq k < (d+j)/6$. So that we are considering all the (Q_1, W_1)'s in the scheme

$$G_1 = G^{\mathrm{s}}\left(\frac{d-2j}{3}; 2, \frac{d+j}{3}, 1\right) = G^{\mathrm{s}}\left(\frac{d+j}{3} - 2k; 2, \frac{d+j}{3}, 1\right).$$

Then formula (15.3) gives the Hodge-Deligne polynomial of G_1. Then we get the Hodge-Deligne polynomial

$$\begin{aligned} q_4^{j \equiv_6 2d} &:= \mathcal{HD}(R_{a,b,c}) = \left(\mathcal{HD}(\mathbb{P}^{4g-5+(2d-4j)/3}) - \mathcal{HD}(\mathbb{P}^{2g-3+(d-2j)/3})\right) \mathcal{HD}(G_1) \mathcal{HD}(R_a) = \\ &= \left(\mathcal{HD}(\mathbb{P}^{4g-5+(2d-4j)/3}) - \mathcal{HD}(\mathbb{P}^{2g-3+(d-2j)/3})\right) \mathcal{HD}(\mathbb{P}^{g-2}) \cdot \\ &\quad \cdot \mathcal{HD}(G_1) \cdot \mathcal{HD}(G_2)\left(\mathcal{HD}(G_2) - 1\right) = \end{aligned}$$

$$= \frac{(uv)^{2g-2+(d-2j)/3}-(uv)^{4g-4+(2d-4j)/3}}{1-uv}\cdot\frac{1-(uv)^{g-1}}{1-uv}\cdot\frac{(1+u)^g(1+v)^g}{1-uv}\cdot$$

$$\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot\left[\frac{(uv)^{j/2}x^{-j/2}}{1-x(uv)^{-1}}-\frac{(uv)^{g+(d-2j)/3+1}x^{1-j/2}}{1-x(uv)^2}-x^{-j/2}\right]\cdot$$

$$\cdot(1+u)^g(1+v)^g((1+u)^g(1+v)^g-1).$$

(1a-ii) Let us suppose that $(Q_2,0)\simeq(Q_3,0)$. Since $k_2=0$, then we get that $\mathrm{Ext}^2((Q_2,0),(Q_1,W_1))=0$. So we can apply proposition 6.1.8 in order to parametrize all the corresponding (E,V)'s. In this case we need to compute the invariants:

$$a=\dim\,\mathrm{Ext}^1((Q_3,0),(Q_2,0))=C_{32}+1=g$$

and

$$b=\dim\,\mathrm{Ext}^1((Q_3,0),(Q_1,W_1))=C_{31}=2g-2+(d-2j)/3.$$

Therefore, we get a projective bundle R_a over $G_2=G_3$ with fibers isomorphic to $\mathbb{P}^{g-1}$; the (E,V)'s we are interested in are parametrized by a bundle $R_{a,b}$ over $G_1\times R_a$ with fibers isomorphic to $\mathbb{C}^{2g-3+(d-2j)/3}\times\mathbb{P}^{2g-3+(d-2j)/3}$. In this case the schemes G_1 and G_2 are as in case (1a-i), so we get the polynomial:

$$q_5^{j\equiv_6 2d}:=\mathcal{HD}(G_2)\mathcal{HD}(G_1)\mathcal{HD}(\mathbb{C}^{2g-3+(d-2j)/3})\mathcal{HD}(\mathbb{P}^{2g-3+(2d-j)/3})\mathcal{HD}(\mathbb{P}^{g-1})=$$

$$=(1+u)^g(1+v)^g\frac{(1+u)^g(1+v)^g}{1-uv}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot$$

$$\cdot\left[\frac{(uv)^{j/2}x^{-j/2}}{1-x(uv)^{-1}}-\frac{(uv)^{g+(d-2j)/3+1}x^{1-j/2}}{1-x(uv)^2}-x^{-j/2}\right]\cdot$$

$$\cdot(uv)^{2g-3+(d-2j)/3}\frac{1-(uv)^{2g-2+(d-2j)/3}}{1-uv}\cdot\frac{1-(uv)^g}{1-uv}.$$

(1b) Let us suppose that the graded is given by $(Q_1,W_1)\oplus(Q_2,0)\oplus(Q_3,0)$ where:

- Q_1 is a line bundle of degree $d_1=j/2$;
- Q_2 is a line bundle of degree $d_2=(2d-j)/6$;
- Q_3 is a vector bundle of rank 2 and degree $d_3=(2d-j)/3$.

Since Q_2 is a line bundle and Q_3 is a vector bundle of rank 2, then these 2 coherent systems cannot be isomorphic; since they are both α_c-stable, we get that $\mathrm{Hom}((Q_3,0),(Q_2,0))=0$. Moreover, we have also that $\mathrm{Hom}\,((Q_2,0),(Q_1,W_1))=0$ because both objects are α_c-stable with the same slope and they are not isomorphic. Then we can apply proposition 6.1.6 in order to parametrize all the corresponding (E,V)'s. In this case the invariant a can only assume the value

$$a = \dim \operatorname{Ext}^1((Q_3,0),(Q_2,0)) = C_{32} = n_2 n_3(g-1) - d_2 n_3 + d_3 n_2 =$$

$$= 2(g-1) - \frac{2d-j}{3} + \frac{2d-j}{3} = 2g-2$$

on the set $U_a = G_2 \times G_3$. So we get a projective bundle R_a over U_a with fibers isomorphic to $\mathbb{P}^{2g-3}$. If we write $E'' = (E'',0)$ for any extension of Q_3 by Q_2, we get that $N'' = 3$ and $D'' = d_2 + d_3 = d - j/2$. Moreover, $\operatorname{Ext}^2((E'',0),(Q_1,W_1)) = 0$ because $k'' = 0$ and also $\operatorname{Hom}(-,-) = 0$; therefore we get that the invariant b can assume only the value:

$$b = \dim \operatorname{Ext}^1((E'',0),(Q_1,W_1)) = n_1 N''(g-1) - d_1 N'' + D'' n_1 =$$

$$= 3(g-1) - 3j/2 + d - j/2 = 3g - 3 + d - 2j.$$

Moreover, the invariant c can only assume the value:

$$c = \dim \operatorname{Ext}^1((Q_3,0),(Q_1,W_1)) = C_{31} = n_1 n_3(g-1) - d_1 n_3 + d_3 n_1 =$$

$$= 2g - 2 - j + (2d-j)/3 = 2g - 2 + (2d-4j)/3.$$

Therefore, we get that $U_{a,b,c} = G_1 \times R_a$ and we get a bundle $R_{a,b,c}$ over $U_{a,b,c}$ with fibers isomorphic to $\mathbb{P}^{b-1} \smallsetminus \mathbb{P}^{c-1} = \mathbb{P}^{3g-4+d-2j} \smallsetminus \mathbb{P}^{2g-3+(2d-4j)/3}$. Now the objects (Q_i, W_i)'s vary in the following sets:

$$(Q_1, W_1) \in G_1 = G(1, j/2, 1),$$

$$(Q_2, 0) \in G_2 = J^{(2d-j)/6}C, \quad (Q_3,0) \in G_3 = M^{\mathrm{s}}(2,(2d-j)/3).$$

Since we are assuming that $j \equiv 2d \bmod 6$, then $(2d-j)/3$ is even, so for the scheme G_3 we need to use formula (8.11). Then we get the Hodge-Deligne polynomial

$$q_6^{j \equiv_6 2d} := \mathcal{HD}(R_{a,b,c}) = \Big(\mathcal{HD}(\mathbb{P}^{3g-4+d-2j}) - \mathcal{HD}(\mathbb{P}^{2g-3+(2d-4j)/3})\Big)\mathcal{HD}(G_1)\mathcal{HD}(R_a) =$$

$$= \frac{(uv)^{2g-2+(2d-4j)/3} - (uv)^{3g-3+d-2j}}{1-uv} \cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)} \cdot$$

$$\cdot \mathcal{HD}(\mathbb{P}^{2g-3}) \cdot \mathcal{HD}(J^{(2d-j)/6}C) \cdot \mathcal{HD}(M^{\mathrm{s}}(2,\mathrm{even})) =$$

$$= \frac{(uv)^{2g-2+(2d-4j)/3} - (uv)^{3g-3+d-2j}}{1-uv} \cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)} \cdot \frac{1-(uv)^{2g-2}}{1-uv} \cdot$$

$$\cdot (1+u)^g(1+v)^g \cdot \frac{1}{2(1-uv)(1-(uv)^2)} \Big(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g +$$

$$-(1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1} - u^2v^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2\Big).$$

(1c) Let us suppose that the graded is given by $(Q_1, W_1) \oplus (Q_2, 0) \oplus (Q_3, 0)$ where:

- Q_1 is a line bundle of degree $d_1 = j/2$;

- Q_2 is a vector bundle of rank 2 and degree $d_2=(2d-j)/3$;
- Q_3 is a line bundle of degree $d_3=(2d-j)/6$.

In this case $\operatorname{Hom}((Q_2,0),(Q_1,W_1))=0$ because both objects are α_c-stable with the same slope and they are not isomorphic. Since Q_2 is a line bundle and Q_2 is a vector bundle of rank 2, then these 2 coherent systems cannot be isomorphic, therefore we can apply again proposition 6.1.6 in order to parametrize all the corresponding (E,V)'s. Since $\operatorname{Hom}((Q_3,0),(Q_2,0))=0$, then the invariant a can only assume the value

$$a=\dim \operatorname{Ext}^1((Q_3,0),(Q_2,0))=C_{32}=n_2n_3(g-1)-d_2n_3+d_3n_2=$$
$$=2(g-1)-(2d-j)/3+(2d-j)/3=2g-2$$

on the set $U_a=G_2\times G_3$. So we will get a projective bundle R_a over U_a with fibers $\mathbb{P}^{2g-3}$. If we write $E''=(E'',0)$ for any non-split extension of Q_3 by Q_2, we get that $N''=3$ and $D''=d_2+d_3=d-j/2$. Moreover, $\operatorname{Ext}^2((E'',0),(Q_1,W_1))=0$ because $K''=0$ and also $\operatorname{Hom}(-,-)=0$; therefore we get that the invariant b can assume only the value:

$$b=\dim \operatorname{Ext}^1((E'',0),(Q_1,W_1))=n_1N''(g-1)-d_1N''+D''n_1=$$
$$=3(g-1)-3j/2+d-j/2=3g-3+d-2j.$$

Moreover, the invariant c can only assume the value:

$$c=\dim \operatorname{Ext}^1((Q_3,0),(Q_1,W_1))=C_{31}=n_1n_3(g-1)-d_1n_3+d_3n_1=$$
$$=g-1-j/2+(2d-j)/6=g-1+(d-2j)/3.$$

Therefore, we get that $U_{a,b,c}=G_1\times R_a$ and we get a bundle $R_{a,b,c}$ over $U_{a,b,c}$ with fibers isomorphic to $\mathbb{P}^{b-1}\smallsetminus\mathbb{P}^{a-1}=\mathbb{P}^{3g-4+d-2j}\smallsetminus\mathbb{P}^{g-2+(d-2j)/3}$. Now the objects (Q_i,W_i)'s vary in the following sets:

$$(Q_1,W_1)\in G_1=G(1,j/2,1),$$
$$(Q_2,0)\in G_2=M^{\mathrm{s}}(2,(2d-j)/3),\quad (Q_3,0)\in G_3=J^{(2d-j)/6}C.$$

Since we are assuming that $j\equiv 2d \bmod 6$, then $(2d-j)/3$ is even, so for the scheme G_2 we need to use formula (8.11). Then we get the Hodge-Deligne polynomial

$$q_7^{j\equiv_6 2d}:=\mathcal{HD}(R_{a,b,c})=\Big(\mathcal{HD}(\mathbb{P}^{3g-4+d-2j})-\mathcal{HD}(\mathbb{P}^{g-2+(d-2j)/3})\Big)\mathcal{HD}(G_1)\mathcal{HD}(R_a)=$$
$$=\frac{(uv)^{g-1+(d-2j)/3}-(uv)^{3g-3+d-2j}}{1-uv}\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}\cdot$$
$$\cdot\mathcal{HD}(\mathbb{P}^{2g-3})\cdot\mathcal{HD}(M^{\mathrm{s}}(2,\text{even}))\cdot\mathcal{HD}(J^{(2d-j)/6}C)=$$
$$=\frac{(uv)^{g-1+(d-2j)/3}-(uv)^{3g-3+d-2j}}{1-uv}\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}\cdot\frac{1-(uv)^{2g-2}}{1-uv}\cdot$$

$$\cdot(1+u)^g(1+v)^g \cdot \frac{1}{2(1-uv)(1-(uv)^2)}\Big(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g+$$

$$-(1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1}-u^2v^2)-(1-u^2)^g(1-v^2)^g(1-uv)^2\Big).$$

This concludes the computations for the case when (E,V) has unique $\alpha(j)$-Jordan-Hölder filtration of length 3.

(2) Not unique $\alpha(j)$-Jordan-Hölder filtration. In this case the $\alpha(j)$-canonical filtration has necessarily length 2 (it cannot be equal to 1 because in that case this would imply that the corresponding (E,V)'s are semistable only at the critical value and they are not stable either on the left or on the right of any such value). If the length of the canonical filtration is $s=2$, by using the same argument used before we get that the canonical filtration of (E,V) is given by

$$0\subset(Q_1,W_1)\subset(E,V)$$

with $(E,V)/(Q_1,W_1)\simeq(Q_2,0)\oplus(Q_3,0)$. In this case the 2 cases denoted by (b) and (c) before coincide since the order of $(Q_2,0)$ and $(Q_3,0)$ is not important. Therefore, we have only to consider cases (a) and (b).

(2a) Let us suppose that the (Q_i,W_i)'s are described as in (a). Then $(Q_2,0)$ and $(Q_3,0)$ are of the same type, so we need to consider 2 subcases.

(2a-i) Let us suppose that $(Q_2,0)\not\simeq(Q_3,0)$; since these objects are of the same type, then we can use proposition 7.1.2 in order to have a global parametrization. In this case the invariant a can assume only the value

$$a=\dim\,\mathrm{Ext}^1\left((Q_2,0),(Q_1,W_1)\right)=C_{21}=n_1n_2(g-1)-d_1n_2+d_2n_1=$$
$$=2(g-1)-\frac{d+j}{3}+\frac{2d-j}{3}=2g-2+(d-2j)/3$$

and analogously also b can only assume the value $b=g-1+(d-2j)/3$. Therefore the only schemes we are interested in are those described in (c) and (d) in that proposition. From the point of view of Hodge-Deligne polynomials, we can assume that there is a unique index $i=j$, so that there is only one scheme of type (d). Using the last part of proposition 7.1.2, we get that the (E,V)'s we are interested in are parametrized by a scheme $M/\mathbb{Z}_2$ and from the point of view of Hodge-Deligne polynomials we can assume that M is the scheme

$$G_1\times(G_2\times G_3\smallsetminus\Delta_{23})\times\mathbb{P}^{2g-3+(d-2j)/3}\times\mathbb{P}^{2g-3+(d-2j)/3},$$

where $\mathbb{Z}_2$ acts by:

$$\Big((Q_1,W_1),Q_2,Q_3,\mu_2,\mu_3\Big)\mapsto\Big((Q_1,W_1),Q_3,Q_2,\mu_3,\mu_2\Big).$$

Let us write $M':=G_2\times\mathbb{P}^{2g-3+(d-2j)/3}=J^{(2d-j)/6}C\times\mathbb{P}^{2g-3+(d-2j)/3}$; then

$$\mathcal{HD}(M')(u,v)=\frac{1-(uv)^{2g-2+(d-2j)/3}}{1-uv}(1+u)^g(1+v)^g.$$

Therefore we can compute:

$$\begin{aligned}A:=\mathcal{HD}((M'\times M')/\mathbb{Z}_2)(u,v)=\\=\frac{1}{2}\Big((\mathcal{HD}(M')(u,v))^2+\mathcal{HD}(M')(-u^2,-v^2)\Big)=\end{aligned}$$

$$=\frac{1}{2}\left(\frac{(1-(uv)^{2g-2+(d-2j)/3})^2}{(1-uv)^2}(1+u)^{2g}(1+v)^{2g}+\frac{1-(uv)^{4g-4+(2d-4j)/3}}{1-(uv)^2}(1-u^2)^g(1-v^2)^g\right)$$

and

$$\begin{aligned}B:=\mathcal{HD}\Big((\Delta_{23}\times\mathbb{P}^{2g-3+(d-2j)/3}\times\mathbb{P}^{2g-3+(d-2j)/3})/\mathbb{Z}_2\Big)=\\=\mathcal{HD}(\Delta_{23})\cdot\mathcal{HD}\Big((\mathbb{P}^{2g-3+(d-2j)/3}\times\mathbb{P}^{2g-3+(d-2j)/3})/\mathbb{Z}_2\Big)=\\=\frac{1}{2}(1+u)^g(1+v)^g\left(\frac{(1-(uv)^{2g-2+(d-2j)/3})^2}{(1-uv)^2}+\frac{1-(uv)^{4g-4+(2d-4j)/3}}{1-(uv)^2}\right).\end{aligned}$$

Since $j\equiv 2d \bmod 6$, then it makes sense to define $k:=j/2\in\mathbb{N}_0$ and we get that (Q_1,W_1) varies in the scheme

$$G_1=G^{\mathrm{s}}\left(\frac{d-2j}{3};2,\frac{d+j}{3},1\right)=G^{\mathrm{s}}\left(\frac{d+j}{3}-2k;2,\frac{d+j}{3},1\right).$$

So formula (15.3) gives the Hodge-Deligne polynomial of G_1. Finally, we can compute:

$$\begin{aligned}q_8^{j\equiv_6 2d}:=\mathcal{HD}(M/\mathbb{Z}_2)=\mathcal{HD}(G_1)\cdot(A-B)=\\=\frac{(1+u)^g(1+v)^g}{2(1-uv)}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot\\\cdot\left[\frac{(uv)^{j/2}x^{-j/2}}{1-x(uv)^{-1}}-\frac{(uv)^{g+(d-2j)/3+1}x^{1-j/2}}{1-x(uv)^2}-x^{-j/2}\right]\cdot\end{aligned}$$

$$\begin{aligned}\Bigg\{\frac{(1-(uv)^{2g-2+(d-2j)/3})^2}{(1-uv)^2}(1+u)^{2g}(1+v)^{2g}+\frac{1-(uv)^{4g-4+(2d-4j)/3}}{1-(uv)^2}(1-u^2)^g(1-v^2)^g+\\-(1+u)^g(1+v)^g\left(\frac{(1-(uv)^{2g-2+(d-2j)/3})^2}{(1-uv)^2}+\frac{1-(uv)^{4g-4+(2d-4j)/3}}{1-(uv)^2}\right)\Bigg\}.\end{aligned}$$

(2a-ii) If we are in case (a) and $(Q_2,0)\simeq(Q_3,0)$, then the corresponding (E,V)'s are parametrized using proposition 7.1.3. From the point of view of Hodge-Deligne polynomials, we can assume that there is a single index i. Also in this case, there is only one value for the invariant a, namely $a=2g-2+(d-2j)/3$ as in (2a-i). So we can assume that the (E,V)'s we are interested in are parametrized by a grassmannian $\mathrm{Grass}(2,R_a)$ where R_a is a vector bundle over $U_a = G_1 \times G_2$ with fibers isomorphic to $\mathbb{C}^{2g-3+(d-2j)/3}$. Here G_1 and G_2 are as in (2a-i), so we get the polynomial:

$$q_9^{j\equiv_6 2d} := \mathcal{HD}(\mathrm{Grass}(2, R_a)) =$$
$$= \mathcal{HD}\Big(\mathrm{Grass}(2, 2g-2+(d-2j)/3)\Big) \cdot \mathcal{HD}(G_1) \cdot \mathcal{HD}(G_2) =$$
$$= \frac{(1-(uv)^{2g-3+(d-2j)/3})(1-(uv)^{2g-2+(d-2j)/3})}{(1-uv)^2(1-(uv)^2)}(1+u)^{2g}(1+v)^{2g})\cdot$$
$$\cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} \cdot \left[\frac{(uv)^{j/2}x^{-j/2}}{1-x(uv)^{-1}} - \frac{(uv)^{g+(d-2j)/3+1}x^{1-j/2}}{1-x(uv)^2} - x^{-j/2}\right].$$

(2b) We have also to consider case (b) (that coincides with case (c)). In that case Q_2 and Q_3 have different ranks, so in particular their types are different, so we can simply apply proposition 7.1.1 in order to parametrize the corresponding (E, V)'s. In this case the invariants b can only assume the value:

$$b = \dim\ \mathrm{Ext}^1((Q_3, 0), (Q_1, W_1)) = 2g - 2 + (2d-4j)/3$$

(this is computed as the invariant c is computed in (1b)), and a can only assume the value:

$$a = \dim\ \mathrm{Ext}^1((Q_2, 0), (Q_1, W_1)) = C_{21} = n_1 n_2 (g-1) - d_1 n_2 + d_2 n_1 =$$
$$= g - 1 - j/2 + (2d-j)/6 = g - 1 + (d-2j)/3.$$

From the point of view of Hodge-Deligne polynomials we can assume that there is a single index i. So we can assume that the (E, V)'s we are interested in are parametrized by a scheme R that comes with a sequence of two projective fibrations

$$R \longrightarrow A \longrightarrow U = G_1 \times G_2 \times G_3;$$

the first fibration has fibers isomorphic to $\mathbb{P}^{b-1} = \mathbb{P}^{2g-3+(2d-4j)/3}$, while the second fibration has fibers isomorphic to $\mathbb{P}^{a-1} = \mathbb{P}^{g-2+(d-2j)/3}$. Here the schemes G_i for $i = 1, 2, 3$ coincide with those described in case (1b). So we get the polynomial:

$$q_{10}^{j\equiv_6 2d} = \mathcal{HD}(\mathbb{P}^{2g-3+(2d-4j)/3})\mathcal{HD}(\mathbb{P}^{g-2+(d-2j)/3})\mathcal{HD}(G_1)\mathcal{HD}(G_2)\mathcal{HD}(G_3) =$$
$$= \frac{1-(uv)^{2g-2+(2d-4j)/3}}{1-uv} \cdot \frac{1-(uv)^{g-1+(d-2j)/3}}{1-uv}(1+u)^g(1+v)^g\cdot$$
$$\cdot \frac{1}{2(1-uv)(1-(uv)^2)}\Big(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g +$$
$$-(1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1}-u^2v^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2\Big)\cdot$$
$$\cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}.$$

15.2.3 Case $r = 4$

In this case the graded is necessarily made of 4 objects of the form $(Q_1, W_1), (Q_i, 0)_{i=2,3,4}$ where (Q_1, W_1) must be necessarily the first object of the graded in order not to destabilize (E, V) for $\alpha(j)^-$. Moreover, every Q_i for $i = 2, 3, 4$ must be a line bundle and the stability conditions prove that the Q_i's for $i = 2, 3, 4$ must all have the same degree $d_2 = d_3 = d_4 = (2d - j)/6$. Therefore, *this case is possible only when* $j \equiv 2d \bmod 6$. Moreover, Q_1 is a line bundle of degree $d_1 = d - 3d_2 = d - (2d - j)/2 = j/2$.

Then we need to consider several different subcases according to the various $\alpha(j)$-canonical filtrations. The cases we will consider are those when the canonical filtration is one of the following types: $(1,1,1,1)$ (unique $\alpha(j)$-Jordan-Hölder filtration), $(1,3)$, $(1,2,1)$ and $(1,1,2)$. A priori we should also consider the cases $(2,1,1)$, $(3,1)$ and (4); none of these 3 cases is actually possible since in each case we will have a subobject $(Q_i, 0) \subset (E, V)$ for some $i \in \{2, 3, 4\}$ and this will prove that (E, V) is not α_c^--stable, so these 3 cases do not occur in the description of $G^-(\alpha_c; 4, d, 1)$.

(1) Canonical filtration of type (1,1,1,1) (unique $\alpha(j)$-Jordan-Hölder filtration). Since the $(Q_i, 0)$'s for $i = 2, 3, 4$ are all of the same type, we need to consider 4 subcases according to the various relations between them:

(a) $(Q_2, 0) \not\simeq (Q_3, 0) \simeq (Q_4, 0)$;

(b) $(Q_2, 0) \not\simeq (Q_3, 0) \not\simeq (Q_4, 0)$;

(c) $(Q_2, 0) \simeq (Q_3, 0) \simeq (Q_4, 0)$;

(d) $(Q_2, 0) \simeq (Q_3, 0) \not\simeq (Q_4, 0)$.

(1a) Let us suppose that $(Q_2, 0) \not\simeq (Q_3, 0) \simeq (Q_4, 0)$. Then we can apply proposition 6.2.6. In this case we need to compute the invariants a, b, c, d, e, f. In order to do that, let (E_2, V_2) be any non-split extension of $(Q_2, 0)$ by (Q_1, W_1) and let $(E'', 0)$ be any non-split extension of $(Q_4, 0)$ by $(Q_3, 0)$. Then E_2 is a vector bundle of rank $N_2 = 2$ and degree $D_2 = (2d - j)/6 + j/2 = (d + j)/3$; moreover, the dimension of V_2 is $K_2 = 1$. E'' is a vector bundle of rank $N'' = 2$ and degree $D'' = d_3 + d_4 = 2d_3 = (2d - j)/3$. Since $(Q_1, W_1) \not\simeq (Q_2, 0)$, we have:

$$a = \dim \operatorname{Ext}^1((Q_2, 0), (Q_1, W_1)) = C_{21} =$$

$$= n_1 n_2 (g - 1) - d_1 n_2 + d_2 n_1 = g - 1 - j/2 + (2d - j)/6 = g - 1 + (d - 2j)/3.$$

Since $(Q_3, 0) \simeq (Q_4, 0)$, we have

$$b = \dim \operatorname{Ext}^1((Q_4, 0), (Q_3, 0)) = C_{43} + 1 = n_3 n_4 (g - 1) - d_3 n_4 + d_4 n_3 + 1 = g.$$

Moreover, since $(Q_3, 0)$ is of the same type of $(Q_2, 0)$ and $(Q_2, 0) \not\simeq (Q_1, W_1)$, we have that

$$f = \dim \operatorname{Ext}^1((Q_3,0),(Q_1,W_1)) = a = g-1+(d-2j)/3.$$

Now $(E'',0)$ is a non-split extension of $(Q_3,0)$ by itself and (E_2,V_2) is a non-split extension of $(Q_2,0)$ by (Q_1,W_1). So as in (6.71) we get:

$$\operatorname{Hom}((E'',0),(E_2,V_2)) = 0 = \operatorname{Hom}((Q_4,0),(E_2,V_2)).$$

Moreover, we have also that $\operatorname{Hom}((E'',V''),(Q_1,W_1)) = 0$ because the graded of (E'',V'') does not contain any object isomorphic to (Q_1,W_1). Then we can compute also the following invariants.

$$c = \dim \operatorname{Ext}^1((E'',V''),(E_2,0)) = N_2N''(g-1) - D_2N'' + D''N_2 =$$
$$= 4(g-1) - 2(d+j)/3 + 2(2d-j)/3 = 4g-4+(2d-4j)/3.$$

$$d = \dim \operatorname{Ext}^1((Q_4,0),(E_2,V_2)) = N_2n_4(g-1) - D_2n_4 + d_4N_2 =$$
$$= 2(g-1) - (d+j)/3 + (2d-j)/3 = 2g-2+(d-2j)/3.$$

$$e = \dim \operatorname{Ext}^1((E'',0),(Q_1,W_1)) = n_1N''(g-1) - d_1N'' + D''n_1 =$$
$$= 2(g-1) - j + (2d-j)/3 = 2g-2+(2d-4j)/3.$$

So each invariant can only assume one value. From the point of view of Hodge-Deligne polynomials, we can ignore the indices i, j, l of proposition 6.2.6, so without loss of generality we can assume the following description:

- $U_a = G_1 \times G_2$ and there is a projective bundle R_a over it with fibers isomorphic to $\mathbb{P}^{a-1} = \mathbb{P}^{g-2+(d-2j)/3}$;
- $U^b = G_3 = G_4$ and there is a projective bundle R^b over it with fibers isomorphic to $\mathbb{P}^{b-1} = \mathbb{P}^{g-1}$;
- from the point of view of Hodge-Deligne polynomials we can suppose that $R_a \times R^b = G_1 \times G_2 \times G_3 \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-1}$. So we can assume that

  $$U_{a,b,c,d,e,f} = G_1 \times (G_2 \times G_3 \smallsetminus \Delta_{23}) \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-1};$$

  there is a bundle $R_{a,b,c,d,e,f}$ over it with fibers isomorphic to $\mathbb{C}^{d-1} \times (\mathbb{P}^{c-d-1} \smallsetminus \mathbb{P}^{e-f-1}) = \mathbb{C}^{2g-3+(d-2j)/3} \times (\mathbb{P}^{2g-3+(d-2j)/3} \smallsetminus \mathbb{P}^{g-2+(d-2j)/3})$

The (E,V)'s we are interested in are parametrized by the scheme $R_{a,b,c,d,e,f}$. In this case

$$G_1 = G(1, j/2, 1), \quad G_2 = G_3 = G_4 = J^{(2d-j)/6}C.$$

So we get the polynomial

$$q_{11}^{j\equiv_6 2d}=\mathcal{HD}(R_{a,b,c,d,e,f})=\mathcal{HD}(G_1)(\mathcal{HD}(G_2)^2-\mathcal{HD}(G_2))\mathcal{HD}(\mathbb{P}^{g-1})\mathcal{HD}(\mathbb{P}^{g-2+(d-2j)/3})\cdot$$
$$\cdot\mathcal{HD}(\mathbb{C}^{2g-3+(d-2j)/3}\times(\mathbb{P}^{2g-3+(d-2j)/3}\smallsetminus\mathbb{P}^{g-2+(d-2j)/3}))=$$
$$=(1+u)^g(1+v)^g((1+u)^g(1+v)^g-1)\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}\cdot\frac{1-(uv)^g}{1-uv}\cdot$$
$$\cdot\frac{1-(uv)^{g-1+(d-2j)/3}}{1-uv}\cdot(uv)^{2g-3+(d-2j)/3}\cdot\frac{(uv)^{g-1+(d-2j)/3}-(uv)^{2g-2+(d-2j)/3}}{1-uv}.$$

(1b) Let us suppose that $(Q_2,0)\not\simeq(Q_3,0)\not\simeq(Q_4,0)$; we don't fix any additional condition on the relations between $(Q_2,0)$ and $(Q_4,0)$. Then we can apply proposition 6.2.8. Also in this case we need to compute invariants a,b,c,d,e,f. In order to do that, let (E_2,V_2) and $(E'',0)$ be as in the case (1a). Then we get the same invariants computed before, except for the invariant b that now has value $g-1$ (instead of g). In particular, each invariant can only assume one value. From the point of view of Hodge-Deligne polynomials, we can ignore the indices i,j,l of that proposition, so we can assume that:

- $U_a=G_1\times G_2$ and there is a projective bundle R_a over it with fibers isomorphic to $\mathbb{P}^{a-1}=\mathbb{P}^{g-2+(d-2j)/3}$;
- $U^b=G_3\times G_4\smallsetminus\Delta_{34}$ and there is a bundle R^b over it with fibers isomorphic to $\mathbb{P}^{b-1}=\mathbb{P}^{g-2}$;
- from the point of view of Hodge-Deligne polynomials we can suppose that $R_a\times R^b=G_1\times G_2\times(G_3\times G_4\smallsetminus\Delta_{34})\times\mathbb{P}^{g-2+(d-2j)/3}\times\mathbb{P}^{g-2}$. So we can assume that

$$U_{a,b,c,d,e,f}=G_1\times(G_2\times(G_3\times G_4\smallsetminus\Delta_{34})\smallsetminus\Delta_{23}\times G_4)\times\mathbb{P}^{g-2+(d-2j)/3}\times\mathbb{P}^{g-2};$$

there is a bundle $R_{a,b,c,d,e,f}$ over it with fibers isomorphic to

$$\mathbb{P}^{c-1}\smallsetminus\mathbb{P}^{d+e-f-1}=\mathbb{P}^{4g-5+(2d-4j)/3}\smallsetminus\mathbb{P}^{3g-4+(2d-4j)/3}.$$

The (E,V)'s we are interested in are parametrized by the scheme $R_{a,b,c,d,e,f}$. Also in this case

$$G_1=G(1,j/2,1),\quad G_2=G_3=G_4=J^{(2d-j)/6}C.$$

Now as in (15.4) we get that

$$\mathcal{HD}(G_2\times(G_3\times G_4\smallsetminus\Delta_{34})\smallsetminus\Delta_{23}\times G_4)=\mathcal{HD}(G_2)(\mathcal{HD}(G_2)-1)^2;$$

So we get the polynomial

$$q_{12}^{j\equiv_6 2d}=\mathcal{HD}(R_{a,b,c,d,e,f})=\mathcal{HD}(G_1)\mathcal{HD}(G_2)(\mathcal{HD}(G_2)-1)^2\cdot$$

$$\cdot\mathcal{HD}(\mathbb{P}^{g-2})\mathcal{HD}(\mathbb{P}^{g-2+(d-2j)/3})\mathcal{HD}(\mathbb{P}^{4g-5+(2d-4j)/3}\smallsetminus\mathbb{P}^{3g-4+(2d-4j)/3})=$$

$$=(1+u)^g(1+v)^g((1+u)^g(1+v)^g-1)^2\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}\cdot$$

$$\cdot\frac{1-(uv)^{g-1}}{1-uv}\cdot\frac{1-(uv)^{g-1+(d-2j)/3}}{1-uv}\cdot\frac{(uv)^{3g-3+(2d-4j)/3}-(uv)^{4g-4+(2d-4j)/3}}{1-uv}.$$

(1c)-(1d) As we stated in remark 6.2.3, we are still not able give a geometric description of these 2 cases. We simply denote the corresponding polynomials by $q_{13}^{j\equiv_6 2d}$ and $q_{14}^{j\equiv_6 2d}$ respectively.

(2) Canonical filtration of type (1,3). Since the $(Q_i,0)$'s for $i=2,3,4$ are all of the same type, we need to consider 3 subcases as follows:

(a) there are no pairs of isomorphic objects among the $(Q_i,0)$'s for $i=2,3,4$;

(b) exactly 2 objects among the $(Q_i,0)$'s are isomorphic; without loss of generality we can assume that they are $(Q_2,0)$ and $(Q_3,0)$;

(c) $(Q_2,0)\simeq(Q_3,0)\simeq(Q_4,0)$.

(2a) Let us suppose that there are no pairs of isomorphic objects among the $(Q_i,0)$'s for $i=2,3,4$. Then we can apply proposition 7.3.4. In this case we need to compute the invariants a,b,c; the same computation that gives the invariants a and f in case (1a) proves that we have:

$$b=\dim\operatorname{Ext}^1((Q_3,0),(Q_1,W_1))=g-1+(d-2j)/3.$$

Analogously, since $(Q_2,0)$, $(Q_3,0)$ and $(Q_4,0)$ are all of the same type, we get that

$$a=\dim\operatorname{Ext}^1((Q_2,0),(Q_1,W_1))=g-1+(d-2j)/3=$$
$$=\dim\operatorname{Ext}^1((Q_4,0),(Q_1,W_1))=c.$$

Now $G_1=G(\alpha(j);1;j/2,1)$ and $G_2=J^{(2d-j)/6}C$, so both spaces are irreducible. Therefore the index i appearing in proposition 7.3.4 assumes only one value. Moreover, since $G_2=G_3=G_4$, also the indices j and k can assume only one value. Therefore, we get that

$$U^2_{a;i}=U^3_{b;j}=U^4_{c;k}=G_1\times G_2.$$

Then we get that the only scheme $R_{a,b,c;i,j,k}$ that we will be interested in is $R_{a,a,a;i,i,i}$, that comes with a locally trivial fibration to

$$U_{a,a,a;i,i,i}=U^2_{a;i}\times_{G_1}U^3_{a;i}\times_{G_1}U^4_{a;i}=G_1\times G_2\times G_2\times G_2.$$

with fibers isomorphic to $\mathbb{P}^{g-2+(d-2j)/3}\times\mathbb{P}^{g-2+(2j-d)/3}\times\mathbb{P}^{g-2+(d-2j)/3}$. Since this is the only case, then the only object we need to consider is given by case (j) of that proposition, namely

$$R := (R_{a,a,a;i,i,i}|_{G_1 \times (G_2 \times G_2 \times G_2 \smallsetminus \Delta)})/S_3$$

where Δ is the big diagonal of $G_2 \times G_2 \times G_2$, i.e. the set of all triples of objects such that at least 2 of them are isomorphic. Every $\sigma \in S_3$ acts by permutations on the ordered set $\{2, 3, 4\}$ and it acts as follows on $U_{a,a,a;i,i,i}$ and $R_{a,a,a;i,i,i}$:

- $(Q_i, W_i)_{i=1,2,3,4} \mapsto ((Q_1, W_1), (Q_{\sigma(i)}, W_{\sigma(i)})_{i=2,3,4})$;
- $(\mu_i)_{i=2,3,4} \mapsto (\mu_{\sigma(i)})_{i=2,3,4}$ for every point (μ_2, μ_3, μ_4) in the fiber over a quadruple $(Q_i, W_i)_{i=1,\cdots,4} \in U_{a,a,a;i,i,i}$.

Moreover, there exists a finite disjoint covering of the base space $G_1 \times (G_2 \times G_2 \times G_2 \smallsetminus \Delta)$ by locally closed subschemes T_l that are invariant under the action of S_3 on $G_1 \times G_2 \times G_2 \times G_2$; in addition, there exist trivializations of the fibrations from $R_{a,a,a;i,i,i}$ to $U_{a,a,i;i,i,i}$

$$R|_{T_l} \xrightarrow{\sim} T_l \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3}$$

that are compatible with the natural action of S_3 on $T_l \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3}$. From the point of view of Hodge-Deligne polynomials, we can therefore assume that R coincides with a scheme of the form M/S_3, where M is the scheme

$$G_1 \times (G_2 \times G_2 \times G_2 \smallsetminus \Delta) \times \mathbb{P}^{g-2+(2j-d)/3} \times \mathbb{P}^{g-2+(2j-d)/3} \times \mathbb{P}^{g-2+(2j-d)/3}$$

and every $\sigma \in S_3$ (considered as the set of permutations of $\{2, 3, 4\}$) acts on M as follows:

$$((Q_1, W_1), Q_2, Q_3, Q_4, \mu_2, \mu_3, \mu_4) \mapsto$$
$$\mapsto \big((Q_1, W_1), Q_{\sigma(2)}, Q_{\sigma(3)}, Q_{\sigma(4)}, \mu_{\sigma(2)}, \mu_{\sigma(3)}, \mu_{\sigma(4)}\big).$$

Let us consider the following schemes:

$$\begin{aligned}
M' &:= G_2 \times \mathbb{P}^{g-2+(d-2j)/3} = J^{(2d-j)/6}C \times \mathbb{P}^{g-2+(d-2j)/3}, \\
\Delta_0 &:= \{(Q_2, Q_3, Q_4) \in G_2 \times G_2 \times G_2 \text{ s.t. } Q_2 \simeq Q_3 \simeq Q_4\}, \\
\Delta_1 &:= \{(Q_2, Q_3, Q_4) \in G_2 \times G_2 \times G_2 \text{ s.t. } Q_2 \simeq Q_3 \not\simeq Q_4\}, \\
\Delta_2 &:= \{(Q_2, Q_3, Q_4) \in G_2 \times G_2 \times G_2 \text{ s.t. } Q_2 \simeq Q_4 \not\simeq Q_3\}, \\
\Delta_3 &:= \{(Q_2, Q_3, Q_4) \in G_2 \times G_2 \times G_2 \text{ s.t. } Q_3 \simeq Q_4 \not\simeq Q_2\}.
\end{aligned} \tag{15.8}$$

Then $\Delta_0 \simeq G_2$ and $\Delta_i \simeq G_2 \times G_2 \smallsetminus \Delta'$ for $i = 1, 2, 3$, where Δ' is the diagonal of $G_2 \times G_2$. Moreover, we can write

$$\Delta = \Delta_0 \amalg \Delta_1 \amalg \Delta_2 \amalg \Delta_3.$$

Now we have that

$$M/S_3 \simeq G_1 \times \Big((M' \times M' \times M')/S_3 \smallsetminus$$
$$\big((\Delta_0 \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3})/S_3 \amalg \tag{15.9}$$
$$\amalg((\Delta_1 \amalg \Delta_2 \amalg \Delta_3) \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3})/S_3)\big)\Big).$$

Now

$$\mathcal{HD}(M')(u,v) = (1+u)^g(1+v)^g \frac{1-(uv)^{g-1+(d-2j)/3}}{1-uv}.$$

So we can use lemma 8.0.6 in order to compute

$$A := \mathcal{HD}((M' \times M' \times M')/S_3)(u,v) = \frac{1}{6}(\mathcal{HD}(M')(u,v))^3 +$$
$$+\frac{1}{2}\mathcal{HD}(M')(-u^2,-v^2) \cdot \mathcal{HD}(M')(u,v) + \frac{1}{3}\mathcal{HD}(M')(u^3,v^3) =$$
$$= (1+u)^{3g}(1+v)^{3g}\frac{(1-(uv)^{g-1+(d-2j)/3})^3}{6(1-uv)^3} + (1-u^2)^g(1-v^2)^g(1+u)^g(1+v)^g \cdot$$
$$\cdot \frac{(1-(uv)^{2g-2+(2d-4j)/3})(1-(uv)^{g-1+(d-2j)/3})}{2(1-(uv)^2)(1-uv)} + (1+u^3)^g(1+v^3)^g\frac{1-(uv)^{3g-3+d-2j}}{3(1-(uv)^3)}.$$

Now the action of S_3 on Δ_0 is trivial; moreover, we have

$$(\Delta_1 \amalg \Delta_2 \amalg \Delta_3)/S_3 \simeq \Delta_1/\mathbb{Z}_2 \simeq (G_2 \times G_2 \smallsetminus \Delta')/\mathbb{Z}_2 \simeq (G_2 \times G_2)/\mathbb{Z}_2 \smallsetminus G_2.$$

So we have:

$$(\Delta_0 \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3})/S_3 \simeq$$
$$\simeq G_2 \times (\mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3})/S_3$$

and

$$\Big((\Delta_1 \amalg \Delta_2 \amalg \Delta_3) \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3}\Big)/S_3 \simeq$$
$$\simeq \Big((G_2 \times G_2 \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3})/\mathbb{Z}_2 \smallsetminus$$
$$\smallsetminus G_2 \times (\mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3})/\mathbb{Z}_2\Big) \times \mathbb{P}^{g-2+(d-2j)/3}.$$

So we compute:

$$B_1 := \mathcal{HD}(G_2 \times (\mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3})/S_3)) =$$
$$= (1+u)^g(1+v)^g \cdot \Bigg(\frac{(1-(uv)^{g-1+(d-2j)/3})^3}{6(1-uv)^3} +$$
$$+\frac{(1-(uv)^{2g-2+(2d-4j)/3})(1-(uv)^{g-1+(d-2j)/3})}{2(1-(uv)^2)(1-uv)} + \frac{1-(uv)^{3g-3+d-2j}}{3(1-(uv)^3)}\Bigg);$$

$$B_2 := \mathcal{HD}((G_2 \times G_2 \times \mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3})/\mathbb{Z}_2) \times \mathbb{P}^{g-2+(d-2j)/3}) =$$

$$= \frac{1}{2} \cdot \left((1+u)^{2g}(1+v)^{2g} \frac{(1-(uv)^{g-1+(d-2j)/3})^2}{(1-uv)^2} + (1-u^2)^g(1-v^2)^g \cdot \right.$$

$$\left. \cdot \frac{1-(uv)^{2g-2+(2d-4j)/3}}{1-(uv)^2} \right) \cdot \frac{1-(uv)^{g-1+(d-2j)/3}}{1-uv};$$

$$B_3 = \mathcal{HD}(G_2 \times ((\mathbb{P}^{g-2+(d-2j)/3} \times \mathbb{P}^{g-2+(d-2j)/3})/\mathbb{Z}_2) \times \mathbb{P}^{g-2+(d-2j)/3}) =$$

$$= \frac{1}{2}(1+u)^g(1+v)^g \left(\frac{(1-(uv)^{g-1+(d-2j)/3})^2}{(1-uv)^2} + \frac{1-(uv)^{2g-2+(2d-4j)/3}}{1-(uv)^2} \right) \cdot$$

$$\cdot \frac{1-(uv)^{g-1+(d-2j)/3}}{1-uv}.$$

Then by considering everything together, we have:

$$q_{15}^{j\equiv_6 2d} := (A - B_1 - B_2 + B_3) \cdot \mathcal{HD}(G_1) =$$

$$= \left\{ ((1+u)^{3g}(1+v)^{3g} \frac{(1-(uv)^{g-1+(d-2j)/3})^3}{6(1-uv)^3} + (1-u^2)^g(1-v^2)^g(1+u)^g(1+v)^g \cdot \right.$$

$$\cdot \frac{(1-(uv)^{2g-2+(2d-4j)/3})(1-(uv)^{g-1+(d-2j)/3})}{2(1-(uv)^2)(1-uv)} + (1+u^3)^g(1+v^3)^g \frac{1-(uv)^{3g-3+d-2j}}{3(1-(uv)^3)} +$$

$$-(1+u)^g(1+v)^g \cdot \left(\frac{(1-(uv)^{g-1+(d-2j)/3})^3}{6(1-uv)^3} + \right.$$

$$\left. + \frac{(1-(uv)^{2g-2+(2d-4j)/3})(1-(uv)^{g-1+(d-2j)/3})}{2(1-(uv)^2)(1-uv)} + \frac{1-(uv)^{3g-3+d-2j}}{3(1-(uv)^3)} \right) +$$

$$- \frac{1}{2} \cdot \left((1+u)^{2g}(1+v)^{2g} \frac{(1-(uv)^{g-1+(d-2j)/3})^2}{(1-uv)^2} + (1-u^2)^g(1-v^2)^g \cdot \right.$$

$$\left. \cdot \frac{1-(uv)^{2g-2+(2d-4j)/3}}{1-(uv)^2} \right) \cdot \frac{1-(uv)^{g-1+(d-2j)/3}}{1-uv} +$$

$$+ \frac{1}{2}(1+u)^g(1+v)^g \left(\frac{(1-(uv)^{g-1+(d-2j)/3})^2}{(1-uv)^2} + \frac{1-(uv)^{2g-2+(2d-4j)/3}}{1-(uv)^2} \right) \cdot$$

$$\left. \cdot \frac{1-(uv)^{g-1+(d-2j)/3}}{1-uv} \right\} \cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}.$$

(2b) Let us suppose that $(Q_2, 0) \simeq (Q_3, 0) \not\simeq (Q_4, 0)$. Then we can apply proposition 7.3.5. In this case we need to compute the invariants a and b. The same analysis of case (2a) proves that $a=b=g-1+(d-2j)/3$ and that the indices i and j can only assume one value. Therefore, we get that

$$U^2_{a;i}=U^4_{b;j}=G_1\times G_4.$$

and

$$V_{a,b;i,j}=(U^2_{a;i}\times_{G_1}U^4_{b;j})\cap(G_1\times(G_2\times G_2\smallsetminus\Delta))=G_1\times(G_2\times G_2\smallsetminus\Delta).$$

The scheme we are looking at is $R_{a,b;i,j}$, that comes with a morphism

$$R_{a,b;i,j}\stackrel{\phi_2\circ\phi_1}{\longrightarrow}V_{a,b;i,j},$$

where ϕ_1 is a fibration with fibers isomorphic to $\mathbb{P}^{b-1}=\mathbb{P}^{g-2+(d-2j)/3}$ and ϕ_2 is the grassmannian fibration of 2-planes associated to a vector bundle $Q_{a,b;i,j}$ over $V_{a,b;i,j}$ with rank $a=g-1+(2j-d)/3$. So we get:

$$\begin{aligned}q_{16}^{j\equiv_6 2d}&:=\mathcal{HD}(R_{a,b;i,j})=\\
=\mathcal{HD}(G_2\times G_2\smallsetminus\Delta)&\mathcal{HD}(G_1)\mathcal{HD}(\mathbb{P}^{g-2+(d-2j)/3})\mathcal{HD}(\mathrm{Grass}(2,g-1+(d-2j)/3))=\\
&=\left((1+u)^{2g}(1+v)^{2g}-(1+v)^g(1+v)^g\right)\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^gx^{-j/2}}{(1-x)(1-uvx)}\cdot\\
&\cdot\frac{1-(uv)^{g-1+(d-2j)/3}}{1-uv}\cdot\frac{(1-(uv)^{g-2+(d-2j)/3})(1-(uv)^{g-1+(d-2j)/3})}{(1-uv)(1-(uv)^2)}.\end{aligned}$$

(2c) Let us suppose that $(Q_2,0)\simeq(Q_3,0)\simeq(Q_4,0)$. Then we can apply proposition 7.3.6. In this case the only invariant that we need is a. As before, $a=g-1+(d-2j)/3$ and the index i can only assume one value. Therefore, we get that $U_{a;i}=G_1\times G_2$. The scheme we are looking at is the grassmannian of 3-planes associated to a locally free sheaf $R_{a;i}$ of rank $a=g-1+(d-2j)/3$ over $U_{a;i}$. So we get:

$$\begin{aligned}q_{17}^{j\equiv_6 2d}&:=\mathcal{HD}(G_1)\mathcal{HD}(G_4)\mathcal{HD}(\mathrm{Grass}(3,g-1+(d-2j)/3))=\\
&=(1+u)^g(1+v)^g\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^gx^{-j/2}}{(1-x)(1-uvx)}\cdot\\
&\cdot\frac{(1-(uv)^{g-3+(d-2j)/3})(1-(uv)^{g-2+(d-2j)/3})(1-(uv)^{g-1+(d-2j)/3})}{(1-uv)(1-(uv)^2)(1-(uv)^3)}.\end{aligned}$$

(3) Canonical filtration of type (1,1,2).

Since the $(Q_i,0)$'s for $i=2,3,4$ are all of the same type and since the order of $(Q_3,0)$ and $(Q_4,0)$ is not important, we need to consider 4 cases as follows:

(a) $(Q_2,0)\simeq(Q_3,0)\not\simeq(Q_4,0)$;

(b) $(Q_i,0)\not\simeq(Q_j,0)$ for all $i\neq j\in\{2,3,4\}$;

(c) $(Q_2,0)\not\simeq(Q_3,0)\simeq(Q_4,0)$;

(d) $(Q_2,0)\simeq(Q_3,0)\simeq(Q_4,0)$.

(3a) Let us suppose that $(Q_2,0)\simeq(Q_3,0)\not\simeq(Q_4,0)$. Then we can apply proposition 7.7.1. In this case we need to compute the invariants a,b,c,d. In order to do that, let (E_2,V_2) be any non-split extension of $(Q_2,0)$ by (Q_1,W_1); then E_2 is a vector bundle of rank $N_2=2$ and degree $D_2=d_1+d_2=j/2+(2d-j)/6=(d+j)/3$; moreover, the dimension of V_2 is $K_2=1$. Since $(Q_2,0)\not\simeq(Q_1,W_1)$, we have:

$$a=\dim\ \mathrm{Ext}^1((Q_2,0),(Q_1,W_1))=C_{21}=n_1n_2(g-1)-d_1n_2+d_2n_1=$$

$$=g-1-j/2+(2d-j)/6=g-1+(d-2j)/3.$$

By the same computation we get that

$$c=\dim\ \mathrm{Ext}^1((Q_4,0),(Q_1,W_1))=g-1+(d-2j)/3.$$

Since (E_2,V_2) is a non-split extension of $(Q_2,0)$ by (Q_1,W_1) and since $(Q_i,0)\not\simeq(Q_1,W_1)$ for $i=2,3,4$, then

$$\mathrm{Hom}((Q_i,0),(E_2,V_2))=0 \text{ for } i=2,3,4.$$

Therefore, we have that

$$b=\dim\ \mathrm{Ext}^1((Q_4,0),(E_2,V_2))=N_2n_4(g-1)-D_2n_4+d_4N_2=$$

$$=2(g-1)-(d+j)/3+(2d-j)/3=2g-2+(d-2j)/3.$$

By the same computation we get that

$$d=\dim\ \mathrm{Ext}^1((Q_2,0),(E_2,V_2))=2g-2+(d-2j)/3.$$

So each invariant can only assume one value. From the point of view of Hodge-Deligne polynomials, we can ignore the indices i and j of proposition 7.7.1, so we can assume that we have the following description.

- $U_a=G_1\times G_2$ and there is a projective bundle R_a over it with fibers isomorphic to $\mathbb{P}^{a-1}=\mathbb{P}^{g-2+(d-2j)/3}$; from the point of view of Hodge-Deligne polynomials we can assume that $R_a=G_1\times G_2\times\mathbb{P}^{g-2+(d-2j)/3}$;
- from the point of view of Hodge-Deligne polynomials we can assume that $U_{a,b,c,d}=G_1\times(G_2\times G_4\smallsetminus\Delta_{24})\times\mathbb{P}^{g-2+(d-2j)/3}$; there is a scheme $R_{a,b,c,d}$ together with a fibration to $U_{a,b,c,d}$ with fibers isomorphic to

$$(\mathbb{P}^{d-1}\smallsetminus\mathbb{P}^{a-2})\times(\mathbb{P}^{b-1}\smallsetminus\mathbb{P}^{c-1})=$$

$$=(\mathbb{P}^{2g-3+(d-2j)/3}\smallsetminus\mathbb{P}^{g-3+(d-2j)/3})\times(\mathbb{P}^{2g-3+(d-2j)/3}\smallsetminus\mathbb{P}^{g-2+(d-2j)/3}).$$

In this case

$$G_1 = G(1, j/2, 1), \quad G_2 = G_4 = J^{(2d-j)/6}C.$$

So we get the polynomial

$$q_{18}^{j\equiv_6 2d} = \mathcal{HD}(R_{a,b,c,d}) = \mathcal{HD}(G_1)(\mathcal{HD}(G_2)^2 - \mathcal{HD}(G_2))\mathcal{HD}(\mathbb{P}^{g-2+(d-2j)/3})\cdot$$
$$\cdot(\mathcal{HD}(\mathbb{P}^{2g-3+(d-2j)/3} \smallsetminus \mathbb{P}^{g-3+(d-2j)/3-1}) \cdot \mathcal{HD}(\mathbb{P}^{2g-3+(d-2j)/3} \smallsetminus \mathbb{P}^{g-2+(d-2j)/3-1}) =$$
$$= \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}(1+u)^g(1+v)^g((1+u)^g(1+v)^g - 1)\cdot$$
$$\cdot\frac{1-(uv)^{g-1+(d-2j)/3}}{1-uv}\cdot\frac{(uv)^{g-2+(d-2j)/3}-(uv)^{2g-2+(d-2j)/3}}{1-uv}\cdot\frac{(uv)^{g-1+(d-2j)/3}-(uv)^{2g-2+(d-2j)/3}}{1-uv}.$$

(3b) Let us suppose that $(Q_i, 0) \not\simeq (Q_j, 0)$ for all $i \neq j \in \{2, 3, 4\}$. Then we can apply proposition 7.7.2. In this case we need to compute the invariants a, b, c, d, e. Also in this case, we denote by (E_2, V_2) any non-split extension of $(Q_2, 0)$ by (Q_1, W_1); again we have $(N_2, D_2, K_2) = (2, (d+j)/3, 1)$. Since $(Q_1, W_1) \not\simeq (Q_i, 0)$ for $i = 2, 3, 4$, we have as before

$$a = \dim \operatorname{Ext}^1((Q_2, 0), (Q_1, W_1)) = g - 1 + (d-2j)/3,$$
$$c = \dim \operatorname{Ext}^1((Q_4, 0), (Q_1, W_1)) = g - 1 + (d-2j)/3,$$
$$e = \dim \operatorname{Ext}^1((Q_3, 0), (Q_1, W_1)) = g - 1 + (d-2j)/3$$

and

$$b = \dim \operatorname{Ext}^1((Q_4, 0), (E_2, V_2)) = 2g - 2 + (d-2j)/3,$$
$$d = \dim \operatorname{Ext}^1((Q_3, 0), (E_2, V_2)) = 2g - 2 + (d-2j)/3.$$

So each invariant can only assume one value. From the point of view of Hodge-Deligne polynomials, we can ignore the indices i and j of that proposition, so we can assume that we have the following description.

- $U_a = G_1 \times G_2$ and there is a projective bundle $\varphi_a : R_a \to U_a$ with fibers isomorphic to $\mathbb{P}^{a-1} = \mathbb{P}^{g-2+(d-2j)/3}$; from the point of view of Hodge-Deligne polynomials we can assume that $R_a = G_1 \times G_2 \times \mathbb{P}^{g-2+(d-2j)/3}$.
- The scheme $U_{a,b,c,d,e}$ coincides with

$$\{((E_2, V_2), (Q_3, W_3), (Q_4, W_4)) \in R_a \times G_3 \times G_4 \text{ s.t.}$$
$$(Q_l, W_l) \not\simeq \overline{\varphi}_a(E_2, V_2) \text{ for } l = 3, 4, \ (Q_3, W_3) \not\simeq (Q_4, W_4)\}$$

where $\overline{\varphi}_a$ is the composition of φ_a with the projection to G_2. From the point of view of Hodge-Deligne polynomials we can assume that $U_{a,b,c,d,e} = G_1 \times (G_2 \times G_3 \times G_4 \smallsetminus \Delta) \times \mathbb{P}^{g-2+(d-2j)/3}$, where Δ is the "big" diagonal of $G_2 \times G_3 \times G_4 = G_2 \times G_2 \times G_2$. Since

$(b,c)=(d,e)$, then the we need only to consider case (b) in proposition 7.7.2, so the (E,V)'s we are considering are parametrized by a scheme of the form $R_{a,b,c,b,c}/\mathbb{Z}_2$; by proposition 7.7.2 from the point of view of Hodge-Deligne polynomials we can assume that $R_{a,b,c,b,c}/\mathbb{Z}_2$ coincides with the scheme $(M/\mathbb{Z}_2)\times\mathbb{P}^{g-2+(d-2j)/3}\times G_1$, where

$$M:=(G_2\times G_3\times G_4\smallsetminus\Delta)\times(\mathbb{P}^{2g-3+(d-2j)/3}\smallsetminus\mathbb{P}^{g-2+(d-2j)/3})^2$$

and where $\mathbb{Z}_2$ acts on M by permutations on $G_2\times G_3$ and on $(\mathbb{P}^{2g-3+(d-2j)/3}\smallsetminus\mathbb{P}^{g-2+(d-2j)/3})^2$.

Also in this case

$$G_2=G_3=G_4=J^{(2d-j)/6}C,\quad G_1=G(1,j/2,1).$$

Moreover, we have that $\Delta\simeq\Delta_0\amalg\Delta_1\amalg\Delta_2\amalg\Delta_3$, where the Δ_i's are as in (15.8). Let us define the scheme

$$M':=G_2\times(\mathbb{P}^{2g-3+(d-2j)/3}\smallsetminus\mathbb{P}^{g-2+(d-2j)/3}).$$

By construction the action of $\mathbb{Z}_2$ is trivial on $\Delta_0\amalg\Delta_1\simeq G_1\times G_1$ and it exchanges Δ_2 and Δ_3. Then

$$\begin{aligned}M/\mathbb{Z}_2=\Big(M'\times M'\times G_2\Big)/\mathbb{Z}_2\smallsetminus\Big((\Delta_0\amalg\Delta_1\times(\mathbb{P}^{2g-3+(d-2j)/3}\smallsetminus\mathbb{P}^{g-2+(d-2j)/3})^2)/\mathbb{Z}_2\amalg\\ \amalg(\Delta_2\amalg\Delta_3\times(\mathbb{P}^{2g-3+(d-2j)/3}\smallsetminus\mathbb{P}^{g-2+(d-2j)/3})^2)/\mathbb{Z}_2\Big)=\\ =\Big(M'\times M'\Big)/\mathbb{Z}_2\times G_2\smallsetminus\Big(\Delta_0\amalg\Delta_1\times(\mathbb{P}^{2g-3+(d-2j)/3}\smallsetminus\mathbb{P}^{g-2+(d-2j)/3})^2/\mathbb{Z}_2\amalg\\ \amalg\Delta_2\times(\mathbb{P}^{2g-3+(d-2j)/3}\smallsetminus\mathbb{P}^{g-2+(d-2j)/3})^2\Big).\end{aligned}$$

Now

$$\begin{aligned}\mathcal{HD}(M')=\mathcal{HD}(G_2)\times\mathcal{HD}(\mathbb{P}^{2g-3+(d-2j)/3}\smallsetminus\mathbb{P}^{g-2+(d-2j)/3})=\\ =(1+u)^g(1+v)^g\frac{(uv)^{g-1+(d-2j)/3}-(uv)^{2g-2+(d-2j)/3}}{1-uv}.\end{aligned}$$

Therefore, by [MOVG2, lemma 2.6] we get the following polynomial:

$$\begin{aligned}\mathcal{HD}((M'\times M')/\mathbb{Z}_2)=\\ =\frac{1}{2}\Bigg((1+u)^{2g}(1+v)^{2g}\frac{((uv)^{g-1+(d-2j)/3}-(uv)^{2g-2+(d-2j)/3})^2}{(1-uv)^2}+\\ +(1-u^2)^g(1-v^2)^g\frac{(uv)^{2g-2+(2d-4j)/3}-(uv)^{4g-4+(2d-4j)/3}}{1-(uv)^2}\Bigg).\end{aligned}$$

Moreover, we can compute

$$\mathcal{HD}((\mathbb{P}^{2g-3+(d-2j)/3} \smallsetminus \mathbb{P}^{g-2+(d-2j)/3})^2/\mathbb{Z}_2) =$$
$$= \frac{1}{2}\left(\frac{((uv)^{g-1+(d-2j)/3} - (uv)^{2g-2+(d-2j)/3})^2}{(1-uv)^2} + \frac{(uv)^{2g-2+(2d-4j)/3} - (uv)^{4g-4+(2d-4j)/3}}{1-(uv)^2}\right).$$

So we get that:

$$\mathcal{HD}(M/\mathbb{Z}_2) = \frac{1}{2}\left((1+u)^{2g}(1+v)^{2g}\frac{((uv)^{g-1+(d-2j)/3} - (uv)^{2g-2+(d-2j)/3})^2}{(1-uv)^2} + \right.$$
$$\left. + (1-u^2)^g(1-v^2)^g\frac{(uv)^{2g-2+(2d-4j)/3} - (uv)^{4g-4+(2d-4j)/3}}{1-(uv)^2}\right)(1+u)^g(1+v)^g +$$
$$-\frac{1}{2}(1+u)^{2g}(1+v)^{2g}\left(\frac{((uv)^{g-1+(d-2j)/3} - (uv)^{2g-2+(d-2j)/3})^2}{(1-uv)^2} + \right.$$
$$\left. + \frac{(uv)^{2g-2+(2d-4j)/3} - (uv)^{4g-4+(2d-4j)/3}}{1-(uv)^2}\right) +$$
$$-(1+u)^g(1+v)^g((1+u)^g(1+v)^g - 1)\frac{((uv)^{g-1+(d-2j)/3} - (uv)^{2g-2+(d-2j)})^2}{(1-uv)^2}.$$

So we get the polynomial:

$$q_{19}^{j\equiv_6 2d} = \mathcal{HD}(R_{a,b,c,b,c}/\mathbb{Z}_2) = \mathcal{HD}(M/\mathbb{Z}_2)\mathcal{HD}(G_1)\mathcal{HD}(\mathbb{P}^{g-2+(d-2j)/3}) =$$
$$= \left\{\frac{1}{2}\left((1+u)^{2g}(1+v)^{2g}\frac{((uv)^{g-1+(d-2j)/3} - (uv)^{2g-2+(d-2j)/3})^2}{(1-uv)^2} + \right.\right.$$
$$\left. + (1-u^2)^g(1-v^2)^g\frac{(uv)^{2g-2+(2d-4j)/3} - (uv)^{4g-4+(2d-4j)/3}}{1-(uv)^2}\right)(1+u)^g(1+v)^g +$$
$$-\frac{1}{2}(1+u)^{2g}(1+v)^{2g}\left(\frac{((uv)^{g-1+(d-2j)/3} - (uv)^{2g-2+(d-2j)/3})^2}{(1-uv)^2} + \right.$$
$$\left. + \frac{(uv)^{2g-2+(2d-4j)/3} - (uv)^{4g-4+(2d-4j)/3}}{1-(uv)^2}\right) +$$
$$\left. -(1+u)^g(1+v)^g((1+u)^g(1+v)^g - 1)\frac{((uv)^{g-1+(d-2j)/3} - (uv)^{2g-2+(d-2j)/3})^2}{(1-uv)^2}\right\}\cdot$$
$$\cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)} \cdot \frac{1-(uv)^{g-1+(d-2j)/3}}{1-uv}.$$

(3c) Let us suppose that $(Q_2, 0) \not\simeq (Q_3, 0) \simeq (Q_4, 0)$. Then we can apply proposition 7.7.3. In this case we need to compute the invariants a, b, c. Also in this case we get

$$a = \dim \operatorname{Ext}^1((Q_2,0),(Q_1,W_1)) = g-1+(d-2j)/3,$$
$$b = \dim \operatorname{Ext}^1((Q_3,0),(E_2,V_2)) = 2g-2+(d-2j)/3,$$
$$c = \dim \operatorname{Ext}^1((Q_3,0),(Q_1,W_1)) = g-1+(d-2j)/3,$$

so each invariant can only assume one value. From the point of view of Hodge-Deligne polynomials, we can assume that we have the following description.

- $U_a = G_1 \times G_2$ and there is a projective bundle $\varphi_a : R_a \to U_a$ with fibers isomorphic to $\mathbb{P}^{a-1} = \mathbb{P}^{g-2+(d-2j)/3}$;
- $U_{a,b,c}$ is the set

$$\{((E_2,V_2),(Q_3,0)) \in R_a \times G_3 \text{ s.t. } (Q_3,0) \not\simeq \overline{\varphi}_a(E_2,V_2)\},$$

where $\overline{\varphi}_a$ is the composition of φ_a with the projection to G_2. From the point of view of Hodge-Deligne polynomials we can assume that $U_{a,b,c} = G_1 \times (G_2 \times G_3 \smallsetminus \Delta) \times \mathbb{P}^{g-2+(d-2j)/3}$, where Δ is the diagonal of $G_2 \times G_3 = G_2 \times G_2$. The (E,V)'s we are interested in are parametrized by a scheme $R_{a,b,c}$. Such a scheme comes with a fibration $R_{a,b,c} \to U_{a,b,c}$ with fibers isomorphic to $\mathbb{C}^{2c} \times Grass(2,b-c) = \mathbb{C}^{2g-2+(2d-4j)/3} \times Grass(2,g-1)$.

Then we get the polynomial

$$q_{20}^{j\equiv_6 2d} = \mathcal{HD}(R_{a,b,c}) = (\mathcal{HD}(G_2)^2 - \mathcal{HD}(G_2)) \times \mathcal{HD}(G_1) \times \mathcal{HD}(\mathbb{P}^{g-2+(d-2j)/3})\cdot$$
$$\cdot \mathcal{HD}(\mathbb{C}^{2g-2+(2d-4j)/3})\mathcal{HD}(Grass(2,g-1)) =$$
$$= (1+u)^g(1+v)^g((1+u)^g(1+v)^g-1) \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}\cdot$$
$$\cdot \frac{1-(uv)^{g-1+(d-2j)/3}}{1-uv}\cdot (uv)^{2g-2+(2d-4j)/3}\cdot \frac{(1-(uv)^{g-2})(1-(uv)^{g-1})}{(1-uv)(1-(uv)^2)}.$$

(3d) Let us suppose that $(Q_2,0) \simeq (Q_3,0) \simeq (Q_4,0)$. Then we can apply proposition 7.7.4. In this case the invariants a,b and the scheme R_a are as in (3c). Moreover, the scheme $U_{a,b}$ coincides with R_a and the (E,V)'s we are interested in are parametrized by a scheme $R_{a,b}$. Such a scheme comes with a fibration $R_{a,b} \to U_{a,b}$ with fibers isomorphic to $\mathbb{C}^{2a-2} \times Grass(2,b-a+1) = \mathbb{C}^{2g-4+(2d-4j)/3} \times Grass(2,g)$. Then we get the polynomial

$$q_{21}^{j\equiv_6 2d} = \mathcal{HD}(R_{a,b}) = \mathcal{HD}(G_2)\mathcal{HD}(G_1) \times \mathcal{HD}(\mathbb{P}^{g-2+(d-2j)/3})\cdot$$
$$\cdot \mathcal{HD}(\mathbb{C}^{2g-4+(2d-4j)/3})\mathcal{HD}(Grass(2,g)) =$$
$$= (1+u)^g(1+v)^g \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)} \cdot \frac{1-(uv)^{g-1+(d-2j)/3}}{1-uv}\cdot$$

$$\cdot (uv)^{2g-4+(2d-4j)/3} \frac{(1-(uv)^{g-1})(1-(uv)^g)}{(1-uv)(1-(uv)^2)}.$$

(4) Canonical filtration of type (1,2,1).
In this situation we need to consider the following subcases:

(a) $(Q_2, 0) \not\simeq (Q_3, 0) \simeq (Q_4, 0)$;

(b) $(Q_i, 0) \not\simeq (Q_j, 0)$ for all $j \neq j \in \{2, 3, 4\}$;

(c) $(Q_2, 0) \simeq (Q_3, 0) \not\simeq (Q_4, 0)$;

(d) $(Q_2, 0) \simeq (Q_3, 0) \simeq (Q_4, 0)$.

As we said in remark 12.2.2 we are able to describe completely cases (a),(c) and (d) but not case (b).

(4a) If we suppose that $(Q_2, 0) \not\simeq (Q_3, 0) \simeq (Q_4, 0)$, then we can apply proposition 7.6.2. In this case we need to compute the invariants a, b, c. In order to do that, first of all let us fix any pair of non-split extensions of the form

$$0 \to (Q_1, W_1) \longrightarrow (E_{i1}, V_{i1}) \longrightarrow (Q_i, 0) \to 0$$

for $i = 2, 3$ and let us denote by

$$0 \to (Q_1, W_1) \longrightarrow (E_2, V_2) \longrightarrow (Q_2, 0) \oplus (Q_3, 0) \to 0$$

their sum. Then (E_2, V_2) is a coherent system of rank $N_2 = 3$, degree $D_2 = d_1 + d_2 + d_3 = (4d + j)/6$ and with $K_2 = 1$. Since $(Q_2, 0)$ and $(Q_3, 0)$ are not isomorphic to (Q_1, W_1), then as in case (3) we have that:

$$a = \dim \operatorname{Ext}^1((Q_2, 0), (Q_1, W_1)) = g - 1 + (d - 2j)/3,$$
$$b = \dim \operatorname{Ext}^1((Q_3, 0), (Q_1, W_1)) = g - 1 + (d - 2j)/3.$$

Moreover,

$$c = \dim \operatorname{Ext}^1((Q_3, 0), (E_2, V_2)) = N_2 n_3 (g - 1) - D_2 n_3 + d_3 N_2 =$$
$$= 3(g - 1) - \frac{4d + j}{6} + \frac{2d - j}{2} = 3g - 3 + (d - 2j)/3.$$

So each invariant can assume only one value. By proposition 7.6.2 we can therefore suppose that we have the following description.

- $U_a^2 = G_1 \times G_2$, $U_b^3 = G_1 \times G_3$, $U_{a,b} = G_1 \times G_2 \times G_3$ and $V_{a,b} = G_1 \times (G_2 \times G_3 \smallsetminus \Delta)$; there is a fibration $R_{a,b} \to V_{a,b}$ with fibers isomorphic to $\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} = (\mathbb{P}^{g-2+(d-2j)/3})^2$;
- $U_{a,b,c} = R_{a,b}$ and there is a fibration $R_{a,b,c} \to U_{a,b,c}$ with fibers isomorphic to $\mathbb{P}^{c-1} \smallsetminus \mathbb{P}^{b-2} = \mathbb{P}^{3g-4+(d-2j)/3} \smallsetminus \mathbb{P}^{g-3+(d-2j)/3}$.

So we get:

$$q_{22}^{j\equiv_6 2d} := \mathcal{HD}(R_{a,b,c}) = \mathcal{HD}(G_1)(\mathcal{HD}(G_2)^2 - \mathcal{HD}(G_2))\cdot$$
$$\cdot(\mathcal{HD}(\mathbb{P}^{g-2+(d-2j)/3}))^2(\mathcal{HD}(\mathbb{P}^{3g-4+(d-2j)/3}) - \mathcal{HD}\mathbb{P}^{g-3+(d-2j)/3})) =$$
$$= (1+u)^g(1+v)^g((1+u)^g(1+v)^g-1)\frac{(1-(uv)^{g-1+(d-2j)/3})^2}{(1-uv)^2}\cdot$$
$$\cdot\frac{(uv)^{g-2+(d-2j)/3}-(uv)^{3g-3+(d-2j)/3}}{1-uv}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}.$$

(4b) As we said before, currently we are not able to compute the polynomial for this case. For simplicity we name it as $q_{23}^{j\equiv_6 2d}$.

(4c) If we suppose that $(Q_2,0)\simeq(Q_3,0)\not\simeq(Q_4,0)$, then we can apply proposition 7.6.3. In this case we need to compute the invariants a,b,c. Let (E_2,V_2) be as in (4a); then it is a coherent system of rank $N_2=3$, degree $D_2=(4d+j)/6$ and with $K_2=1$. By the same computations of case (4a) we have that:

$$a = \dim\ \mathrm{Ext}^1((Q_2,0),(Q_1,W_1)) = g-1+(d-2j)/3,$$
$$b = \dim\ \mathrm{Ext}^1((Q_4,0),(E_2,V_2)) = 3g-3+(d-2j)/3,$$
$$c = \dim\ \mathrm{Ext}^1((Q_4,0),(Q_1,W_1)) = g-1+(d-2j)/3.$$

So each invariant can assume only one value. By proposition 7.6.3 we can therefore suppose that we have the following description.

- $U_a=G_1\times G_2$ and there is a fibration $R_a\to U_a$ with fibers isomorphic to $Grass(2,a)=Grass(2,g-1+(d-2j)/3)$. From the point of view of Hodge-Deligne polynomials we can assume that $R_a=G_1\times G_2\times Grass(2,g-1+(d-2j)/3)$.
- From the point of view of Hodge-Deligne polynomials we can assume that

$$U_{a,b,c}=G_1\times(G_2\times G_4\smallsetminus\Delta)\times Grass(2,g-1+(d-2j)/3)$$

  and that there is a fibration $R_{a,b,c}\to U_{a,b,c}$ with fibers isomorphic to $\mathbb{P}^{b-1}\smallsetminus\mathbb{P}^{c-1}=\mathbb{P}^{3g-4+(d-2j)/3}\smallsetminus\mathbb{P}^{g-2+(d-2j)/3}$.

So we get:

$$q_{24}^{j\equiv_6 2d} := \mathcal{HD}(R_{a,b,c}) = \mathcal{HD}(G_1)(\mathcal{HD}(G_2)^2 - \mathcal{HD}(G_2))\cdot$$
$$\cdot\mathcal{HD}(Grass(2,g-1+(d-2j)/3))(\mathcal{HD}(\mathbb{P}^{3g-4+(d-2j)/3}) - \mathcal{HD}(\mathbb{P}^{g-2+(d-2j)/3})) =$$
$$= (1+u)^g(1+v)^g((1+u)^g(1+v)^g-1)\frac{(1-(uv)^{g-2+(d-2j)/3})(1-(uv)^{g-1+(d-2j)/3})}{(1-uv)(1-(uv)^2)}\cdot$$

$$\cdot\frac{(uv)^{g-1+(d-2j)/3}-(uv)^{3g-3+(d-2j)/3}}{1-uv}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}.$$

(4d) If we suppose that $(Q_2,0)\simeq(Q_3,0)\simeq(Q_4,0)$, then we can apply proposition 7.6.4. In this case we need to compute the invariants a,b. Let (E_2,V_2) be as in (4a); then as before we have that:

$$a=\dim\operatorname{Ext}^1((Q_2,0),(Q_1,W_1))=g-1+(d-2j)/3,$$
$$b=\dim\operatorname{Ext}^1((Q_2,0),(E_2,V_2))=3g-3+(d-2j)/3.$$

So each invariant can assume only one value. By proposition 7.6.4 we can therefore suppose that we have the following description.

- $U_a=G_1\times G_2$ and there is a fibration $R_a\to U_a$ with fibers isomorphic to $Grass(2,a)=Grass(2,g-1+(d-2j)/3)$.
- $U_{a,b}=R_a$ and there is a fibration $R_{a,b}\to U_{a,b}$ with fibers isomorphic to $\mathbb{P}^{b-1}\smallsetminus\mathbb{P}^{a-3}=\mathbb{P}^{3g-4+(d-2j)/3}\smallsetminus\mathbb{P}^{g-4+(d-2j)/3}$.

So we get:

$$\begin{aligned}q_{25}^{j\equiv_6 2d}&:=\mathcal{HD}(R_{a,b})=\mathcal{HD}(G_1)\mathcal{HD}(G_2)\mathcal{HD}(Grass(2,g-1+(d-2j)/3))\cdot\\&\cdot(\mathcal{HD}(\mathbb{P}^{3g-4+(d-2j)/3})-\mathcal{HD}(\mathbb{P}^{g-4+(d-2j)/3}))=\\&=(1+u)^g(1+v)^g\frac{(1-(uv)^{g-2+(d-2j)/3})(1-(uv)^{g-1+(d-2j)/3})}{(1-uv)(1-(uv)^2)}\cdot\\&\cdot\frac{(uv)^{g-3+(d-2j)/3}-(uv)^{3g-3+(d-2j)/3}}{1-uv}\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}.\end{aligned}$$

15.3 Crossing a critical value $\alpha(j)$

According to the description given at the beginning of the chapter, the non-zero effective critical values are contained in the set of those $\alpha(j)$'s such that $[j]_6\in\{0,2,4,2d+3\}_{\text{mod }6}$. The computations given in the previous 2 sections actually prove that each such j gives rise to an actual critical value. To be more precise:

(1) if $j\equiv 2d+3$ mod 6, then

$$\mathcal{HD}(G^-(\alpha(j);4,d,1))-\mathcal{HD}(G^+(\alpha(j);4,d,1))=q_2^{j\equiv_6 2d+3}-p_2^{j\equiv_6 2d+3};$$

(2) if $j\equiv 2d+2$ mod 6 or $j\equiv 2d+4$ mod 6, then

$$\begin{aligned}&\mathcal{HD}(G^-(\alpha(j);4,d,1))-\mathcal{HD}(G^+(\alpha(j);4,d,1))=\\&=q_1^{j\equiv_6 2d+2}-p_1^{j\equiv_6 2d+2}=q_1^{j\equiv_6 2d+4}-p_1^{j\equiv_6 2d+4};\end{aligned}$$

(3) if $j \equiv 2d \bmod 6$, then

$$\begin{aligned}&\mathcal{HD}(G^-(\alpha(j);4,d,1)) - \mathcal{HD}(G^+(\alpha(j);4,d,1)) = \\ &= q_1^{j\equiv_6 2d} + \cdots + q_{25}^{j\equiv_6 2d} - (p_1^{j\equiv_6 2d} + \cdots + p_{25}^{j\equiv_6 2d}).\end{aligned}$$

Actually, we can compute explicitly those quantities only in the first 2 cases, as shown below. In the third case we are not able to get an explicit result for 2 different problems:

- first of all, there are no formulae in the literature for the Hodge-Deligne polynomials of $M^s(3,e)$ when $e \equiv 0 \bmod 3$; therefore it is not possible to write explicitly the polynomials $p_1^{j\equiv_6 2d}$ and $q_1^{j\equiv_6 2d}$;
- secondly, as we said before, at the moment we are not able to compute the 8 polynomials $p_{13}^{j\equiv_6 2d}, p_{14}^{j\equiv_6 2d}, p_{23}^{j\equiv_6 2d}, p_{24}^{j\equiv_6 2d}, p_{25}^{j\equiv_6 2d}, q_{13}^{j\equiv_6 2d}, q_{14}^{j\equiv_6 2d}$ and $q_{23}^{j\equiv_6 2d}$.

15.3.1 j equivalent to $2d+3$ modulo 6

If $j \equiv_3 2d+3$ modulo 6, then we get:

$$\begin{aligned}&\mathcal{HD}(G^-(\alpha(j);4,d,1)) - \mathcal{HD}(G^+(\alpha(j);4,d,1)) = q_2^{j\equiv_6 2d+3} - p_2^{j\equiv_6 2d+3} = \\ &= \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^3(1-u^2v^2)} \cdot \\ &\cdot (1+u)^{2g}(1+v)^{2g} \cdot \left[(uv)^{2g-2+j} - (uv)^{4g-4+(2d-4j)/3}\right] \cdot \\ &\cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} \cdot \left[\frac{(uv)^{(j-1)/2}x^{(1-j)/2}}{1-x(uv)^{-1}} - \frac{(uv)^{g+(d-2j)/3}x^{(1-j)/2}}{1-x(uv)^2}\right]. \end{aligned} \tag{15.10}$$

15.3.2 j equivalent to $2d+2$ or to $2d+4$ modulo 6

Let us suppose that $j \equiv_6 2d+2$ or $j \equiv_6 2d+4$; then in both cases we have that $d-j/2 \not\equiv_3 0$. Therefore we can use formula (8.12) in order to compute the Hodge-Deligne polynomial of $M^s(3,d-j/2) = M(3,d-j/2)$. So we get:

$$\begin{aligned}&\mathcal{HD}(G^-(\alpha(j);4,d,1)) - \mathcal{HD}(G^+(\alpha(j);4,d,1)) = q_1^{j\equiv_6 2d+2} - p_1^{j\equiv_6 2d+2} = \\ &= \mathcal{HD}(M(3,d-j/2))\frac{(uv)^{3j/2} - (uv)^{3g-3+d-2j}}{1-uv} \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)} = \\ &= (1+u)^g(1+v)^g \frac{(uv)^{3j/2} - (uv)^{3g-3+d-2j}}{(1-uv)^2(1-u^2v^2)^2(1-u^3v^3)} \cdot \\ &\cdot \left[(1+u^2v^3)^g(1+u^3v^2)^g(1+uv^2)^g(1+u^2v)^g + \right. \\ &-u^{2g-1}v^{2g-1}(1+uv)^2(1+u)^g(1+v)^g(1+uv^2)^g(1+u^2v)^g + \\ &\left. +u^{3g-1}v^{3g-1}(1+uv+u^2v^2)(1+u)^{2g}(1+v)^{2g}\right] \cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-j/2}}{(1-x)(1-uvx)}. \end{aligned} \tag{15.11}$$

15.4 The "last" moduli space for d non-equivalent to 0 modulo 3

The moduli spaces $G(\alpha; 4, d, 1)$ are non-empty only if $\alpha < d/3$. Therefore the last moduli space $G_L(4, d, 1)$ is the one for $\alpha = d/3 - \varepsilon = \alpha(0)^-$, so

$$\begin{aligned}\mathcal{HD}(G_L(4,d,1)) &= \mathcal{HD}(G^-(\alpha(0);4,d,1)) = \\ &= \mathcal{HD}(G^-(\alpha(0);4,d,1)) - \mathcal{HD}(G^+(\alpha(0);4,d,1)).\end{aligned} \tag{15.12}$$

The value $j = 0$ can never be obtained in case (1); it can be obtained in case (2) if $d \equiv_3 1$ or $d \equiv_3 2$ and it can be obtained in case (3) if $d \equiv_3 0$. As we said before, we are not able to compute (15.12) in case (3). In case (2), if we use (15.11) we get:

Corollary 15.4.1. *Let us suppose that $d \not\equiv_3 0$. Then the Hodge-Deligne polynomial of $G(\alpha(0)^-; 4, d, 1) = G(d/3 - \varepsilon, 4, d, 1) = G_L(4, d, 1)$ is given by*

$$\begin{aligned}\mathcal{HD}(G(\alpha(0)^-;4,d,1)) = (1+u)^g(1+v)^g \frac{1-(uv)^{3g-3+d}}{(1-uv)^2(1-u^2v^2)^2(1-u^3v^3)} \cdot \\ \cdot \big[(1+u^2v^3)^g(1+u^3v^2)^g(1+uv^2)^g(1+u^2v)^g + \\ -u^{2g-1}v^{2g-1}(1+uv)^2(1+u)^g(1+v)^g(1+uv^2)^g(1+u^2v)^g + \\ +u^{3g-1}v^{3g-1}(1+uv+u^2v^2)(1+u)^{2g}(1+v)^{2g}\big].\end{aligned} \tag{15.13}$$

If we denote by $p(u, v)$ that polynomial, a direct check proves that

$$p(u^{-1}, v^{-1}) = (uv)^{12-12g-d} p(u, v).$$

We know that the dimension of the moduli spaces $G(\alpha; 4, d, 1)$ is given by $\beta(4, d, 1) = 12g - 12 + d$; therefore the previous polynomial satisfies Poincaré duality.

15.5 Crossing the critical value $\alpha(1)$ for every d

As we said before, the non-zero critical values $\alpha(j)$ are all such that $[j]_6 \in \{0, 2, 4, 2d + 3\}_{\text{mod } 6}$. Therefore, if $\alpha(1)$ is an actual critical value then necessarily $1 = j \equiv_6 2d + 3$, so we are in case (1). Actually, this is equivalent to imposing that $d \equiv_3 2$. If this happens, then by using formula (15.10) we get that

$$\begin{aligned}\mathcal{HD}(G^-(\alpha(1);4,d,1)) - \mathcal{HD}(G^+(\alpha(1);4,d,1)) = \\ = \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^3(1-u^2v^2)} \cdot \\ \cdot (1+u)^{2g}(1+v)^{2g} \cdot \left[(uv)^{2g-1} - (uv)^{4g-5+(2d-1)/3}\right] \cdot \\ \cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} \cdot \left[\frac{1}{1-x(uv)^{-1}} - \frac{(uv)^{g+(d-2)/3}}{1-x(uv)^2}\right].\end{aligned}$$

So we have the following lemma.

Lemma 15.5.1. *If $d \not\equiv_3 2$, then $\alpha(1)$ is not an actual critical value. If $d \equiv_3 2$, then $\alpha(1)$ is an actual critical value and*

$$\mathcal{HD}(G^-(\alpha(1);4,d,1)) - \mathcal{HD}(G^+(\alpha(1);4,d,1)) =$$
$$= \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^3(1-u^2v^2)} \cdot$$
$$\cdot (1+u)^{2g}(1+v)^{2g} \cdot \left[(uv)^{2g-1} - (uv)^{4g-5+(2d-1)/3}\right] \cdot \left[1-(uv)^{g+(d-2)/3}\right].$$

By combining corollary 15.4.1 and lemma 15.5.1, if $d \equiv_3 2$ we have

$$\mathcal{HD}(G(\alpha(1)^-;4,d,1)) = (1+u)^g(1+v)^g \frac{1-(uv)^{3g-3+d}}{(1-uv)^2(1-u^2v^2)^2(1-u^3v^3)} \cdot$$
$$\cdot \big[(1+u^2v^3)^g(1+u^3v^2)^g(1+uv^2)^g(1+u^2v)^g +$$
$$-u^{2g-1}v^{2g-1}(1+uv)^2(1+u)^g(1+v)^g(1+uv^2)^g(1+u^2v)^g +$$
$$+u^{3g-1}v^{3g-1}(1+uv+u^2v^2)(1+u)^{2g}(1+v)^{2g}\big] +$$
$$+ \frac{(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g}{(1-uv)^3(1-u^2v^2)} \cdot (1+u)^{2g}(1+v)^{2g} \cdot$$
$$\cdot \left[(uv)^{2g-1} - (uv)^{4g-5+(2d-1)/3}\right] \cdot \left[1-(uv)^{g+(d-2)/3}\right].$$

By rearranging, we get:

Corollary 15.5.2. *If $d \equiv_3 1$, then $\alpha(1)$ is not an actual critical value, so $G(\alpha(1)^-;4,d,1) = G(\alpha(0)^-;4,d,1)$; therefore (15.13) gives also the Hodge-Deligne polynomial of $G(\alpha(1)^-;4,d,1)$. If $d \equiv_3 2$, then $\alpha(1)$ is an actual critical value and*

$$\mathcal{HD}(G(\alpha(1)^-;4,d,1)) = \frac{(1+u)^g(1+v)^g}{(1-uv)^3(1-u^2v^2)} \left\{ \frac{1-(uv)^{3g-3+d}}{(1+uv)(1-u^3v^3)} \cdot \right.$$
$$\cdot \big[(1+u^2v^3)^g(1+u^3v^2)^g(1+uv^2)^g(1+u^2v)^g +$$
$$-u^{2g-1}v^{2g-1}(1+uv)^2(1+u)^g(1+v)^g(1+uv^2)^g(1+u^2v)^g +$$
$$+u^{3g-1}v^{3g-1}(1+uv+u^2v^2)(1+u)^{2g}(1+v)^{2g}\big] +$$
$$+ \big[(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g\big] \cdot (1+u)^g(1+v)^g \cdot$$
$$\left. \cdot \left[(uv)^{2g-1} - (uv)^{4g-5+(2d-1)/3}\right] \cdot \left[1-(uv)^{g+(d-2)/3}\right] \right\}.$$

Remark 15.5.1. If $d \equiv_3 0$, then $\alpha(0)$ is not an actual critical value, so also in this case $G(\alpha(1)^-;4,d,1) = G(\alpha(0)^-;4,d,1)$, but since corollary 15.4.1 holds only for $d \not\equiv_3 0$, then we cannot have an explicit formula for the Hodge-Deligne polynomial of such a space.

15.6 Crossing the critical value $\alpha(2)$ for d non-equivalent to 1 modulo 3

The case $j = 2$ corresponds always to an actual critical value. To be more precise, if $d \not\equiv_3 1$, we are in case (2); if $d \equiv_3 1$, this corresponds to case (3) (in this case, as before we cannot say anything explicitly). By using (15.11), if $d \not\equiv_3 1$ then:

$$\begin{gathered}
\mathcal{HD}(G^-(\alpha(2);4,d,1)) - \mathcal{HD}(G^+(\alpha(2);4,d,1)) = \\
= (1+u)^g(1+v)^g \frac{(uv)^3 - (uv)^{3g-7+d}}{(1-uv)^2(1-u^2v^2)^2(1-u^3v^3)} \cdot \\
\cdot \big[(1+u^2v^3)^g(1+u^3v^2)^g(1+uv^2)^g(1+u^2v)^g + \\
-u^{2g-1}v^{2g-1}(1+uv)^2(1+u)^g(1+v)^g(1+uv^2)^g(1+u^2v)^g + \\
+u^{3g-1}v^{3g-1}(1+uv+u^2v^2)(1+u)^{2g}(1+v)^{2g}\big] \cdot \operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-1}}{(1-x)(1-uvx)}.
\end{gathered}$$

By expanding in power series around $x = 0$, we have

$$\operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g x^{-1}}{(1-x)(1-uvx)} = \operatorname*{coeff}_{x^1} \frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)} = 1 + g(u+v) + uv.$$

So we get:

Lemma 15.6.1. *If $d \not\equiv_3 1$, then*

$$\begin{gathered}
\mathcal{HD}(G^-(\alpha(2);4,d,1)) - \mathcal{HD}(G^+(\alpha(2);4,d,1)) = \\
= (1+u)^g(1+v)^g \frac{(uv)^3 - (uv)^{3g-7+d}}{(1-uv)^2(1-u^2v^2)^2(1-u^3v^3)} \cdot \\
\cdot \big[(1+u^2v^3)^g(1+u^3v^2)^g(1+uv^2)^g(1+u^2v)^g + \\
-u^{2g-1}v^{2g-1}(1+uv)^2(1+u)^g(1+v)^g(1+uv^2)^g(1+u^2v)^g + \\
+u^{3g-1}v^{3g-1}(1+uv+u^2v^2)(1+u)^{2g}(1+v)^{2g}\big] \cdot (1 + g(u+v) + uv).
\end{gathered}$$

We can combine corollary 15.5.2 and lemma 15.6.1 only if $d \equiv_3 2$. In this case we get:

$$\begin{gathered}
\mathcal{HD}(G^-(\alpha(2);4,d,1)) = \frac{(1+u)^g(1+v)^g}{(1-uv)^3(1-u^2v^2)} \left\{ \frac{1-(uv)^{3g-3+d}}{(1+uv)(1-u^3v^3)} \cdot \right. \\
\cdot \big[(1+u^2v^3)^g(1+u^3v^2)^g(1+uv^2)^g(1+u^2v)^g + \\
-u^{2g-1}v^{2g-1}(1+uv)^2(1+u)^g(1+v)^g(1+uv^2)^g(1+u^2v)^g + \\
+u^{3g-1}v^{3g-1}(1+uv+u^2v^2)(1+u)^{2g}(1+v)^{2g}\big] + \\
+ \big[(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^g(1+v)^g\big] \cdot (1+u)^g(1+v)^g \cdot \\
\left. \cdot \Big[(uv)^{2g-1} - (uv)^{4g-5+(2d-1)/3}\Big] \cdot \Big[1-(uv)^{g+(d-2)/3}\Big] \right\} +
\end{gathered}$$

$$+(1+u)^g(1+v)^g\frac{(uv)^3-(uv)^{3g-7+d}}{(1-uv)^2(1-u^2v^2)^2(1-u^3v^3)}\cdot$$
$$\cdot\Big[(1+u^2v^3)^g(1+u^3v^2)^g(1+uv^2)^g(1+u^2v)^g+$$
$$-u^{2g-1}v^{2g-1}(1+uv)^2(1+u)^g(1+v)^g(1+uv^2)^g(1+u^2v)^g+$$
$$+u^{3g-1}v^{3g-1}(1+uv+u^2v^2)(1+u)^{2g}(1+v)^{2g}\Big]\cdot(1+g(u+v)+uv).$$

So we have:

Corollary 15.6.2. *If $d\equiv_3 2$, then $\alpha(2)$ is an actual critical value and*

$$\mathcal{HD}(G^-(\alpha(2);4,d,1))=$$
$$=\frac{(1+u)^g(1+v)^g}{(1-uv)^3(1-u^2v^2)}\Bigg\{\frac{1-(uv)^{3g-3+d}+[(uv)^3-(uv)^{3g-7+d}]\cdot[1+g(u+v)+uv]}{(1+uv)(1-u^3v^3)}\cdot$$
$$\cdot\Big[(1+u^2v^3)^g(1+u^3v^2)^g(1+uv^2)^g(1+u^2v)^g+$$
$$-u^{2g-1}v^{2g-1}(1+uv)^2(1+u)^g(1+v)^g(1+uv^2)^g(1+u^2v)^g+$$
$$+u^{3g-1}v^{3g-1}(1+uv+u^2v^2)(1+u)^{2g}(1+v)^{2g}\Big]+$$
$$+\Big[(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g\Big]\cdot(1+u)^g(1+v)^g\cdot$$
$$\cdot\Big[(uv)^{2g-1}-(uv)^{4g-5+(2d-1)/3}\Big]\cdot\Big[1-(uv)^{g+(d-2)/3}\Big]\Bigg\}.\tag{15.14}$$

15.7 Crossing the critical value $\alpha(3)$ for every d

As we said before, the non-zero critical values $\alpha(j)$ are all such that $[j]_6\in\{0,2,4,2d+3\}_{\mathrm{mod}\ 6}$. Therefore, if $\alpha(3)$ is an actual critical value then necessarily $3=j\equiv_6 2d+3$, so we are in case (1). Actually, this is equivalent to imposing that $d\equiv_3 0$. If this happens, then by using formula (15.10) we get that

$$\mathcal{HD}(G^-(\alpha(1);4,d,1))-\mathcal{HD}(G^+(\alpha(1);4,d,1))=$$
$$=\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^3(1-u^2v^2)}\cdot(1+u)^{2g}(1+v)^{2g}\cdot$$
$$\cdot\Big[(uv)^{2g+1}-(uv)^{4g-8+2d/3}\Big]\cdot\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot\left[\frac{uvx^{-1}}{1-x(uv)^{-1}}-\frac{(uv)^{g-2+d/3}x^{-1}}{1-x(uv)^2}\right].$$

Now

$$\operatorname*{coeff}_{x^0}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot\left[\frac{uvx^{-1}}{1-x(uv)^{-1}}-\frac{(uv)^{g-2+d/3}x^{-1}}{1-x(uv)^2}\right]=$$
$$=\operatorname*{coeff}_{x^1}\frac{(1+ux)^g(1+vx)^g}{(1-x)(1-uvx)}\cdot\left[\frac{uv}{1-x(uv)^{-1}}-\frac{(uv)^{g-2+d/3}}{1-x(uv)^2}\right]=$$

$$= \operatorname*{coeff}_{x^1}(1+gux)(1+gvx)(1+x)(1+uvx)\cdot\Big[(uv)(1+x(uv)^{-1})-(uv)^{g-2+d/3}(1+x(uv)^2)\Big]=$$

$$=\operatorname*{coeff}_{x^1}[1+x(1+g(u+v)+uv)]\cdot\Big[(uv)(1+x(uv)^{-1})-(uv)^{g-2+d/3}(1+x(uv)^2)\Big]=$$

$$=(uv)[(uv)^{-1}+1+g(u+v)+uv]-(uv)^{g-2+d/3}[(uv)^2+1+g(u+v)+uv]=$$

$$=1+(uv)+g(uv)(u+v)+(uv)^2-(uv)^{g+d/3}-(uv)^{g-2+d/3}-g(uv)^{g-2+d/3}(u+v)-(uv)^{g-1+d/3}.$$

So we have the following lemma.

Lemma 15.7.1. *If $d\not\equiv_3 0$, then $\alpha(3)$ is not an actual critical value. If $d\equiv_3 0$, then $\alpha(3)$ is an actual critical value and*

$$\mathcal{HD}(G^-(\alpha(3);4,d,1))-\mathcal{HD}(G^+(\alpha(3);4,d,1))=$$
$$=\frac{(1+u^2v)^g(1+uv^2)^g-(uv)^g(1+u)^g(1+v)^g}{(1-uv)^3(1-u^2v^2)}\cdot(1+u)^{2g}(1+v)^{2g}\cdot$$
$$\cdot\Big[(uv)^{2g+1}-(uv)^{4g-8+2d/3}\Big]\cdot\Big[1+(uv)+g(uv)(u+v)+(uv)^2+$$
$$-(uv)^{g+d/3}-(uv)^{g-2+d/3}-g(uv)^{g-2+d/3}(u+v)-(uv)^{g-1+d/3}\Big].$$

We recall that we don't know any of the polynomials for $G(\alpha(j)^-;4,d,1)$ for $d\equiv_3 0$ and $j=0,1,2$. Therefore the formula of the previous lemma is not useful in order to get information for the moduli space $G(\alpha(3)^-;4,d,1)$ when $d\equiv_3 0$. We can get a polynomial for such a space only if $d\equiv_3 2$.

Corollary 15.7.2. *If $d\not\equiv_3 1,2$, then $\alpha(3)$ is not an actual critical value, so $G(\alpha(3)^-;4,d,1)=G(\alpha(2)^-;4,d,1)$; therefore (15.14) gives also the Hodge-Deligne polynomial of $G(\alpha(1)^-;4,d,1)$ whenever $d\equiv_3 2$.*

Remark 15.7.1. In principle, one can go further and try to compute what happens when we cross $\alpha(4)$. In this case, we are always in cases (2) or (3); more precisely, we have a complete formula only in case (2), that corresponds to $d\not\equiv_3 0$. So we are able to get formulae for crossing $\alpha(4)$ only when $d\equiv_3 0$ or $d\equiv_3 1$. But in these 2 cases we have no information about $G(\alpha(2)^-;4,d,1)$. So with the present technique we cannot go any further, until all the polynomials involved in case (3) are known completely.

Chapter 16

Case n=2, k=2 on a Petri curve

In this chapter we want to study the moduli spaces $G(\alpha; 2, d, 2)$. In order to be able to do some computations, we will restrict to any Petri curve of genus $g \geq 2$ with d large (see below for the details). We recall the definition:

Definition 16.0.1. ([BGMN, definition 2.9]) A curve C is called a *Petri curve* if the Petri map

$$H^0(L) \otimes H^0(L^\vee \otimes K) \longrightarrow H^0(K)$$

is injective for every line bundle L over C.

By [BGMN, theorem 8.1], the moduli spaces $G(\alpha; 2, d, 2)$ for α non-critical are non-empty if and only if $d > 2$. In this case they are irreducible and of the expected dimension $2d - 3$.

Let us consider the critical values for the triple $(n, d, k) = (2, d, 2)$. By [BGMN, §2] the non-zero virtual critical values are all in the set

$$\left\{ \frac{nd' - n'd}{n'k - nk'} \text{ s.t. } 0 \leq k' \leq k,\ 0 < n' < n,\ n'k \neq nk',\ d' \in \mathbb{Z} \right\} \cap]0, \infty[.$$

In our case, this gives

$$\left\{ \frac{2d' - d}{2 - 2k'} \text{ s.t. } k' = 0, 2, \quad d' \in \mathbb{Z} \right\} \cap]0, \infty[,$$

that is

$$\left\{ \frac{2d' - d}{2} \text{ s.t. } d' > \frac{d}{2} \right\} \cup \left\{ \frac{d - 2d'}{2} \text{ s.t. } d' < \frac{d}{2} \right\},$$

where the first set corresponds to destabilizing subsystems of the form $(1, d', 0)$ and the second one corresponds to destabilizing subsystems of the form $(1, d', 2)$. Actually, the 2 sets coincide both with the set

$$\left\{ \alpha(j) := \frac{d - 2j}{2} \text{ s.t. } j < \frac{d}{2} \right\}.$$

By [BGMN, proposition 4.6], actually the non-zero effective critical values are only a finite subset of such a set. We will describe such a subset below.

16.1 The moduli spaces $G^+(\alpha(j); 2, d, 2)$

Let $\alpha(j)$ be any virtual critical value for $j < d/2$ and let us suppose that (E, V) belongs to $G^+(\alpha(j); 2, d, 2)$. Then by lemma 1.0.6, we get that (E, V) appears in a non-split exact sequence:

$$0 \to (Q_1, W_1) \to (E, V) \to (Q_2, W_2) \to 0 \tag{16.1}$$

where:

(a) (Q_1, W_1) and (Q_2, W_2) are both $\alpha(j)^+$-stable with $\frac{k_1}{n_1} < \frac{k}{n} = 1 < \frac{k_2}{n_2}$;

(b) (Q_1, W_1) and (Q_2, W_2) are both $\alpha(j)$-semistable with the same $\alpha(j)$-slope as (E, V).

Conversely, it is easy to see that every such (E, V) is actually a point of $G^+(\alpha(j); 2, d, 2)$. Moreover, every such (E, V) is completely determined by the class of the non-split extension (16.1), up to multiplication by invertible scalars. Condition (a) implies that $n_1 = n_2 = 1$ and that $k_1 = 0$, so $k_2 = 2$; condition (b) implies that $d_1 = d - j$, so $d_2 = j$. Now

$$(Q_1, W_1) = (Q_1, 0 \in G(1, d-j, 0) = J^{d-j}C =: G_1, \quad (Q_2, W_2) \in G(1, j, 2) =: G_2. \tag{16.2}$$

We must impose that $j \geq 0$ in order to have that G_2 is non-empty. Therefore, the only interesting (i.e. a priori non-empty) schemes of the form $G^+(\alpha(j); 2, d, 2)$ are those for $0 \leq j < \frac{d}{2}$. So from now on we will work with this setting.

Now let us fix any pair of objects $(Q_1, 0) \in G_1$, $(Q_2, W_2) \in G_2$. By lemma 1.0.4 we have that $\mathrm{Hom}((Q_2, W_2), (Q_1, 0)) = 0$. Moreover, by [BGMN, proposition 3.2] we have that

$$\mathbb{H}^2_{21} = \mathrm{Ext}^2((Q_2, W_2), (Q_1, 0)) = H^0(Q_1^\vee \otimes N_2 \otimes K)^\vee,$$

where N_2 is the kernel of the evaluation morphism $\phi_2 : W_2 \otimes \mathcal{O}_C \to Q_2$. Since (Q_2, W_2) is a coherent system, then $H^0(\phi_2)$ is injective, so in particular we must have that ϕ_2 is non-zero. If $N_2 = 0$, then we conclude directly that $\mathbb{H}^2_{21} = 0$. If $N_2 \neq 0$, then we get that N_2 is a line bundle because dim $W_2 = 2$ and rank $Q_2 = 1$. So we have an exact sequence

$$0 \to N_2 \to W_2 \otimes \mathcal{O}_C \to L \to 0,$$

where L is a line bundle with at least 2 sections. Therefore, deg $N_2 \leq -2$. So

$$\deg(Q_1^\vee \otimes N_2 \otimes K) = -d_1 + \deg N_2 + 2g - 2 \leq -d + j + 2g - 4 < 2g - 2.$$

Then by Clifford theorem we have that if $\mathbb{H}^2_{21}$ is not zero, then

$$\dim \mathbb{H}^2_{21} = \mathrm{h}^0(Q_1^\vee \otimes N_2 \otimes K) \leq \frac{-d+j+2g-4}{2} + 1 = \frac{-d+j+2g-2}{2}.$$

Since we want to compute the Hodge polynomials of some of the moduli spaces $G(\alpha(j)^-;2,d,1)$, then we need to restrict to the case when $\mathbb{H}^2_{21}$ is zero. The previous computation shows that $\mathbb{H}^2_{21}$ is zero if we assume that

$$j < d - 2g + 2.$$

This is only a sufficient condition, we don't know if it is also necessary. By combining this with the previous conditions on j, we are therefore restricting from now on to the case when

$$0 \leq j < \min\left\{\frac{d}{2}, d - 2g + 2\right\}. \tag{16.3}$$

Remark 16.1.1. Such a set is non-empty whenever $d \geq 2g-1$. Moreover, such a set coincides with the whole range of the values of j under consideration (i.e. $0 \leq j < d/2$) if $d \geq 4g-4$. So if $d \geq 4g-4$ we are able to describe the geometry of all the finitely many flips from the last moduli space to the first one.

So from now on let us assume that $d \geq 4g-4$ and let us fix any $0 \leq j < d/2$. Then by proposition 1.0.7 we have that

$$\begin{aligned}\dim \mathrm{Ext}^1((Q_2,W_2),(Q_1,0)) &= C_{21} + \dim \mathbb{H}^0_{21} + \dim \mathbb{H}^2_{21} = C_{21} = \\ &= n_1n_2(g-1) - d_1n_2 + d_2n_1 + k_2d_1 - k_2n_1(g-1) = \\ &= g - 1 - d + j + j + 2d - 2j - 2(g-1) = d - g + 1.\end{aligned}$$

Now the moduli space G_1 is smooth and irreducible for every $0 \leq j \leq d/2$. Since we assumed that C is a Petri curve, then by [BGMN, §2.3], we have that $G(1,j,2)$ is non-empty if and only if $\beta := 2j - g - 2 \geq 0$, i.e. if and only if $j \geq g/2 + 1$. So we get that the only non-zero (virtual) critical values $\alpha(j)$ for which $G^+(\alpha(j),2,d,2)$ is non-empty are those such that

$$\frac{g}{2} + 1 \leq j < \frac{d}{2}.$$

On both the G_i's there are families of coherent systems $(\mathcal{Q}_i, \mathcal{W}_i)$ (because of [BGMMN, proposition A.8]), so we can apply proposition 5.0.5 for $r = 2$ and we get that there is a projective bundle

$$\varphi_j : R_j \longrightarrow G_1 \times G_2$$

with fibers isomorphic to $\mathbb{P}^{d-g}$; there is an injective morphism from R_j to $G(\alpha(j)^+;2,d,2)$, such that the image coincides with $G^+(\alpha(j);2,d,2)$. Therefore, we get:

Lemma 16.1.1. *Let us suppose that* $d \geq 4g-4$. *Then* $G^+(\alpha(j);2,d,2)$ *is non-empty only if* $g/2+1 \leq j < d/2$. *In this case, its Hodge-Deligne polynomial is given by*

$$p^j := \mathcal{HD}\Big(G^+\left(\alpha(j);(d-2j)/2;2,d,2\right)\Big) =$$

$$= \mathcal{HD}(J^{d-j}C)\mathcal{HD}(G(1,j,2))\mathcal{HD}(\mathbb{P}^{d-g}) =$$

$$= (1+u)^g(1+v)^g\frac{1-(uv)^{d-g+1}}{1-uv}\mathcal{HD}(G(1,j,2)). \tag{16.4}$$

16.2 The moduli spaces $G^-(\alpha(j);2,d,2)$

Let us consider now $G^-(\alpha(j);2,d,2)$. By applying again lemma 1.0.6 we get that every $(E,V) \in G^-(\alpha(j);2,d,2)$ sits in a non-split exact sequence (16.1) where:

(a') (Q_1,W_1) and (Q_2,W_2) are both $\alpha(j)^-$-stable with $\frac{k_1}{n_1} > \frac{k}{n} = 1 > \frac{k_2}{n_2}$;

(b') (Q_1,W_1) and (Q_2,W_2) are both $\alpha(j)$-semistable with the same $\alpha(j)$-slope as (E,V).

Conversely, as before it is easy to show that every such (E,V) is actually a point of $G^-(\alpha(j);2,d,2)$; moreover any such (E,V) is uniquely associated to a non-split exact sequence (16.1) with conditions (a') and (b'), up to multiplication by invertible scalars.

Condition (a') implies that $n_1 = n_2 = 1$ and that $k_1 = 2$, so $k_2 = 0$. Moreover, condition (b') implies that $d_2 = d-j$, so $d_1 = j$. Now

$$(Q_1,W_1) \in G(1,j,2) =: G_1, \quad (Q_2,W_2) = (Q_2,0) \in G(1,d-j,0) = J^{d-j}C =: G_2. \tag{16.5}$$

Therefore, also in this case the (virtual) critical values $\alpha(j)$ such that $G^-(\alpha(j);2,d,1)$ is non empty are those such that $g/2+1 \leq j < d/2$.

For every pair of objects (Q_1,W_1), $(Q_2,0)$ in those 2 spaces, we have that $\mu_{\alpha(j)^-}(Q_2,0) > \mu_{\alpha(j)^-}(Q_1,W_1)$ as a consequence of properties (a') and (b'). Therefore, by lemma 1.0.4 there are no morphisms from $(Q_2,0)$ to (Q_1,W_1), so $\mathbb{H}^0_{21} = 0$. Moreover, by [BGMN, equation (11)], we have that also $\mathbb{H}^2_{21} = 0$. Therefore

$$\begin{aligned}\dim \operatorname{Ext}^1((Q_2,0),(Q_1,W_1)) = C_{21} = n_1n_2(g-1) - d_1n_2 + d_2n_1 = \\ = g-1-j+d-j = g+d-1-2j.\end{aligned}$$

Therefore as in the previous section we get that the space $G^-(\alpha(j);2,d,2)$ is given by a projective bundle over $G_1 \times G_2$ with fibers isomorphic to $\mathbb{P}^{g+d-2-2j}$, so we have:

Lemma 16.2.1. *Let us suppose that $d \geq 4g-4$; then $G^-(\alpha(j); 2, d, 2)$ is non-empty only if $g/2+1 \leq j < d/2$. In this case, its Hodge-Deligne polynomial is given by*

$$\begin{aligned} q^j &:= \mathcal{HD}\Big(G^-(\alpha(j); (d-2j)/2; 2, d, 2)\Big) = \\ &= \mathcal{HD}(J^{d-j}C)\mathcal{HD}(G(1, j, 2))\mathcal{HD}(\mathbb{P}^{g+d-2-2j}) = \\ &= (1+u)^g(1+v)^g \frac{1-(uv)^{g+d-1-2j}}{1-uv}\mathcal{HD}(G(1, j, 2)). \end{aligned} \tag{16.6}$$

16.3 The polynomials for $G(\alpha(k)^-; 2, d, 2)$

By combining the previous 2 lemmas we get the following result.

Theorem 16.3.1. *Let us suppose that C is a smooth projective irreducible Petri curve of genus $g \geq 2$ and let $d \geq 4g-4$. Then the actual critical values for $(2, d, 2)$ are all of the form*

$$\alpha(k) = \frac{d-2k}{2}, \quad \frac{g}{2}+1 \leq k < \frac{d}{2}.$$

For each value of k in that range the following formula holds:

$$\begin{aligned} &\mathcal{HD}(G(\alpha(k)^-; 2, d, 2)) = \\ &= \frac{(1+u)^g(1+v)^g}{1-uv} \sum_{j=\lceil g/2\rceil+1}^{k} \Big((uv)^{d-g+1} - (uv)^{g+d-1-2j})\mathcal{HD}(G(1, j, 2))\Big) = \\ &= \frac{(1+u)^g(1+v)^g}{1-uv} \left(\left(k-\frac{g}{2}\right)(uv)^{d-g+1} \sum_{j=\lceil g/2\rceil+1}^{k} \mathcal{HD}(G(1, j, 2)) + \right. \\ &\left. - \sum_{j=\lceil g/2\rceil+1}^{k} (uv)^{g+d-1-2j})\mathcal{HD}(G(1, j, 2))\right). \end{aligned} \tag{16.7}$$

Since we assumed that C is a Petri curve, then by [BGMN, §2.3], we have the following properties for $G(1, j, 2)$:

- if $\beta = 2j-g-2 \geq 0$, then $G(1, j, 2)$ is smooth of dimension β;
- if $\beta > 0$, then $G(1, j, 2)$ is irreducible;
- if $\beta \geq 0$, there is a morphism to the Brill-Noether locus:

$$\gamma : G(1, j, 2) \longrightarrow B(1, j, 2); \tag{16.8}$$

the fiber of γ over any line bundle L is isomorphic to the Grassmannian $Grass(2, h^0(L))$.

Then we have the following lemma

Lemma 16.3.2. *Let us fix any smooth projective irreducible Petri curve of genus $g \geq 2$. Then for every $j \geq 2g-1$ we have*

$$\mathcal{HD}(G(1,j,2)) = \frac{(1+u)^g(1+v)^g}{(1-uv)(1-(uv)^2)}\left[1-(uv)^{j-g}-(uv)^{j+1-g}+(uv)^{2j+1-2g}\right]. \qquad (16.9)$$

Moreover,

$$\mathcal{HD}(G(1,2g-2,2)) = \frac{1}{(1-uv)(1-(uv)^2)}\cdot$$

$$\cdot\left\{((1+u)^g(1+v)^g-1)[1-(uv)^{g-2}-(uv)^{g-1}+(uv)^{2g-3}]+1-(uv)^{g-1}-(uv)^g+(uv)^{2g-1}\right\}. \qquad (16.10)$$

Proof. First of all, let us prove (16.9), so let us assume that $j \geq 2g-1$. Since we are working over a curve of genus $g \geq 2$, then we get that $j \geq g+1$. Let us fix any line bundle L on C with degree j. Then by Riemann-Roch we have

$$h^0(C,L) \geq h^0(C,L) - h^1(C,L) = j+1-g \geq g+1+1-g = 2.$$

Therefore whenever $j \geq g+1$ we have that $B(1,j,2)$ coincides with the Jacobian J^jC.

Since $j \geq 2g-1$, then by Riemann-Roch and [Ha, IV, example 1.3.4], we get that for every line bundle L on C:

$$h^0(C,L) = h^0(C,L) - h^1(C,L) = j+1-g.$$

So if $j \geq 2g-1$, then the fiber of (16.8) over any point $L \in B(1,j,2) = J^jC$ is the grassmannian $Grass(2,j+1-g)$. Then we get that for each $j \geq 2g-1$ we have:

$$\mathcal{HD}(G(1,j,2)) = \mathcal{HD}(B(1,j,2))\mathcal{HD}(Grass(2,j+1-g)) =$$

$$= (1+u)^g(1+v)^g\frac{(1-(uv)^{j-g})(1-(uv)^{j+1-g})}{(1-uv)(1-(uv)^2)}.$$

So we get the first formula. Now let us prove also the second formula. If $g \geq 3$, then by Riemann-Roch we get that for every line bundle L of degree $2g-2$ we get

$$h^0(C,L) \geq h^0(C,L) - h^1(C,L) = 2g-2+1-g = g-1 \geq 2.$$

Therefore for every $g \geq 3$ we have $G(1,2g-2,2) = J^{2g-2}C$. In this case $h^0(C,L) = g$ if L is the canonical bundle, otherwise we have $h^0(C,L) = g-1$. So in this case the fiber of (16.8) is equal to the Grassmannian $Grass(2,g)$ if L is the canonical bundle and it is equal to $Grass(2,g-1)$ in the opposite case. Then for all $g \geq 3$ we have:

$$\mathcal{HD}(G(1,2g-2,2)) = \frac{1}{(1-uv)(1-(uv)^2)}\cdot$$

$$\cdot\left\{((1+u)^g(1+v)^g-1)[1-(uv)^{g-2}-(uv)^{g-1}+(uv)^{2g-3}]+1-(uv)^{g-1}-(uv)^g+(uv)^{2g-1}\right\}. \tag{16.11}$$

When the genus of C is $g=2$ the Brill-Noether locus $B(1,2g-2,2)=B(1,2,2)$ consists of a single point, namely the canonical bundle K on C. Moreover, the fiber of $G(1,2g-2,2)=G(1,2,2)$ over K consists of the Grassmannian $Grass(2,g)=Grass(2,2)=\mathrm{Spec}(\mathbb{C})$. Therefore for $g=2$ the moduli space $G(1,2g-2,2)$ consists of a single point. So its Hodge-Deligne polynomial is equal to 1.

Now if we substitute $g=2$ in (16.11) we get the polynomial 1. Therefore (16.11) is still valid for $g=2$, so we conclude. □

This result is immediately applicable in order to compute

$$\mathcal{HD}(G(\alpha(k);2,d,2))-\mathcal{HD}(G(\alpha(l);2,d,2))$$

for any pair of integers k,l such that

$$2g-2\leq k,\, l<\frac{d}{2}.$$

In particular, we can apply this result starting from the last moduli space and crossing every critical value from right to left (or starting from the first moduli space and crossing from left to right) whenever

$$\frac{g}{2}+1\geq 2g-2 \quad \Leftrightarrow g\leq 4.$$

So one would be able to get complete results for every $G(\alpha(k);2,d,2)$ for $g=2,3,4$, provided that the polynomials of the first moduli space or of the last one are known. We remark that differently from the cases when $k=1$, in the case under consideration the moduli space $G(\alpha;2,d,2)$ is always non-empty also when α is very large. Therefore it is not possible to compute the Hodge-Deligne polynomial of the last moduli space as we did in the previous chapters.

Bibliography

[BGMN] S. B. Bradlow, O. García-Prada, V. Muñoz, P. E. Newstead. *Coherent Systems and Brill-Noether Theory*, International Journal of Mathematics, Vol. 14, No. 7, pp. 683-733 (2003), arXiv: math.AG 0205317v1;

[BGMMN] S. B. Bradlow, O. García-Prada, V. Mercat, V. Munõz, P. E. Newstead. *On the Geometry of Moduli Spaces of Coherent Systems on Algebraic Curves*, International Journal of Mathematics, Vol. 18, No. 4, pp. 411-453 (2007), arXiv: math.AG 0407523v5;

[BO] S. B. Bradlow, O. García-Prada, *An application of coherent systems to a Brill-Noether problem*, J. Reine Angew. Math. No. 551, pp. 123-143 (2002);

[D1] P. Deligne, *Théorie de Hodge I*, Actes du Congrès international des Mathématiciens (Nice, 1970), Gauthier-Villars, 1, pp. 425–430 (1971);

[D2] P. Deligne, *Théorie de Hodge II*, Publications Mathématiques de l'IHÉS, 40, pp. 5-57 (1971);

[D3] P. Deligne, *Théorie de Hodge III*, Publications Mathématiques de l'IHÉS, 44, pp. 5-77 (1974);

[DK] V. I. Danilov, A. G.Khovanskiǐ, *Newton polyhedra and an algorithm for computing Hodge-Deligne numbers*, Math. USSR Izvestiya, 29, pp. 279-298 (1987);

[EK] R. Earl, F. Kirwan *The Hodge numbers of the moduli spaces of vector bundles over a Riemann surface*, Q. J. Math. 51, 465-483 (2000), arXiv: math.AG 0012260v1;

[Ha] R. Hartshorne, *Algebraic Geometry*, Springer (1977);

[He] M. He *Espaces de modules de systèmes cohérents*, International Journal of Mathematics, Vol. 9, pp. 545-598 (1998);

[GM] C. Gonzalez-Martinez, *A stratification of some moduli spaces of coherent systems on algebraic curves and their Hodge-Poincaré polynomials*, arXiv: math.AG 0809.0134v1;

[KN] A. D. King, P. E. Newstead *Moduli of Brill-Noether pairs on algebraic curves*, International Journal of Mathematics, Vol. 6, No. 5, pp. 733-748 (1995) arXiv: math.AG 9408005v1;

[L] H. Lange, *Universal Families of Extensions*, Journal of Algebra, 83, pp. 101-112 (1983);

[LP] J. Le Potier, *Faisceaux semi-stable et systémes cohérents*, in Vector Bundles in Algebraic Geometry, Durham 1993, eds. N.J. Hitchin, P. E. Newstead and W. M. Oxbury, LMS Lecture Notes Series 208, Cambridge University Press, pp. 179-239 (1995);

[LP2] J. Le Potier, *Lectures on vector bundles*, Cambridge Studies in Advanced Mathematics 54, Cambridge University Press, (1995);

[LN] H. Lange, P. E. Newstead, *Coherent Systems of genus 0 III: Computation of Flips for k=1*, International Journal of Mathematics, Vol. 19, No. 9 pp. 1103-1119 (2008), arXiv: math.AG 0707.1466v2;

[LN2] H. Lange, P. E. Newstead *Hodge polynomials and birational types of moduli spaces of coherent systems on elliptic curves*, Manuscripta Mathematica, 130, pp. 1-19 (2009) arXiv: math.AG 0810.1425v2;

[Mac] I. G. Macdonald, *The Poincaré polynomial of a symmetric product*, Proc. Camb. Phil. Soc. No 58, pp. 563-568 (1962);

[M] V. Muñoz, *Hodge polynomials of the moduli spaces of rank 3 pairs*, Geometriae dedicata, Vol. 136. pp. 17-46 (2008), arXiv: math.AG 0706.0593v1;

[MOVG] V. Muñoz, D. Ortega, M.J. Vásquez-Gallo, *Hodge polynomials of the moduli spaces of pairs*, arXiv: math.AG 0606676v2;

[MOVG2] V. Muñoz, D. Ortega, M.J. Vásquez-Gallo, *Hodge polynomials of the moduli spaces of triples of rank (2,2)*, arXiv: math.AG 0701642v1;

[N] P. E. Newstead, *Introduction to moduli problems and orbit spaces*, Lecture Notes, Tata Institute of fundamental research, Springer-Verlag (1978);

[T] M. Thaddeus, *Stable pairs, linear systems and the Verlinde formula* Inventiones Mathematicae, Vol. 117, No. 1, pp. 317-353 (1994), arXiv: math.AG 9210007v1.

Acknowledgements

First of all I want to thank my advisor Peter Newstead. I am deeply grateful for the opportunity to work with him, for suggesting this problem and for his guidance and teaching during the last year and a half. This thesis would not have been possible without his constant advices and support. I would also like to express my gratitude to both Ann and Peter Newstead for their great help both in Liverpool and in Cambridge, for several dinners together, for lots of conversations (not necessarily about math!) and for letting me feel at home in England.

I would also like to thank Professor Vicente Muñoz that was the referee for this Ph.D. thesis (I am sorry about the length!).

I am very grateful to SISSA, that awarded me with 2 fellowships both for the Master's degree and for the Ph.D. and that gave me the opportunity of studying and carrying out my research in such a beautiful environment. I would also like to thank both the Newton Institute in Cambridge and the University of Liverpool for their hospitality in April-June 2011 and January-March 2012 respectively. I was also financially supported by an Erasmus fellowship for 3 months, by the italian Indam-GNSAGA group and by the Organizing Committee of the VBAC Conference 2012 in Barcelona.

And now, let's turn to italian!

Esserci, esser qui, è già un grande risultato
Max Pezzali, Esserci - Time out

Riuscire a scrivere la fatidica parola "ringraziamenti" equivale a dire che questa tesi mostruosa[1] è ormai praticamente finita, e che eventuali errori (inevitabili!) salteranno fuori solo a distanza di tempo, ovvero per un po' posso tirare il fiato (almeno fintantoché non mi metterò a scrivere articoli, e allora avrà inizio un'operazione di taglia/incolla non da poco!).

Ovviamente non so da che parte cominciare i ringraziamenti, e viene la seria tentazione di scrivere semplicemente "ringrazio tutti quelli che vorrebbero essere ringraziati in questa tesi",

[1] sì, me lo dico da solo! :-)

ma penso che mi attirerei parecchi nomi, anche perché questa dovrebbe essere l'unica parte leggibile di questo lavoro infinito.

Quindi comincio ringraziando la mia famiglia: mia mamma Carla, mio papà Moreno le mie sorelle Giulia e Sofia, le nonne Lucia ed Iris, perché mi avete sostenuto nella mia decisione di trasferirmi a Trieste nonostante ci fossero ottimi motivi per rimanere a Crevalcore, per tutto l'aiuto che mi avete dato in questi 5 anni. Penso che anche i miei nonni Iginio e Franco mi avrebbero sostenuto, e forse l'hanno comunque fatto da dove sono ora. Un grazie anche ad Alessandro e a tutti i parenti, in particolare ai miei cugini Serena, Gabriele e Massimo, nonché a tutti gli zii. Un grazie enorme ad Enzo, Bernadette, Claudia, Elena, Luca, Anna, Marco, Rosa, Giovanni, per tutto.

Un grazie va a tutti gli amici di Crevalcore e di Bologna a cui la matematica ricorda solo compiti in classe per i quali il pomeriggio prima bisognava comporre un numero di telefono che qualcuno aveva anche imparato a memoria... In particolare, un grazie sincero a tutti voi che mi sopportate quando parto per la tangente (e te daje!) a parlare e straparlare di matematica; grazie ad Alberto Tugnoli, Anna Moretti, Carlo Alberto "Tino" Tinelli, Carlotta Lolli, Carlotta Montecchi, Gabriele Corti, Germana Miele, Guglielmo "Gus" Zarri, Lorenzo Cremonini, Matteo Barbieri, Peter Turrini e Viviana Simoni. Un grazie sincero anche ad Anna Ardizzoni, Chiara Casamenti, Damiano Montori, Emanuela Pergolizzi, Francesco "Kikko" Mattioli, Michele De Biase, Laura Marcis, Nicola "Clemmo" Clementel e Valentina Lodi.

Un grazie ai compagni della triennale: per voi potrei fare un "copia e incolla" dalla tesi della triennale, perché nonostante la distanza quasi tutto è rimasto lo stesso con voi, perché quando ritornavo al dm era come se ci fossimo salutati il giorno prima, perché eravate ancora lì in laboratorio di informatica a giocare (ed era inutile dire che stavate lavorando per la Piccolomini o per Casciola, tanto non ci credevo!), perché le gag sono sempre quelle e sempre diverse al tempo stesso. Grazie ad Agnese Savini, Alessandra Tosi, Cris Lippi, Eleonora Galassi, Federico Di Dio e Susi Ansaloni, Filippo Gasperini, Francesca Ferlin, Giulio Tralli, Ivan Tomba, Licia Fiorentini[2], Martina Zaganelli, Michela Tomasi e Yannick Biasca. Grazie anche ad Enrico Manfredi, Marcello Seri e Paolo Filipe Bertoldi. Grazie inoltre a chi è stato traviato dalla matematica o dai matematici, ovvero Luca Zambonelli, Marco Torri e Clio Dosi.

Un grazie speciale va ai miei primi coinquilini: la grandissima Sabrina Rivetti e Giovanni "Cresta" Faonte, con cui ho diviso per un anno e mezzo l'interno Faonte e l'area wi-fi 2 (perché il Cresta era sotto la doccia), nonché a Luca con cui ho diviso un appartamento purtroppo solo 6 mesi[3], e che continua imperterrito nel tentativo di convincermi a fare finanza, ogni volta essendo un po' più vicino alla meta (da bravo analista, sa che spesso le serie convergono, anche se i tempi di convergenza possono essere insospettabilmente alti!). Grazie a tutti e 3 perché

[2] a cui si potrebbero prendere in prestito le parole: "è finita, Matte!" (anche se la frase è stata più volte modificata nel corso degli anni)

[3] beh, diviso sì, in parti uguali ovviamente no! :-)

c'eravate sempre quando ho avuto bisogno.

Un grazie particolare va inoltre al mio secondo coinquilino, Alex Massarenti, e soprattutto ai miei favolosi coinquilini Fabio Tanturri, Flaviana[4] Iurlano e Alessandro "Genzianella" (detto "Nella") Gentile, con cui ho condiviso i miei ultimi 2 anni di vita. Sono stati anni intensi, e sono stato incredibilmente fortunato ad avervi come supporto[5] nell'affrontare i mille piccoli e grandi problemi quotidiani.

Un grazie speciale alla mia pazza compagna d'ufficio Alessandra Guazzi, per la quale Elisa ha coniato la bellissima espressione "guazzesco" (aggettivo) = "pazzesco ma assolutamente fattibile dalla Guazzi". Grazie di tutto, nonostante i tuoi mille casini!

Un grazie a tutti i "commerciali": Andrea "Kal Kalo Kalogero" Mondino, Antonio "Larry Bolide" Lerario e Sofia Cazzaniga, Daniela Tonon, Davide Barilari, Francesca De Marchis e Matteo Sbarigia, Roberto "Gobbo" Gobbetti, Nicola "Pagano Pagus Paguro" Pagani e Susha, Nino Passannanti e Leonora Ferraro, Stefano Romano: per le cene e i pranzi, per i film in compagnia, per i lupi.

Un grazie a Francesco Sala, che mi ha guidato attraverso i meandri della burocrazia del Lussemburgo (uno Stato così piccolo, così tanta burocrazia: la densità media di burocrazia deve essere infinitamente superiore a quella italiana), senza di lui sarei stato sommerso da una marea di documenti da compilare, tradurre, autenticare, scannerizzare, ristampare e firmare.

Devo inoltre ringraziare per questi 5 anni tutti gli altri "triestini", anche se in effetti di triestini doc ce ne sono ben pochi; ne conosco comunque almeno uno che vorrebbe la cittadinanza triestina, o meglio ancora il passaporto "padano", ma anche sanmarinese[6]. Il Cresta, sapendo ormai le lingue[7] direbbero di noi che siamo un melting pot, un crogiolo di accenti, idee e modi di essere spesso agli antipodi. Nonostante ciò, o forse proprio grazie a ciò, questi sono stati 5 anni bellissimi, faticosi a volte, ma magnifici. Per parecchi di voi mi sono divertito spesso a cucinare, e alcuni[8] tifano ancora affiché questa diventi la mia attività principale. Chissà che non succeda in futuro...per dirla con le parole del film "Stranger than fiction",

Harold Crick: "So, when did you decide to become a baker?"
Ana Pascal: "In college."
"Oh, like a cooking college?"
"I went to Harvard Law actually."
"Oh, I'm sorry, I just assumed it was..."

[4]Fla, Flu, Tachiflu, Fluviale, Pluviale, etc.. (sorry, non potevo non fare lo scemo!)

[5]rigorosamente nel senso non matematico del termine! :-)

[6]ma bisogna essere sposati da almeno 30 anni, quindi cambia idea Bobby!

[7]Cresta: "Mi serve dell'alcol per pulire la stanza. Come faccio?" Sabri: "Compra dell'alcol in farmacia." Cresta: "Come si dice alcol?"

[8]ad esempio Jamir, che non si dà ancora pace perché non ho organizzato mai una seconda cena bolognese

"No. No, it's fine. I didn't finish."
"Did something happen?"
"No. I was barely accepted. I mean, really barely. The only reason they let me come was because of my essay "How I was gonna make the world a better place with my degree." Anyway, we would have to participate in these study sessions...my classmates and I, sometimes all night long. And so I'd bake so no one would go hungry while we worked. Sometimes I would bake all afternoon in the kitchen in the dorm and then I'd bring my little treats to the study groups and people loved them. I made oatmeal cookies, peanut-butter bars...dark-chocolate, macadamia-nut wedges. And everyone would eat and stay happy...and study harder and do better on the tests. More and more people started coming to the study groups and I'd bring more snacks. I was always looking for better and better recipes, until soon it was ricotta cheese and apricot croissants and mocha bars with an almond glaze and lemon chiffon cake with zesty peach icing. And at the end of the semester I had 27 study partners, eight Mead journals filled with recipes and a D average. So I dropped out. I just figured that if I was gonna make the world a better place, I would do it with cookies."

Non posso quindi non citare ognuno dei triestini acquisiti con cui ho condiviso parte delle mie gioie e dolori, con cui ho discusso di matematica, di politica, di informatica, a cui ho installato Ubuntu[9], a tutti quelli che ormai mi hanno etichettato sotto la categoria "spammer" di facebook, insomma a tutti voi. Spero di non dimenticare nessun nome citando Ada Boralevi, Alberto Boscaggin, Alessandro Zucca, Alessio "Ans" Ansuini, Alessio Lo Giudice, Andrea Pustetto, Aris Marcolongo, Aurora Meroni, Celeste Damiani, Cristiano Guida, Dario Prandi, Diletta Dalzovo, Domenico Monaco, Dragana Vuckovic, Duvan Franco, Elisa Paoli, Emanuele Parodi, Erika Ercolini, Flavia Poma, Francesca Arici, Francesca Mancini, Francesco Solombrino, Francisca Valentina Kesten, Giangiacomo Sanna, Giovanni Ge, Ina Bisha, Jamir "Mufasa" Marino, Laura "la Zucca" Carlini, Lei Yu, Luca Rizzi, Marco Beria, Marco Bonacini, Marco Morandotti, Mattia Pedrini, Maurizio Garrione, Mauro Bardelloni, Olga Puccioni, Pavel Silveira Diaz, Pietro Giavedoni, Pietro Tortella, Riccardo Lena, Riccardo Montalto, Riccardo Scala, Roberta Lanciani, Roberta Lubiana, Serena Dipierro, Simone Racca, Soehila Palma, Stefano Maggiolo, il mio duale Tommaso Matteini, Virginia Agostiniani. Vorrei scrivere qualcosa per ognuno di voi, ma se lo facessi rischierei di scrivere un'altra intera tesi...:-)

E' stato un onore e un piacere conoscere ogni "triestino". Sarà incredibilmente arduo partire da qui e salutare ognuno di voi.

C'è bisogno di aguzzare la vista, per capire quali sono gli amici
Bisognerebbe restare svegli per scoprire tutti i nemici
Ci vorrebbe un paio di scarpe nuove per partire, per scappare lontano
E poi seguire una traccia sbagliata
Perdersi meglio e non tornare più indietro

[9]a volte dopo avervi stressato per anni sui meriti di Linux e i demeriti di Apple e Microsoft, direi che molti di voi si sono convinti per sfinimento! :-)

Non c'è bisogno di una foto ingiallita per vedere quanto siamo cambiati
Non c'è bisogno...no!
Bisognerebbe fermarsi in tempo, non aver fretta ma rallentare
Bisognerebbe solo ascoltare o ancora meglio, cambiare canale
C'è bisogno di stare attenti nell'osservare la nostra storia
Guardarsi indietro e poi capire che c'è bisogno di più memoria
Si, c'è bisogno!
Ci vorrebbe una muta di corde nuove per suonare sempre scordati
C'è bisogno di nuove canzoni con parole per sognare più forte
Bisognerebbe fare sogni grandiosi, oltre la noia e le nevrosi
Avere cura, avere pazienza, di tutta quanta l'intelligenza...
Si, c'è bisogno!
Modena City Ramblers - Ramblers Blues

Un grazie particolare va ad Emilia Mezzetti, che non ringrazierò mai abbastanza per avermi seguito durante la tesi della specialistica, e a Ugo Bruzzo per avermi aiutato più e più volte con i mille casini del mio dottorato, nonché per alcune ottime cene in giro per il mondo. Ad entrambi devo anche un ringraziamento speciale per le mille lettere di presentazione che hanno scritto per me, e domando venia in anticipo per le altre mille che probabilmente vi chiederò a breve!

Un grazie anche a tutti gli anconetani doc e acquisti, con la certezza che mi sarò scordato qualcuno: Alberto Bianchi, Amy, Elena Torta, Francesco "Fiore" Fioranelli, Francesco Salvucci, Gianluca Sciamanna, Graziano De Capua, Laura Sorbi, Matteo "Maaaateo" Belardinelli, Matteo Canzari, Paolo Raspa, Richard Drozdowski, Tanja Albanesi: grazie per le gite, ma di sicuro non per i bagarozzi :-)

Un grazie ad Emilio Franco Gomez, con cui ho condiviso 3 mesi a Cambridge, che mi voleva rapire e portare in Spagna per cucinare per lui, per le splendide conversazioni in uno strano misto di inglese, italiano e spagnolo, perché anche quando si convince che la politica italiana è assurda, c'è sempre qualcosa di ancora più assurdo che non vedo l'ora di raccontargli. Perché a differenza di noi italiani non è assuefatto nel bene o nel male ai vecchi e nuovi protagonisti di questo schifo di politica.

E infine, il grazie più grande di tutti va ad Elisa, senza la quale non avrei mai superato i mille ostacoli della matematica, della burocrazia, della vita. Non sarei mai arrivato fin qui oggi se non fosse stato per lei.

I was just guessing at numbers and figures
Pulling your puzzles apart;
Questions of science, science and progress
Do not speak as loud as my heart.
Coldplay, "The scientist"