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FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI
Corso di Laurea Specialistica in Matematica

ORBIFOLD E GRUPPOIDI

Tesi di Laurea in Geometria Algebrica

Relatore:
Chiar.ma Prof.ssa
BARBARA FANTECHI

Co-relatore:
Chiar.ma Prof.ssa
EMILIA MEZZETTI

Presentata da:
MATTEO TOMMASINI

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Per Elisa...

Daniel Faraday: [...]*Non aveva più un'ancora.*
Desmond Hume: *Cosa vuoi dire con "ancora"?*
Daniel: *Qualcosa di familiare in entrambi i periodi. Tutte queste..le vedi? Tutte queste sono variabili. Sono casuali, caotiche. Ogni equazione ha bisogno di stabilità, di qualcosa di noto: si chiama costante. Desmond: tu non hai alcuna costante. Quando vai nel futuro, niente là ti è familiare. Quindi se vuoi fermare tutto questo, allora devi trovare qualcosa laggiù, qualcosa a cui tieni davvero, davvero tanto...che esista anche qui, nel 1996.*
Desmond: *Questa costante..può anche essere una persona?*
Daniel: *Sì, forse. Ma devi creare una sorta di contatto. Non hai detto che eri su una nave nel bel mezzo del nulla? Chi stai chiamando?*
Desmond: *La mia costante.*

(Lost-puntata 4x05)

[Per il mio nome i miei genitori] si accordarono su Smillaaaraq, che per l'usura a cui il tempo sottopone tutti noi fu abbreviato in Smilla. Che è solo un suono. Se vai oltre il suono, trovi il corpo con la sua circolazione, il suo movimento di liquidi. Il suo amore per il ghiaccio, la sua ira, il suo struggimento, la sua conoscenza dello spazio, le sue debolezze, infedeltà e lealtà.

Dietro questi sentimenti sorgono e svaniscono le forze indefinite, immagini staccate e sconnesse della memoria, suoni senza nome. E la geometria.
In fondo a noi c'è la geometria. I miei professori all'università continuavano a chiedere qual è la realtà dei concetti geometrici. Dove esistono, chiedevano, un cerchio perfetto, una vera simmetria, un parallelismo assoluto, se non possono essere costruiti in questo mondo imperfetto?

Io non gli rispondevo perché non avrebbero compreso l'ovvietà della risposta e le sue incalcolabili conseguenze. La geometria esiste come fenomeno innato nella nostra coscienza. Nel mondo esterno non esisterà mai un cristallo di neve dalla forma perfetta. Ma nella nostra coscienza c'è l'idea scintillante e impeccabile del ghiaccio perfetto.

(*da "Il senso di Smilla per la neve" di Peter Høeg*)

Contents

> “La matematica non è una marcia per un’autostrada ben tenuta, ma piuttosto un viaggio per uno strano territorio, dove gli esploratori spesso si perdono. Il rigore dovrebbe essere un segnale per lo storico che le mappe sono state tracciate, e i veri esploratori si sono spostati altrove.”
>
> William Sherron Anglin
> "Mathematics and History"

Introduzione

Scopo di questa tesi è costruire le basi per una teoria degli orbifold ridotti complessi come 2-categoria, colmando parzialmente alcune lacune presenti nella letteratura attuale e cercando di uniformizzare le notazioni e le ipotesi usate per analizzare tali oggetti in contesti diversi.

Il concetto di orbifold è stato per la prima volta formalizzato da Ichiro Satake nel 1956 con il nome di V-manifold nell’articolo “On a generalization of the notion of manifold”, sebbene con alcune ipotesi leggermente diverse da quelle che useremo nella tesi, mentre il nome “orbifold” è stato per la prima volta introdotto da William Thurston nelle sue note di geometria del 1979 intitolate “The geometry and topology of 3-manifolds”. Tuttavia l’idea di orbifold, sebbene non formalizzata, risale almeno all’articolo “Thèorie des groupes fuchsiens” pubblicato da Henri Poincaré nel 1882.

Il nome “orbifold” deriva dalla contrazione delle parole “orbit” e “manifold” in quanto i primi esempi noti di orbifold in letteratura derivano dal considerare lo spazio delle orbite ottenute dall’azione di un gruppo finito di automorfismi su una varietà (manifold) liscia o olomorfa. In generale l’insieme risultante con la topologia quoziente non ha più una struttura di manifold, ma una abbastanza simile. Storicamente la teoria degli orbifold è nata per descrivere oggetti di questo tipo, che erano studiati come “varietà (algebriche o analitiche) con singolarità quoziente finite” ben prima della formalizzazione dovuta a Satake e Thurston.

Ricordiamo che un manifold complesso può essere descritto come uno spazio topologico X (a base numerabile e Hausdorff) dotato di una classe di atlanti "equivalenti" dove ogni atlante è una collezione di carte $\{(\widetilde{U}_i, \phi_i)\}_{i\in I}$ tali che gli $\widetilde{U}_i$ siano aperti di $\mathbb{C}^n$ e ogni mappa ϕ_i sia un omeomorfismo da $\widetilde{U}_i$ sopra un aperto U_i di X, in maniera tale che la famiglia $\{U_i\}_{i\in I}$ sia un ricoprimento di X e sia verificata una condizione di compatibilità sulle intersezioni di due qualunque aperti U_i e U_j in X.

Analogamente, un orbifold complesso si può descrivere come uno spazio topologico X dotato di una classe di atlanti "equivalenti", dove ogni atlante è una collezione di "carte" (che in questo caso sono chiamate "sistemi uniformizzanti") della forma $(\widetilde{U}_i, G_i, \pi_i)$ dove ogni $\widetilde{U}_i$ è un aperto di $\mathbb{C}^n$, G_i è un gruppo finito che agisce su $\widetilde{U}_i$ come gruppo di automorfismi complessi e $\pi_i : \widetilde{U}_i \longrightarrow \pi_i(\widetilde{U}_i) =: U_i$ è una mappa continua che induce un omeomorfismo tra $\widetilde{U}_i/G_i$ e U_i, in maniera tale che la famiglia degli U_i sia un ricoprimento aperto di X e sia soddisfatta una condizione di compatibilità locale simile a quella descritta nei manifold. Per la definizione precisa, si veda il capitolo 2.

Detta in altri termini, i manifold complessi sono modellati localmente da aperti di $\mathbb{C}^n$, mentre gli orbifold complessi sono modellati localmente da aperti di tale forma, modulo un'azione di gruppo finito. In letteratura è ben descritta la nozione di atlante, mentre la nozione di equivalenza di atlanti per orbifold è soltanto accennata senza verificare che sia effettivamente una relazione di equivalenza, quindi questo è stato uno degli argomenti su cui si è focalizzata la nostra attenzione.

Come nel caso dei manifold, anche per gli orbifold esiste una nozione di "mappa" tra orbifold. Nel caso dei manifold questa è definita come una mappa continua tra i corrispondenti spazi topologici, che può essere localmente "sollevata" ad una mappa olomorfa tra opportune carte in dominio e codominio; dato che le mappe ϕ_i (usate nella definizione degli atlanti) sono omeomorfismi, quindi in particolare biettive, è chiaro che ogni mappa continua può ammettere al più un sollevamento come mappa tra manifold.

Per gli orbifold si trova in letteratura (ad esempio, [Pe]) un'analoga definizione ("sistemi compatibili"), ma a causa dell'azione dei gruppi sulle carte in dominio e codominio, la definizione risulta molto più complicata. Inoltre, a differenza del caso dei manifold, per gli orbifold non è più assicurata l'unicità del sollevamento per una generica mappa continua tra gli spazi topologici soggiacenti.

Oltre alle definizioni di oggetti (gli atlanti) e di morfismi (i sistemi compatibili), nel contesto degli orbifold si può dare anche la definizione di "trasformazione naturale" tra morfismi. La definizione che ho usato in questa tesi è quella di "trasformazione naturale" ottenuta come piccola modifica di una simile definizione descritta nella tesi di dottorato di Fabio Perroni ("Orbifold Cohomology of ADE-singularities").

Il nome transformazione naturale non è casuale, ma deriva dal contesto della teoria delle 2-categorie. Questa teoria (che sarà richiamata brevemente nel primo capitolo) si è dimostrata utile in molteplici contesti matematici, quindi pare naturale chiedersi se si possano definire opportune operazioni di "composizione" tra mappe tra orbifold e tra trasformazioni naturali, in maniera tale da soddisfare gli assiomi di una 2-categoria. Ciò che effettivamente ho trovato è che si può definire una 2-categoria, denotata con (**Pre-Orb**) dove gli oggetti sono gli atlanti, i morfismi i sistemi compatibili e i 2-morfismi sono le trasformazioni naturali tra sistemi compatibili. A tale 2-categoria non è stato dato il nome di (**Orb**) come sarebbe ragionevole aspettarsi, in quanto quest'ultima dovrebbe avere come oggetti le classi di equivalenza di atlanti. In tal caso sarebbe però necessario anche ridefinire in maniera compatibile i morfismi e i 2-morfismi; nell'ultimo capitolo è esposta un'idea di come ciò potrebbe essere realizzato, insieme ai problemi ancora aperti al riguardo.

Quanto descritto finora è il tipico approccio della geometria differenziale al concetto di orbifold, in cui gli oggetti hanno un'interpretazione geometrica abbastanza semplice, mentre i morfismi e i 2-morfismi risultano difficili da definire. Una visione alternativa di questi oggetti è stata introdotta da Ieke Moerdijk nell'articolo "Orbifolds as groupoids: an introduction" del 2002, usando anche la tesi di dottorato di Dorette Pronk "Groupoid Representation for Sheaves on Orbifolds" del 1997. In questi lavori si dimostra come è naturale pensare agli orbifolds nei termini di gruppoidi, una nozione che descriveremo in dettaglio nel terzo capitolo.

In particolare, i lavori di Ieke Moerdijk e di Dorette Pronk mostrano come sia possibile associare ad ogni atlante un gruppoide sopra la categoria (**Manifolds**) dei manifold complessi con alcune proprietà aggiuntive. Tali oggetti saranno descritti e analizzati approfonditamente nel terzo capitolo, prima nel caso di una categoria qualunque, poi nel caso della categoria (**Manifolds**). Anche in questo contesto sono note in letteratura le definizioni di morfismo tra gruppoidi e di trasformazione naturale tra morfismi; in questo caso si riesce a descrivere una struttura di 2-categoria, che nella tesi è denotata con (**Grp**).

Apparentemente la descrizione dei gruppoidi come oggetti risulta essere meno intuitiva di quella degli atlanti di un orbifold, ma in compenso è molto più semplice definire e lavorare con i morfismi e i 2-morfismi. Di conseguenza in letteratura esistono due approcci complementari agli orbifold: il primo è quello della geometria differenziale, in cui in generale si ignorano i 2-morfismi in quanto eccessivamente difficili da maneggiare, e si lavora solo con la categoria costituita da atlanti e morfismi. Il secondo è quello che, assumendo i lavori di Moerdijk e Pronk, associa ad ogni atlante un gruppoide e poi si limita a lavorare in questa 2-categoria, ignorando le definizioni di morfismi e di 2-morfismi date nel contesto della geometria differenziale.

Lo scopo principale di questa tesi è dunque quello di mostrare che i due modi di procedere sono "compatibili" anche per quanto riguarda i morfismi e i 2-morfismi, Per essere più precisi, il quarto capitolo mostrerà come costruire un 2-funtore (analogo del concetto di funtore nel contesto delle 2-categorie) da (**Pre-Orb**) in (**Grp**). In altri termini, si descriverà come associare ad ogni sistema compatibile tra atlanti un morfismo di gruppoidi e come assegnare ad ogni tranformazione naturale in (**Pre-Orb**) una trasformazione naturale in (**Grp**), in maniera tale da conservare le composizioni e le identità. Questo permette di rendere coerenti i due approcci agli orbifold appena descritti.

In questa trattazione rimangono aperti due problemi: il primo è verificare se è possibile descrivere una relazione di equivalenza anche sui morfismi e sui 2-morfismi di (**Pre-Orb**) in maniera tale da poter definire una nuova 2-categoria (**Orb**); il secondo, strettamente collegato al primo, consiste nel verificare se, una volta costruita (**Orb**), sia ancora possibile descrivere un 2-funtore "indotto" dal precedente da tale 2-categoria in un'altra "simile" a (**Grp**), cioè ottenuta da essa tramite una qualche relazione di equivalenza sugli oggetti, sui morfismi e sui 2-morfismi. Se questo fosse effettivamente possibile, sembrerebbe ragionevole aspettarsi che la relazione di equivalenza sugli oggetti sia quella nota in letteratura con il nome di "Morita equivalence", e questo risulterebbe compatibile con la definizione di orbifold come "classe di equivalenza di gruppoidi mediante Morita equivalence", come descritto ad esempio in alcuni lavori di Moerdijk ([M]).

Questa tesi è organizzata nel modo seguente:

Capitolo 1: Categorie e 2-categorie Lo scopo di questo capitolo è fornire le basi per lo studio delle categorie e delle 2-categorie. La trattazione di questa parte segue strettamente il libro "Handbook of categorical algebra 1 - Basic Category Theory" di Francis Borceux. In particolare, la prima parte è dedicata alla definizione di categoria e funtore, corredata da esempi, seguita dalla definizione di trasformazione naturale nel contesto delle categorie (**Cat**) e (**Manifolds**). Gli esempi descritti in questa sezione motivano l'introduzione dei concetti di 2-categoria e di 2-funtore, i cui assiomi sono enunciati nella terza parte. La quarta e ultima sezione riguarda la definizione di prodotto fibrato in una categoria arbitraria e la descrizione della sua costruzione esplicita nelle categorie (**Sets**) e (**HOM**). Quest'ultima definizione sarà utilizzata nel capitolo 3 per definire i gruppoidi su una categoria qualunque.

Capitolo 2: La 2-categoria degli orbifold complessi Nella prima sezione viene descritto il concetto di "sistema uniformizzante", ovvero l'analogo nel contesto degli orbifold del concetto di "carta" per un manifold; viene data anche la definizione di "embedding" tra due sistemi uniformizzanti, di gruppo stabilizzante e di atlante, usando i risultati classici sull'argomento (esposti, ad esempio, in [Pe]). Nella seconda e nella terza parte si danno le definizioni di morfismi (sistemi compatibili) e 2-morfismi (trasformazioni naturali) per orbifold; inoltre si descrive come definire le composizioni di morfismi e di 2-morfismi. Nella quarta sezione si verifica che è possibile definire la 2-categoria (**Pre-Orb**); infine nell'ultima sezione si introduce la relazione di "equivalenza" tra atlanti e si verifica che essa è effettivamente una relazione di equivalenza sull'insieme degli atlanti per uno spazio X fissato. Nel corso di questo capitolo si mostra anche come gli orbifold costituiscano una naturale generalizzazione della categoria costituita dai manifold complessi e dalle mappe olomorfe tra di essi.

Capitolo 3: La 2-categoria dei gruppoidi interni in una categoria $\mathscr{C}$ Nelle prime 3 sezioni si descrivono gli oggetti, i morfismi e i 2-morfismi di $\mathscr{C}$-(**Groupoids**), che si verifica essere una 2-categoria nella quarta parte. Tutto questo è costruito su una qualunque categoria $\mathscr{C}$ che ammetta prodotti fibrati. La quinta sezione descrive la 2-categoria (**Grp**) dei gruppoidi interni in (**Manifolds**) con mappe source e target étale e diagonale relativa propria. Per questa parte, si veda ad esempio [M].

Capitolo 4: Dagli orbifold ai gruppoidi Le prime 3 sezioni descrivono come associare ad ogni oggetto, morfismo e 2-morfismo in (**Pre-Orb**) un corrispondente oggetto, morfismo e 2-morfismo in (**Grp**). La quarta e ultima sezione dimostra che questa costruzione soddisfa gli assiomi di 2-funtore. La prima sezione si basa essenzialmente sul lavoro di Dorette Pronk ([Pr]), mentre le ultime tre sono originali.

Capitolo 5: Problemi aperti Si discute la possibilità di definire sulla 2-categoria (**Pre-Orb**) una relazione di equivalenza sui morfismi e sui 2-morfismi, compatibile con quella sugli oggetti definita nel capitolo 2, in modo tale da ottenere ancora una 2-categoria. Si tratta inoltre la possibilità di indurre un 2-funtore avente tale nuova 2-categoria come dominio e si analizza la possibilità che il codominio di tale 2-funtore sia ottenuto a partire da (**Grp**) usando la relazione di equivalenza di Morita (descritta, ad esempio, in [M]). Viene infine suggerito che sia possibile provare che il 2-funtore descritto in precedenza (o il funtore indotto da questo dopo essere passati alle opportune relazioni di equivalenza in dominio e codominio) sia essenzialmente suriettivo e rappresenti un'equivalenza di categorie se ristretto alle categorie costituite da morfismi e 2-morfismi.

> “A un matematico che fa manipolazioni formali capita spesso di avere la sensazione sconfortante che la propria matita lo sorpassi in intelligenza.”
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>
> Howard W. Eves
> “Mathematical Circles”

Chapter 1

Categories and 2-categories

1.1 Categories and functors

Let us start with some basic definitions about categories and functors, taken almost under verbatim from [B]. Although we are mainly interested in 2-categories, I think it is useful to put here also these definitions, in order to compare them with the others.

Definition 1.1. A *category* $\mathscr{C}$ consists of the following data:

(1) a *class* $\mathscr{C}_0$ whose elements are called “objects of the category”;

(2) for every pair A and B of objects, a *set* $\mathrm{Hom}_{\mathscr{C}}(A,B)=\mathscr{C}(A,B)$, whose elements are called “morphisms” or “arrows” from A to B;

(3) for every triple A, B, C of objects, a “composition” law:

$$\mathscr{C}(A,B)\times\mathscr{C}(B,C)\to\mathscr{C}(A,C);$$

the composite of a pair (f,g) will be usually denoted as $g\circ f$ or gf;

(4) for every object A, a morphism $1_A\in\mathscr{C}(A,A)$, called the “identity” on A.

These data must satisfy the following axioms:

(i) (associativity) given any triple $f \in \mathscr{C}(A, B)$, $g \in \mathscr{C}(B, C)$, $h \in \mathscr{C}(C, D)$, the following identity holds in the set $\mathscr{C}(A, D)$:

$$h \circ (g \circ f) = (h \circ g) \circ f;$$

(ii) (identity) for all $f \in \mathscr{C}(A, B)$ and $g \in \mathscr{C}(B, C)$, we have:

$$1_B \circ f = f \quad \text{and} \quad g \circ 1_B = g.$$

We will say that $\mathscr{C}$ is a *small* category if $\mathscr{C}_0$ is a set.

A morphism $f \in \mathscr{C}(A, B)$ will be often represented by the notation $f : A \to B$ and A and B will called "source" and "target" of f. We remark that *in general this does not mean that f is a function from A to B*, as we will see in some of the following examples.

Whenever we write diagrams of this form:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\curvearrowright"] & B \arrow[d, "g"] \\
D \arrow[r, "h"'] & C
\end{tikzcd}$$

we will mean that A, B, C, D are objects and f, g, h, k are morphisms in $\mathscr{C}$ such that $g \circ f = h \circ k$; in this case we will also say that such a diagram is commutative or commutes.

Definition 1.2. A morphism $f : X \to Y$ in a category $\mathscr{A}$ is called an *isomorphism* iff there exists another morphism g *in the same category* such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. In this case the objects X and Y are called *isomorphic.*

Example 1.1. Let us see some basic examples of categories:

- sets and set maps: we refer to this category as (**Sets**);
- topological spaces and continuos maps: (**Top**);
- complex manifolds and holomorphic maps between them: (**Manifolds**);
- groups and group homomorphisms: (**Gr**);
- in the same way we can describe the category of $\mathbb{K}-$vector spaces (**Vect**$_{\mathbb{K}}$), of commutative rings (**Rng**), and so on.

In this list of examples we have always used sets (with additional properties) as objects and set maps (with additional properties) as morphisms of the category. These categories are called *concrete categories*. However, this is not the general case, as one can see from the following examples:

- any group $(G, \cdot)$ can be considered as a category in the following way: the space of objects contains just one element p and any arrow from p to itself corresponds to an element $g \in G$. The composition of arrows is just given by the multiplication in G and the identity corresponds to the identity of the group G. Note that in this case we have obtained a category where all morphisms are isomorphisms;
- a partially ordered set $(X, \leq)$ gives rise to a category $\mathscr{C}$ whose objects are elements of X and where we say that for any pair of objects x and y:

$$\mathscr{C}(x,y) := \begin{cases} \begin{array}{c} \text{a set with a single element,} \\ \text{denoted by } x \overset{\leq}{\to} y \end{array} & \text{if } x \leq y \\ \varnothing & \text{otherwise.} \end{cases}$$

  Note that in this case the existence of the identity and the possibility to compose are a direct consequence of the reflexivity and transitivity of $\leq$;

- let us fix a category $\mathscr{C}$ and an object I in it. Then we define the category $\mathscr{C}/I$ ("$\mathscr{C}$ over I") as follows:
  - objects: all morphisms of $\mathscr{C}$ with target I;
  - morphisms: a morphisms from the object $(f : A \to I)$ to the object $(g : B \to I)$ is any morphism $h : A \to B$ such that $g \circ h = f$, i.e:

$$\begin{tikzcd} A \arrow[d, "f"', ""{name=F}] & B \arrow[d, "g", ""'{name=G}] \\ I & I \arrow[from=F, to=G, "h"] \end{tikzcd} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & \curvearrowright & B \arrow[dl, "g"] \\ & I & \end{tikzcd}$$

  where the composition and the identities are obviously defined using the axioms of $\mathscr{C}$;

- in the same way we define the category "I over $\mathscr{C}$" $(I/\mathscr{C})$:
  - objects: all morphisms of $\mathscr{C}$ with source I
  - morphisms: a morphisms from the object $(f : I \to A)$ to the object $(g : I \to B)$ is any morphism $h : A \to B$ such that $h \circ f = g$, i.e:

$$\begin{tikzcd} I \arrow[d, "f"', ""{name=F}] & I \arrow[d, "g", ""'{name=G}] \\ A & B \arrow[from=F, to=G, "h"] \end{tikzcd} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \begin{tikzcd} & I \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A \arrow[rr, "h"'] & \curvearrowright & B; \end{tikzcd}$$

The last two are special cases of a general construction, called *comma category* (see, for example, [B], §1.6),

- for any category $\mathscr{C}$ we can define its *opposite category* $\mathscr{C}^{\mathrm{op}}$: the objects are the same of $\mathscr{C}$ and for any pair of objects A, B we define $\mathscr{C}^{\mathrm{op}}(A, B) := \mathscr{C}(B, A)$. The identities of $\mathscr{C}^{\mathrm{op}}$ are the same of $\mathscr{A}$; for any pair of morphisms $f \in \mathscr{C}^{\mathrm{op}}(A, B)$ and $g \in \mathscr{C}^{\mathrm{op}}(B, C)$ we define the composition of them as $f \circ g$. In this way the axioms of a category are easily satisfied. In other words, if we consider the morphisms of $\mathscr{C}$ as arrows, those of $\mathscr{C}^{\mathrm{op}}$ are just the same arrows of $\mathscr{C}$, but formally reversed.

Definition 1.3. An object $\bullet$ of a category $\mathscr{C}$ is called *terminal* (or *final*) if for any other object C of the category there exists exactly one morphism $f : C \to \bullet$.

Using this property, one can easily prove that the terminal object of $\mathscr{C}$, if it exists, is unique up to isomorphisms. For example, in (**Sets**) a final object is any of the sets with exactly one element; two of them are obviously in bijection, i.e. they are isomorphic in (**Sets**).

Definition 1.4. The *product of two categories* $\mathscr{A}$ and $\mathscr{B}$ is the category $\mathscr{A} \times \mathscr{B}$ defined as follows:

(1) the objects are the pairs (A, B) with $A \in \mathscr{A}_0$, $B \in \mathscr{B}_0$;

(2) the morphisms with source (A, B) and target (A', B') are the pairs (f, g) where $f \in \mathscr{A}(A, A')$ and $g \in \mathscr{B}(B, B')$;

(3) the composition is made ”component by component“, in other words:

$$(f', g') \circ (f, g) := (f' \circ f, g' \circ g);$$

(4) for every object $(A, B) \in (\mathscr{A} \times \mathscr{B})_0$ we set:

$$1_{(A,B)} = (1_A, 1_B).$$

Clearly the product of two small categories is again a small category.

Definition 1.5. A *(covariant) functor* F from a category $\mathscr{A}$ to a category $\mathscr{B}$ consists of the following data:

(1) a map:

$$\mathscr{A}_0 \to \mathscr{B}_0$$

between the classes of objects; the image of any $A \in \mathscr{A}_0$ is usually denoted by $F(A)$;

(2) for every pair of objects A, A' of $\mathscr{A}$ a set map:

$$\mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'));$$

the image of any $f \in \mathscr{A}(A, A')$ will be always denoted by $F(f)$.

In order to have a functor, we require that the following axioms are satisfied:

(i) for every pair of morphisms $f \in \mathscr{A}(A, A')$ and $g \in \mathscr{A}(A', A'')$

$$F(g \circ f) = F(g) \circ F(f);$$

(ii) for every object A of $\mathscr{A}$

$$F(1_A) = 1_{F(A)}.$$

Note that here $\circ$ denotes both the composition in $\mathscr{A}$ and the composition in $\mathscr{B}$. In general, it will be clear from the context in which category we are working; the same holds also for unities on $\mathscr{A}$ and $\mathscr{B}$.

Definition 1.6. A functor $F : \mathscr{A} \to \mathscr{B}$ is *essentially surjective* if for every object $b \in \mathscr{B}_0$ there exists a (not necessarily unique) object $a \in \mathscr{A}_0$ such that b is isomorpic to $F(a)$.

F is called *full* if for every pair of objects A, A' of $\mathscr{A}$ we have that:

$$F(\mathscr{A}(A, A')) = \mathscr{B}(F(A), F(A'))$$

i.e. if for every morphism g from $F(A)$ to $F(A')$ in $\mathscr{B}$ there exists a (not necessarily unique) morphism f from A to A' in $\mathscr{A}$ such that $F(f) = g$.

F is called *faithful* if for every pair of objects A, B in $\mathscr{A}$ the map:

$$F : \mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'))$$

is injective.

A functor F is called *fully faithful* if it is full and faithful.

A functor which is fully faithful and essentially surjective is called an *equivalence of categories* . A list of equivalent conditions for a functor F to be an equivalence of categories can be found in [B], proposition 3.4.3

Given two functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$, a pointwise composition immediately produces a new functor $G \circ F : \mathscr{A} \to \mathscr{C}$, called *composition* of F and G Then it easy to show that *small* categories and functors between them constitute a new category, denoted with (**Cat**). This is no longer true if we don't restrict to small categories.

Remark 1.1. Until now we have used only the so called *covariant functors*, namely functors which preserve composition. Sometimes one can also find examples of *contravariant functors* $F : \mathscr{A} \to \mathscr{B}$, that still preserve identities, but which reverse the order of compositions, i.e. such that:

$$F(f \circ g) = F(g) \circ F(f)$$

for every pair of composable morphisms f and g in $\mathscr{A}$. To any functor of this form we can associate a covariant functor $F^{\mathbf{op}} : \mathscr{A}^{\mathbf{op}} \to \mathscr{B}$ defined in this way:

- on the level of objects it coincides with F;

- for every morphism $f : A \to B$ in $\mathscr{C}^{\mathbf{op}}$, (i.e. $f : B \to A$ in $\mathscr{C}$) we define:
$$F^{\mathbf{op}}(A \xrightarrow{f} B) := F(B \xrightarrow{f} A).$$

In this way $F^{\mathbf{op}}$ is a covariant functor. For this reason, we will always deal only with covariant functors $F : \mathscr{A} \to \mathscr{B}$ by substituting $\mathscr{A}$ with its opposite category if necessary. Note that the previous construction is invertible and allows us also to associate to every covariant functor a contravariant one, if necessary.

Example 1.2. A very useful functor is the *forgetful* functor. Consider any of the previous categories where the objects (and the morphisms) are sets (and set maps) with additional properties (i.e. a concrete category), for example let us consider the category (**Top**) of topological spaces and continuos maps between them, Then we can define the functor $F : (\mathbf{Top}) \to (\mathbf{Sets})$ as follows:

- for any topological space $X \in (\mathbf{Top})_0$ we define $F(X) := X$, considered just as a set;
- for any continuos map $f : X \to Y$ between topological spaces, we define $F(f) := f$ considered as a set map from X to Y.

Clearly F preserves composition and identities, hence it is a functor from (**Top**) to (**Sets**); it is called “forgetful” functor because it forgets the topological properties of the sets and sets maps it is applied to. The same construction holds whenever we have 2 categories and the first one is constructed from the second one by requiring some additional properties on the level of objects and/or on the level of morphisms.

Example 1.3. In category theory it is often used the notion of *representable* functor: let us fix a category $\mathscr{C}$ and an object C in it. Then we can define the functor:

$$\mathscr{C}(C, -) : \mathscr{C} \to (\mathbf{Sets})$$

as follows: for any object $A \in \mathscr{C}_0$, we set:

$$\mathscr{C}(C,-)(A) := \mathscr{C}(C,A) = \{\text{all morphisms from } C \text{ to } A \text{ in } \mathscr{C}\}.$$

Note that this is a set because of the definition of categories. Now for every morphism $f : A \to B$ in $\mathscr{C}$ we have to define $\mathscr{C}(C,-)(f) =: \mathscr{C}(C,f)$ as a set map from $\mathscr{C}(C,-)(A)$ to $\mathscr{C}(C,-)(B)$, so we proceed as follows:

$$\begin{array}{rccc} \mathscr{C}(C,f): & \mathscr{C}(C,A) & \to & \mathscr{C}(C,B) \\ & (g : C \to A) & \to & (f \circ g : C \to B). \end{array}$$

It is almost immediate to check that this is actually a functor; we will say that this functor is *representable* and that the object C is a *representative* of it. In general, given any functor F from a fixed category $\mathscr{C}$ to (**Sets**) we will say that F is representable if there exists $C \in \mathscr{C}_0$ such that $F = \mathscr{C}(C,-)$; note that if such an object exists, it is necessarily unique.

Example 1.4. ([Lee],problem 6.2) For any smooth map $f : M \to N$ between smooth manifolds, we can define its pullback: $f^* : T^*N \to T^*M$. Then it is easy to prove that the assignement $M \mapsto T^*M$ and $f \mapsto f^*$ defines a *contravariant* functor from the category of smooth manifolds to the category of smooth vector bundles.

1.2 Natural tranformations

Definition 1.7. Given two functors $F, G : \mathscr{A} \to \mathscr{B}$, a *natural tranformation* or *morphism of functors* α from F to G is the datum of a class of morphisms:

$$\alpha := \{\alpha_A : F(A) \to G(A)\}_{A \in \mathscr{A}_0}$$

in $\mathscr{B}$ indexed by the objects of $\mathscr{A}$ and such that for every morphism $f : A \to A'$ in $\mathscr{A}$ we have that the following diagram is commutative in $\mathscr{B}$:

$$\begin{tikzcd}
F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\curvearrowright"] & G(A) \arrow[d, "G(f)"] \\
F(A') \arrow[r, "\alpha_{A'}"'] & G(A').
\end{tikzcd}$$

Whenever we have a natural transformation α from F to G, we will denote it as $\alpha : F \Rightarrow G$ or:

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathscr{B}.
\end{tikzcd}$$

The idea behind the use of this diagram is that we can think to objects as points, functors as oriented segments and natural transformations as oriented 2-cells. Using such a picture, one can think to generalize this theory to n-cells for arbitrary $n \in \mathbb{N}$. This can be done, see for example [Lei].

Now we can compose natural transformations in two different ways. First of all, if we consider a triple of functors F, G, H from $\mathscr{A}$ to $\mathscr{B}$ and two natural transformations: $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, then we define $\beta \odot \alpha : F \Rightarrow H$ as follows: for every object A in $\mathscr{A}$ we set:

$$(\beta \odot \alpha)_A := \beta_A \circ \alpha_A.$$

In order to prove that this is again a natural tranformation it suffices to fix any morphism $f : A \to A'$ in $\mathscr{A}$ and compose the following commutative squares:

$$\begin{tikzcd}
F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\curvearrowright"] & G(A) \arrow[r, "\beta_A"] \arrow[d, "G(f)"] \arrow[dr, phantom, "\curvearrowright"] & H(A) \arrow[d, "H(f)"] \\
F(A') \arrow[r, "\alpha_{A'}"'] & G(A') \arrow[r, "\beta_{A'}"'] & H(A').
\end{tikzcd}$$

So we have described a *vertical composition* of natural transformations:

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=60, "F"{name=U}] \arrow[r, "G"'{name=M}] \arrow[r, bend right=60, "H"'{name=D}] & \mathscr{B} \arrow[r, phantom, "\mapsto"] & \mathscr{A} \arrow[r, bend left=40, "F"{name=U2}] \arrow[r, bend right=40, "H"'{name=D2}] & \mathscr{B}.
\arrow[Rightarrow, from=U, to=M, "\alpha"]
\arrow[Rightarrow, from=M, to=D, "\beta"]
\arrow[Rightarrow, from=U2, to=D2, "\beta \odot \alpha"]
\end{tikzcd}$$

Remark 1.2. Note that if we fix $\mathscr{A}$ and $\mathscr{B}$, we can define a new category $\mathbf{HOM}(\mathscr{A}, \mathscr{B})$ as follows:

$$\mathbf{HOM}(\mathscr{A}, \mathscr{B}) = \begin{cases} \text{objects:} & \text{functors from } \mathscr{A} \text{ to } \mathscr{B} \\ \text{morphisms from } F \text{ to } G: & \begin{array}{c} \text{natural transformations} \\ \alpha : F \Rightarrow G. \end{array} \end{cases}$$

Here the composition is just the vertical composition $\odot$ we have just defined and which is clearly associative; for every object $F : \mathscr{A} \to \mathscr{B}$ the identity on it is the natural transformation $i_F : F \Rightarrow F$ given by $(i_F)_A = 1_{F(A)}$ for every object A in $\mathscr{A}$.

Definition 1.8. A *natural equivalence* α is a natural transformation which is invertible in **HOM**, i.e. which is invertible with respect to the vertical composition. In other words, a natural transformation $\alpha : F \Rightarrow G$ is a natural equivalence iff there exists $\beta : G \Rightarrow F$ such that $\alpha \odot \beta = i_G$ and $\beta \odot \alpha = i_F$.

Now let us consider a triple $\mathscr{A},\mathscr{B},\mathscr{C}$ of categories, together with 4 functors $F, G : \mathscr{A} \to \mathscr{B}$, $H, K : \mathscr{B} \to \mathscr{C}$ and 2 natural tranformations $\alpha : F \Rightarrow G$ and $\beta : H \Rightarrow K$. Then we define $\beta * \alpha : H \circ F \Rightarrow K \circ G$ as follows: for every object A in $\mathscr{A}$, we set:

$$(\beta * \alpha)_A := \beta_{G(A)} \circ H(\alpha_A) : H \circ F(A) \to K \circ G(A).$$

Let us prove that this family actually defines a natural transformation from $H \circ F$ to $K \circ G$: if we fix any morphism $f : A \to A'$ in $\mathscr{A}$, then using α we get a commutative diagram in $\mathscr{B}$:

$$\begin{tikzcd}
F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] \arrow[rd, phantom, "\curvearrowright"] & G(A) \arrow[d, "G(f)"] \\
F(A') \arrow[r, "\alpha_{A'}"'] & G(A').
\end{tikzcd}$$

If we apply to it the functor H we get a commutative diagram in $\mathscr{C}$:

$$\begin{tikzcd}
H \circ F(A) \arrow[r, "H(\alpha_A)"] \arrow[d, "H \circ F(f)"'] \arrow[rd, phantom, "\curvearrowright"] & H \circ G(A) \arrow[d, "H \circ G(f)"] \\
H \circ F(A') \arrow[r, "H(\alpha_{A'})"'] & H \circ G(A').
\end{tikzcd}$$

Now $G(f) : G(A) \to G(A')$ is a morphism in $\mathscr{B}$, hence using the fact that β is a natural transformation from H to K we obtain:

$$\begin{tikzcd}
H \circ G(A) \arrow[r, "\beta_{G(A)}"] \arrow[d, "H \circ G(f)"'] \arrow[rd, phantom, "\curvearrowright"] & K \circ G(A) \arrow[d, "K \circ G(f)"] \\
H \circ G(A') \arrow[r, "\beta_{G(A')}"'] & K \circ G(A').
\end{tikzcd}$$

If we glue together the last two diagrams, we get:

$$\begin{tikzcd}
H \circ F(A) \arrow[r, "(\beta*\alpha)_A"] \arrow[d, "H\circ F(f)"'] \arrow[dr, phantom, "\curvearrowright"] & K \circ G(A) \arrow[d, "K\circ G(f)"] \\
H \circ F(A') \arrow[r, "(\beta*\alpha)_{A'}"'] & K \circ G(A')
\end{tikzcd}$$

i.e. $\beta * \alpha$ is a natural transformation from $H \circ F$ to $K \circ G$. We call such a transformation the *horizontal composition* of α and β; the name is well explained by this diagram:

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathscr{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow \beta"] & \mathscr{C} \arrow[r, phantom, "\mapsto"] & \mathscr{A} \arrow[r, bend left, "H\circ F"] \arrow[r, bend right, "K\circ G"'] \arrow[r, phantom, "\Downarrow \beta * \alpha"] & \mathscr{C}.
\end{tikzcd}$$

Consider also the following example:

Example 1.5. Let us consider the category (**Top**) where the objects are the topological spaces and the morphisms are the continuous maps between them. Given two such maps $f, g : A \to B$, we recall that an *homotopy* α from f to g is a continuous map:

$$\alpha : I \times A \to B$$

where $I = [0, 1]$, such that $\alpha(0, a) = f(a)$ and $\alpha(1, a) = g(a)$ $\forall a \in A$; we write $\alpha : f \Rightarrow g$ whenever α is an homotopy from f to g.

We consider now a triple of continuous maps $f, g, h : A \to B$ together with 2 homotopies $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$. Then we can define a composition of homotopies as follows:

$$\beta \odot \alpha : I \times A \to B$$

$$(\beta \odot \alpha)(t,a) := \begin{cases} \alpha(2t,a) & \text{if } t \leq 1/2 \\ \beta(2t-1,a) & \text{if } t > 1/2. \end{cases}$$

Hence we get a continuous map which coincides with f at $t=0$ and with h at $t=1$, i.e. $\beta \odot \alpha : f \Rightarrow h$. This result appears similar to the definition of vertical composition of natural tranformations given before. The only problem is that this composition of homotopies is not associative in general. Indeed, let us consider 3 homotopies between 4 continuos maps from A to B as follows:

$$\alpha : f \Rightarrow g, \quad \beta : g \Rightarrow h, \quad \text{and} \quad \gamma : h \Rightarrow k.$$

Then the composition:

$$\eta := \gamma \odot (\beta \odot \alpha) : f \Rightarrow k$$

satifies $\eta(\frac{1}{2}, a) = h(a) \quad \forall a \in A$, while the composition:

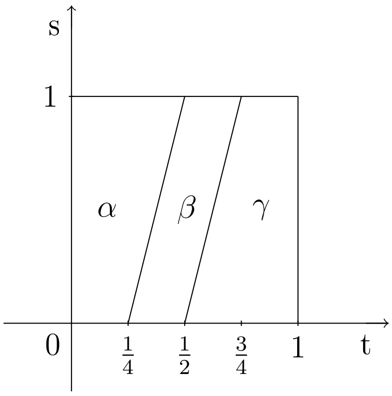
$$\mu := (\gamma \odot \beta) \odot \alpha : f \Rightarrow k$$

is such that $\mu(\frac{1}{2}, a) = g(a) \quad \forall a \in A$.

However, if we consider η and μ as continuous maps: $I \times A \to B$, we can prove that they are homotopic using the homotopy $\Delta : I \times I \times A \to B$ given by:

$$\Delta(s,t,a) = \begin{cases} \alpha(\frac{4t}{s+1}) & \text{if } t \in [0, \frac{1}{4}(s+1)] \\ \beta(4t-(s+1)) & \text{if } t \in [\frac{1}{4}(s+1), \frac{1}{4}(s+2)] \\ \gamma(\frac{4t-(s+2)}{4-(s+2)}) & \text{if } t \in [\frac{1}{4}(s+2), 1]. \end{cases}$$

The idea behind this construction is explained by the following diagram:



Hence the vertical composition of homotopies is associative if we agree to consider not the homotopies, but classes of homotopy of homotopies. A standard but very long check (that we omit) will prove that this gives rise to a composition map on the set of classes of homotopic homotopies.

There is also an analogue of the horizontal composition of natural transformations, described for example on [B], example 7.1.4.b. Again this construction is associative if we pass to equivalence classes of homotopic homotopies.

1.3 2-categories

The construction of vertical and horizontal compositions of natural tranformations in (**Cat**) and of classes of homotopies in (**Top**) suggests the idea to define a notion of “2-category”, where we have not only objects and morphisms, but also morphisms of morphisms between them. In order not to make confusion, we will call morphisms or 1-morphisms the usual morphisms between objects, while the abstract equivalent of natural tranformations and homotopies will be called 2-morphisms.

Clearly, we want to have some compatibility conditions not only on the level of 1-morphisms, but also on 2-morphisms. This leads to give the following definition:

Definition 1.9. ([B], def. 7.1.1) A 2-*category* $\mathscr{A}$ consists of the following data:

(1) a *class* $\mathscr{A}_0$, whose elements are called objects;

(2) for every pair of objects A, B, a *small category* $\mathscr{A}(A,B)$; the objects of this category are called arrows or 1-morphisms and will be denoted by $f : A \to B$. The morphisms of this category between any pair of 1-morphisms f and g are called 2-morphisms and are denoted by $\alpha : f \Rightarrow g$.
According to the previous examples, the composition of 2 composable morphisms (i.e. 2-morphisms) α, β in the category $\mathscr{A}(A,B)$ will be called *vertical* composition and denoted with $\beta \odot \alpha$.

(3) for each triple A, B, C of objects of $\mathscr{A}$, a functor :

$$c_{ABC} : \mathscr{A}(A,B) \times \mathscr{A}(B,C) \to \mathscr{A}(A,C).$$

The composition $c_{ABC}(f,g)$ of two objects $f : A \to B$ with $g : B \to C$ will be denoted by $g \circ f$. The composition $c_{A,B,C}(\alpha,\beta)$ of two morphisms $\alpha : f \Rightarrow f'$ in $\mathscr{A}(A,B)$ and $\beta : g \Rightarrow g'$ in $\mathscr{A}(B,C)$ will be called *horizontal* composition and denoted by $\beta * \alpha$;

(4) for each object A of $\mathscr{A}$, a functor:

$$u_A : \mathbf{1} \to \mathscr{A}(A,A)$$

where $\mathbf{1}$ is the category with a single object x and a single morphism identity 1_x (this is a terminal object in (**Cat**)).

These data are required to satify the following axioms:

(i) (associativity axiom) given four objects A, B, C, D of $\mathscr{A}$, we have:

$$c_{ABD} \circ (1 \times c_{BCD}) = c_{ACD} \circ (c_{ABC} \times 1)$$

where $\circ$ here denotes the usual composition of functors; in other words, we have a commutative diagram of categories and functors:

$$\begin{tikzcd}
\mathscr{A}(A,B) \times \mathscr{A}(B,C) \times \mathscr{A}(C,D) \arrow[r, "1\times c_{BCD}"] \arrow[d, "c_{ABC}\times 1"'] \arrow[dr, phantom, "\curvearrowright"] & \mathscr{A}(A,B) \times \mathscr{A}(B,D) \arrow[d, "c_{ABD}"] \\
\mathscr{A}(A,C) \times \mathscr{A}(C,D) \arrow[r, "c_{ACD}"'] & \mathscr{A}(A,D);
\end{tikzcd}$$

(ii) (unit axiom) for any pair of objects A, B of $\mathscr{A}$ the following isomorphisms hold:

$$c_{AAB} \circ (u_A \times 1) \cong 1_{\mathscr{A}(A,B)} \cong c_{ABB} \circ (1 \times u_B)$$

where we don't use identities because the categories $\mathbf{1} \times \mathscr{A}(A,B)$ and $\mathscr{A}(A,B) \times \mathbf{1}$ are just isomorphic to $\mathscr{A}(A,B)$, but not equal. In other words, we want the following diagram to be commutative:

$$\begin{tikzcd}
\mathbf{1} \times \mathscr{A}(A,B) \arrow[d, "u_A\times 1"'] \arrow[dr, phantom, "\curvearrowright"] & \mathscr{A}(A,B) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, equal] \arrow[dr, phantom, "\curvearrowright"] & \mathscr{A}(A,B) \times \mathbf{1} \arrow[d, "1\times u_B"] \\
\mathscr{A}(A,A) \times \mathscr{A}(A,B) \arrow[r, "c_{AAB}"'] & \mathscr{A}(A,B) & \mathscr{A}(A,B) \times \mathscr{A}(B,B); \arrow[l, "c_{ABB}"]
\end{tikzcd}$$

Remark 1.3. Let us call $u_A(x) =: 1_A$ and $u_A(1_x) =: i_A$ (instead of 1_{1_A}, just for simplicity); note that in this way we have: $i_A : 1_A \Rightarrow 1_A$. Then we can restate the previous axioms in terms of objects, 1-morphisms, 2-morphisms and compositions $\circ, \odot, *$. In this way we get the following axioms:

(a) for every triple of 1-morphisms of the form:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

we have:

$$(h \circ g) \circ f = h \circ (g \circ f);$$

(b) for every diagram:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow \alpha"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow \beta"] & C \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[r, phantom, "\Downarrow \gamma"] & D \end{tikzcd}$$

we have:

$$(\gamma * \beta) * \alpha = \gamma * (\beta * \alpha);$$

(c) for each 1-morphism $A \xrightarrow{f} B$, we have:

$$f \circ 1_A = f = 1_B \circ f;$$

(d) for each 2-morphism $\alpha : (f : A \to B) \Rightarrow (g : A \to B)$ we get:

$$\alpha * i_A = \alpha = i_B * \alpha.$$

(a) and (b) are obtained from (i) applied to objects and functors, and the same for (c) and (d) which come from (ii). Actually, it is clear that these new 4 axioms are equivalent to the previous two, so we will always verify this list instead of the previous one.

Definition 1.10. A *2-isomorphism* is a 2-morphism which is invertible with respect to the vertical composition $\odot$. This is the abstract equivalent of natural equivalences described previously.

Remark 1.4. Let us consider the following diagram:

$$\begin{tikzcd}
A \arrow[r, bend left=50, "f", ""{name=U, below}] \arrow[r, "f'"', ""{name=M}] \arrow[r, bend right=50, "f''"', ""{name=D, above}] & B \arrow[r, bend left=50, "g", ""{name=U2, below}] \arrow[r, "g'"', ""{name=M2}] \arrow[r, bend right=50, "g''"', ""{name=D2, above}] & C
\arrow[Rightarrow, from=U, to=M, "\alpha"]
\arrow[Rightarrow, from=M, to=D, "\beta"]
\arrow[Rightarrow, from=U2, to=M2, "\gamma"]
\arrow[Rightarrow, from=M2, to=D2, "\delta"]
\end{tikzcd}$$

and let us recall the definition of product of categories (definition 1.4): the composition $\odot$ in $\mathscr{A}(A,B)\times\mathscr{A}(B,C)$ is defined “component by component”, i.e $(\beta,\delta)\odot(\alpha,\gamma)=(\beta\odot\alpha,\delta\odot\gamma)$; we recall also that for any pair θ,η of composable 2-morphisms we use $\eta*\theta$ to denote $c_{ABC}(\theta,\eta)$. Hence we get that:

$$(\delta*\beta)\odot(\gamma*\alpha)=c_{ABC}(\beta,\delta)\odot c_{ABC}(\alpha,\gamma)=$$

$$=c_{ABC}((\beta,\delta)\odot(\alpha,\gamma))=c_{ABC}(\beta\odot\alpha,\delta\odot\gamma)=(\beta\odot\alpha)*(\delta\odot\gamma).$$

This formula is known as *interchange law*.

While in category theory we are mainly interested in commutative diagrams, in 2-category theory in general we will use diagrams that only 2-commute. For example, whenever we write:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "k"'] & B \arrow[d, "g"] \\
D \arrow[r, "h"'] & C
\arrow[Rightarrow, from=1-2, to=2-1, shorten=12mm, "\alpha"]
\end{tikzcd}$$

we mean that there exists a 2-morphism $\alpha: g\circ f\Rightarrow h\circ k$. Now let us give some examples of 2-categories.

Example 1.6. The first basic example of 2-category is the one we described previously, where the objects are small categories, the 1-morphisms are the functors and the 2-morphisms are the natural transformations between them. Note that in this case the 2-isomorphisms are exactly the natural equivalences.

Also the second example we described before with topological spaces as objects, continuous maps as 1-morpisms and classes of homotopies as 2-morphisms is an example of 2-category. This is not very hard to prove, but too long for our purposes.

With a little abuse of notation, we will refer to these 2-categories as (**Cat**) and (**Top**) as the corresponding categories previously defined.

Example 1.7. Every category $\mathscr{A}$ can be considered as a 2-category, just saying that given any pair A, B of objects in it, the category $\mathscr{A}(A, B)$ is just a set, i.e. a category where the only morphisms are the identities. In other words, we add to the category only the trivial 2-arrows that we have to put in it because of the fourth point of definition 1.9. The composition on this 2-category is the usual one on 1-morphisms and is trivial on 2-morphisms.

Example 1.8. Conversely, if we consider any 2-category $\mathscr{A}$, we can associate to it a category $\widetilde{\mathscr{A}}$ just ignoring the 2-morphisms. If we do so, the axioms (a) and (c) for a 2-category just coincide with axioms (i) and (ii) for a category (see definition 1.1). We will call $\widetilde{\mathscr{A}}$ the *underlying category* of the 2-category $\mathscr{A}$.

Example 1.9. One can also make the category (**Gr**) of groups and groups homomorphisms into a 2-category in a non trivial way. This is described in [B], example 7.1.4.c. The main idea is that given any group homomorphism $f : G \to H$ and any element $h \in H$, we can define a new group homomorphism $g := h \cdot f \cdot h^{-1} : G \to H$ and we can consider h as a natural transformation between f and g.

Definition 1.11. As in the case of categories, we can easily define the *product* $\mathscr{A} \times \mathscr{B}$ of two 2-categories: the objects are pair of objects (A, B) with $A \in$

$\mathscr{A}_0$, $B \in \mathscr{B}_0$; the 1-morphisms between (A, B) and (A', B') are pairs of 1-morphisms (f, g) with $f \in \mathscr{A}(A, A')_0$ and $g \in \mathscr{B}(B, B')_0$; the 2-morphisms are pairs of 2-morphism (α, β). The compositions of 1-morphisms and of 2-morphisms are defined "component by component".

In the case of category we have already seen the notion of functor; now we are also interested in how to pass from a 2-category to another. Hence we give the following:

Definition 1.12. Given two 2-categories $\mathscr{A}$ and $\mathscr{B}$, a *(covariant) 2-functor* $F : \mathscr{A} \to \mathscr{B}$ consists of the following data:

(1) for each object A in $\mathscr{A}$, an object $F(A)$ in $\mathscr{B}$;

(2) for each pair of objects A, A' in $\mathscr{A}$, a functor:

$$F_{A,A'} : \mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'));$$

with a little abuse of notation, sometimes we will denote this functor only with F as the corresponding 2-functor. These data must satisfy the following axioms:

(i) (compatibility with composition) for any triple A, A', A'' of objects in $\mathscr{A}$, the following diagram of categories and functors commutes:

$$\begin{tikzcd}
\mathscr{A}(A, A') \times \mathscr{A}(A', A'') \arrow[r, "c_{AA'A''}"] \arrow[d, "F_{AA'} \times F_{A'A''}"'] \arrow[dr, phantom, "\curvearrowright"] & \mathscr{A}(A, A'') \arrow[d, "F_{A,A''}"] \\
\mathscr{B}(F(A), F(A')) \times \mathscr{B}(F(A'), F(A'')) \arrow[r, "c_{F(A)F(A')F(A'')}"'] & \mathscr{B}(F(A), F(A''));
\end{tikzcd}$$

(ii) for every object A in $\mathscr{A}$ the following diagram commutes:

$$\begin{tikzcd}
& \mathbf{1} \arrow[dl, "u_A"'] \arrow[dr, "u_{F(A)}"] \arrow[d, phantom, "\curvearrowright"] & \\
\mathscr{A}(A, A) \arrow[rr, "F_{AA}"'] & & \mathscr{B}(F(A), F(A)).
\end{tikzcd}$$

Remark 1.5. We can use also the following equivalent conditions:

(a) for every pair of morphisms $f : A \to A'$ and $g : A' \to A''$ we have:

$$F(g \circ f) = F(g) \circ F(f);$$

(b) for every diagram in $\mathscr{A}$ of the form:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow \alpha"] & A' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow \beta"] & A'' \end{tikzcd}$$

we have that:

$$F(\beta * \alpha) = F(\beta) * F(\alpha);$$

(c) for every object A of $\mathscr{A}$ we have

$$F(1_A) = 1_{F(A)} \quad \text{and} \quad F(i_A) = i_{F(A)}.$$

Here (a) and (b) together are equivalent to (i), while (c) is equivalent to condition (ii).

Remark 1.6. In particular, using axioms (a) and the first part of (c), we get that a 2-functor $F : \mathscr{A} \to \mathscr{B}$ induces an ordinary functor $\widetilde{F} : \widetilde{\mathscr{A}} \to \widetilde{\mathscr{B}}$ between the underlying categories. This functor will be called the *underlying functor* of the 2-functor F.

Note also that given a functor between categories, it is easy to induce a 2-functor between the corresponding 2-categories: it suffices to define it in the trivial way on the level of 2-morphism; at this level there is nothing to check, since we have only the 2-identities.

1.4 Fibered products of categories

Definition 1.13. Let us fix a category $\mathscr{C}$; a commutative diagram:

$$\begin{tikzcd}
V \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[rd, phantom, "\curvearrowright"] & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

in $\mathscr{C}$ is called *cartesian* if it has the following universal property:

UP: for any object U and for any pair of *morphisms* $a : U \to X$ and $b : U \to Y$ in $\mathscr{C}$ such that $f \circ a = g \circ b$, *there exists a unique morphism* $h : U \to V$ such that $a = g' \circ h$ and $b = f' \circ h$, i.e:

$$\begin{tikzcd}
U \arrow[rd, dashed, "h"] \arrow[rrd, bend left, "b"] \arrow[rdd, bend right, "a"'] \arrow[rrd, phantom, "\curvearrowright", pos=0.35] \arrow[rdd, phantom, "\curvearrowright", pos=0.35] & & \\
& V \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[rd, phantom, "\curvearrowright"] & Z \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

We will always denote a cartesian diagram with the notation:

$$\begin{tikzcd}
V \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[rd, phantom, "\square"] & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Definition 1.14. If such a diagram is cartesian, we will say that V is a *fiber product* of X and Z over Y and we will denote it with:

$$X\ {}_f\!\times_g\ Z$$

or also with:

$$X\ \times_Y\ Z$$

if there is no ambiguity on the morphisms f and g used.

Remark 1.7. Let us fix a category $\mathscr{C}$ and a pair of morphisms $f : X \to Y$ and $g : Z \to Y$. Then the fiber product $X\ \times_Y\ Z$, if it exists, is *unique up to isomorphisms.* Indeed, let us suppose that we have two cartesian diagrams with the same lower-right corner:

$$\begin{tikzcd}
V_1 \arrow[r, "f'_1"] \arrow[d, "g'_1"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "g"] \\
X \arrow[r, "g"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
V_2 \arrow[r, "f'_2"] \arrow[d, "g'_2"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Then using the **UP** of the first diagram and the fact that the second one is commutative, we get that there exists a unique $h : V_2 \to V_1$ such that:

$$\begin{tikzcd}
V_2 \arrow[drr, bend left, "f'_2"] \arrow[dr, dashed, "h"] \arrow[ddr, bend right, "g'_2"'] & & \\
& V_1 \arrow[r, "f'_1"] \arrow[d, "f'_1"'] \arrow[dr, phantom, "\curvearrowright"] \arrow[ur, phantom, "\curvearrowright"] \arrow[l, phantom, "\curvearrowright"] & Z \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Y.
\end{tikzcd} \tag{1.1}$$

Conversely, using the **UP** of the second diagram and the commutativity of the first one, we get that there exists a unique $k : V_1 \to V_2$ such that:

$$\begin{tikzcd}
V_1 \arrow[dr, dashed, "k"] \arrow[drr, bend left, "f_1'"] \arrow[ddr, bend right, "g_1'"'] & \curvearrowright & \\
\curvearrowright & V_2 \arrow[r, "f_2'"] \arrow[d, "g_2'"'] \arrow[dr, phantom, "\curvearrowright"] & Z \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Y.
\end{tikzcd} \tag{1.2}$$

Now using together (1.1) and (1.2) we get that:

$$f_1' = f_2' \circ k = f_1' \circ (h \circ k) \quad \text{and} \quad f_2' = f_2' \circ (h \circ k).$$

Now if we use the **UP** of the first diagram together with its commutativity, we get that there exists a *unique* morphism $l : V_1 \to V_1$ such that

$$f_1' = f_1' \circ l \quad \text{and} \quad f_2' = f_2' \circ l.$$

Now since 1_{V_1} has this property, we conclude that $l = 1_{V_1}$; hence:

$$h \circ k = 1_{V_1}.$$

In the same way we get that $k \circ h = 1_{V_2}$, hence h and k are isomorphisms, one the inverse of the other, and V_1 is isomorphic to V_2.

So we are allowed to talk of “the” fiber product (up to isomorphisms) instead of “a” fiber product. This is a general property connected to the fact that the fiber product is a particular case of a construction known as “limit” in category theory (see, for example, [B], chapter 2).

Definition 1.15. Let us fix a category $\mathscr{C}$ and suppose it has a terminal object $\bullet$. If we fix two objects A and B in $\mathscr{C}$, we can consider the unique pair of morphisms $f : A \to \bullet$ and $g : B \to \bullet$. Then if the fiber product $A \times_{\bullet} B$ exists, we refer to it as the *product* of A and B and we denote it with $A \times B$.

Example 1.10. While in general it is difficult to prove if a category has fiber products or not, the definition of fiber products and cartesian diagrams is very simple in the case we work in (**Sets**). Indeed, let us consider a diagram of set and set maps as follows:

$$\begin{tikzcd}
 & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

We want to complete it to a cartesian diagram; a standard way to do this is to consider the set:

$$X \times_Y Z := \{(x, z) \text{ s.t. } x \in X, z \in Z \text{ and } f(x) = g(z)\} \subseteq X \times Z.$$

Let us call pr_1 and pr_2 the two projections from $X \times_Y Z$ to X and Z respectively, i.e. $pr_1(x, z) = x$ and $pr_2(x, z) = z$; then we can consider the diagram:

$$\begin{tikzcd}
X \times_Y Z \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\curvearrowright"] & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

which is clearly commutative since for any $(x, z) \in X \times_Y Z$ we have:

$$g \circ pr_2(x, z) = g(z) = f(x) = f \circ pr_1(x, z).$$

Now we want to prove that this diagram is cartesian, i.e. we want to verify that it has the **UP** described before. Then consider any set U together with a pair of set maps $a : U \to X$ and $b : U \to Z$, such that $f \circ a = g \circ b$.

Hence for any element $u \in U$ we have $f(a(u)) = g(b(u))$; in other words $(a(u), b(u))$ belongs to $X \times_Y Z$; so we can define a set map $h : U \to X \times_Y Z$:

$$h(u) := (a(u), b(u)).$$

Now for every $u \in U$, we get that $pr_1 \circ h(u) = pr_1(a(u), b(u)) = a(u)$, hence $pr_1 \circ h = a$; similary, we get that $pr_2 \circ h = b$. Moreover, one can easily verify that h is the only set map which verifies these conditions. So we have proved that the previous diagram is cartesian. Hence we have proved the well-known fact that:

Proposition 1.4.1. (**Sets**) *is a category where the fiber products always exist.*

Example 1.11. Let us fix a category $\mathscr{C}$ and let us consider the category:

$$\mathscr{D} := \mathbf{HOM}(\mathscr{C}, (\mathbf{Sets}))$$

as described in remark 1.2. Here the objects are functors from $\mathscr{C}$ to (**Sets**), while the morphisms are natural transformations between them; the composition in this category is the vertical composition $\odot$ defined in §1.2. Let us fix in $\mathscr{D}$ any pair of morphisms with common target:

$$\alpha : F \Rightarrow G \quad \text{and} \quad \beta : H \Rightarrow G;$$

we want to define a fiber product K of them in $\mathscr{D}$. Since we work in $\mathscr{D}$, this object must be a functor from $\mathscr{C}$ to (**Sets**); in particular, for any object $A \in \mathscr{C}_0$ we have to define a set $K(A)$. In (**Sets**) we have just computed the fiber product, so it makes sense to define $K(A)$ as the fiber product in (**Sets**) of the set maps α_A and β_A:

$$\begin{tikzcd}
K(A) \arrow[r, "pr_2^A"] \arrow[d, "pr_1^A"'] \arrow[dr, phantom, "\square"] & H(A) \arrow[d, "\beta_A"] \\
F(A) \arrow[r, "\alpha_A"'] & G(A).
\end{tikzcd}$$

Now in order to define a functor, we have also to define the images of morphisms in $\mathscr{C}$, so let us fix any morphism $f : A \to B$ in $\mathscr{C}$. Then we have the following situation:

$$\begin{tikzcd}
K(A) \arrow[rr, "pr_2^A"] \arrow[dd, "pr_1^A"'] \arrow[dr, dashed, "\varphi"] & & H(A) \arrow[d, "G(f)"] \\
& K(B) \arrow[r, "pr_2^B"] \arrow[d, "pr_1^B"'] \arrow[dr, phantom, "\square"] & H(B) \arrow[d, "\beta_B"] \\
F(A) \arrow[r, "F(f)"'] & F(B) \arrow[r, "\alpha_B"'] & G(B).
\end{tikzcd} \tag{1.3}$$

Here we want to prove that the external square is commutative in order to prove the existence of the dashed map φ. So let us take any element $(x, z) \in K(A)$, i.e.

$$x \in F(A), \quad z \in H(A) \quad \text{with} \quad \alpha_A(x) = \beta_A(z);$$

we recall that α and β are natural transformations, hence we have commutative squares:

$$\begin{tikzcd}
F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\curvearrowright"] & G(A) \arrow[d, "G(f)"] \arrow[dr, phantom, "\curvearrowright"] & H(A) \arrow[l, "\beta_A"'] \arrow[d, "H(f)"] \\
F(B) \arrow[r, "\alpha_B"'] & G(B) & H(B). \arrow[l, "\beta_B"]
\end{tikzcd}$$

Hence:

$$\alpha_B \circ F(f) \circ pr_1^A(x,z) = \alpha_B \circ F(f)(x) = G(f) \circ \alpha_A(x) =$$

$$= G(f) \circ \beta_A(z) = \beta_B \circ H(f)(z) = \beta_B \circ H(f) \circ pr_2^A(x,z)$$

so the external square in (1.3) is commutative. Using the fact that the internal square is cartesian in (**Sets**) by definition of $K(B)$, we get that there exists a unique set map $\varphi : K(A) \to K(B)$ such that:

$$F(f) \circ pr_1^A = pr_1^B \circ \varphi \quad \text{and} \quad G(f) \circ pr_2^A = pr_2^B \circ \varphi. \tag{1.4}$$

Now let us define $K(f) := \varphi$; we want to prove that with this definition K is a functor from $\mathscr{C}$ to (**Sets**).

First of all, let us consider any pair of morphisms in $\mathscr{C}$: $A \xrightarrow{f} B \xrightarrow{g} C$ and let us apply the functor K to them. By definition of $K(f)$ and $K(g)$ we get the following commutative diagram:

$$\begin{tikzcd}
K(A) \arrow[rrr, "pr_2^A"] \arrow[dr, "K(f)"] \arrow[dddd, "pr_1^A"'] \arrow[drrr, phantom, "\curvearrowright"] & & & H(A) \arrow[d, "G(f)"'] \arrow[dd, bend left=50, "G(g\circ f)"] \\
& K(B) \arrow[rr, "pr_2^B"] \arrow[dr, "K(g)"] \arrow[ddd, "pr_1^B"'] \arrow[drr, phantom, "\curvearrowright"] & & H(B) \curvearrowright \arrow[d, "G(g)"'] \\
& & K(C) \arrow[r, "pr_2^C"] \arrow[dd, "pr_1^C"'] \arrow[ddr, phantom, "\square"] & H(C) \arrow[dd, "\beta_C"] \\
\arrow[r, phantom, "\curvearrowright"] & \arrow[dr, phantom, "\curvearrowright"] & & \\
F(A) \arrow[r, "F(f)"] \arrow[rr, bend right=40, "F(g\circ f)"'] \arrow[rr, phantom, shift right=5, "\curvearrowright"'] & F(B) \arrow[r, "F(g)"] & F(C) \arrow[r, "\alpha_C"'] & G(C).
\end{tikzcd}$$

Hence if we define $\psi := K(g) \circ K(f)$, we get that:

$$F(g \circ f) \circ pr_1^A = pr_1^C \circ \psi \quad \text{and} \quad G(g \circ f) \circ pr_2^A = pr_2^C \circ \psi; \tag{1.5}$$

but we recall that by definition of $K(g \circ f)$, this is the *unique* map ψ such that (1.5) holds. Hence:

$$K(g \circ f) = \psi = K(g) \circ K(f);$$

since this holds for every pair of composable arrows f, g in $\mathscr{C}$, we have proved that K *preserves compositions.*

Moreover, for any object $A \in \mathscr{C}$ we get that the following diagram is commutative:

$$\begin{tikzcd}
K(A) \arrow[rr, "pr_2^A"] \arrow[dr, "1_{K(A)}"] \arrow[dd, "pr_1^A"'] \arrow[ddr, phantom, "\curvearrowright"] & \arrow[dr, phantom, "\curvearrowright"] & H(A) \arrow[d, "G(1_A)=1_{G(A)}"] \\
 & K(A) \arrow[r, "pr_2^A"] \arrow[d, "pr_1^A"'] \arrow[dr, phantom, "\square"] & H(A) \arrow[d, "\beta_A"] \\
F(A) \arrow[r, "F(1_A)=1_{F(A)}"'] & F(A) \arrow[r, "\alpha_A"'] & G(A)
\end{tikzcd}$$

hence using again the uniqueness part of the **UP** we get that $K(1_A) = 1_{K(A)}$, so we have proved that K *preserves also the identities, hence it is a functor from* $\mathscr{C}$ *to* $(\mathbf{Sets})$.

Now let us define the natural transformations:

$$pr_1 : K \Rightarrow F \quad \text{and} \quad pr_2 : K \Rightarrow H$$

as follows: for every object A in $\mathscr{C}$, we define

$$(pr_1)_A := pr_1^A : K(A) \to F(A)$$

and analogously for $(pr_2)_A$. These are clearly natural transformations because of (1.4).

Now our aim is to prove that the diagram of functors and natural transformations:

$$\begin{tikzcd}
K \arrow[r, Rightarrow, "pr_2"] \arrow[d, Rightarrow, "pr_1"'] & H \arrow[d, Rightarrow, "\beta"] \\
F \arrow[r, Rightarrow, "\alpha"'] & G
\end{tikzcd} \tag{1.6}$$

is cartesian in $\mathscr{D}$. First of all, it is commutative; indeed for any object $A \in \mathscr{C}$ we have that:

$$\alpha_A * (pr_1)_A = \beta_A * (pr_2)_A$$

by definition of $K(A)$. Hence $\alpha \odot pr_1 = \beta \odot pr_2$.

So we have only to prove that the **UP** of fiber products is satisfied, so let us fix any commutative diagram in $\mathscr{D}$ with the same lower-right corner of (1.6):

$$\begin{tikzcd}
L \arrow[r, Rightarrow, "\delta"] \arrow[d, Rightarrow, "\gamma"'] \arrow[dr, phantom, "\curvearrowright"] & H \arrow[d, Rightarrow, "\beta"] \\
F \arrow[r, Rightarrow, "\alpha"'] & G;
\end{tikzcd} \tag{1.7}$$

hence, for any fixed object A in $\mathscr{C}$ we get the following diagram in (**Sets**):

$$\begin{tikzcd}
L(A) \arrow[dr, dashed, "\psi_A"] \arrow[drr, bend left, "\delta_A"] \arrow[ddr, bend right, "\gamma_A"'] & & \\
& K(A) \arrow[r, "pr_2^A"] \arrow[d, "pr_1^A"'] \arrow[dr, phantom, "\square"] & H(A) \arrow[d, "\beta_A"] \\
& F(A) \arrow[r, "\alpha_A"'] & G(A)
\end{tikzcd} \tag{1.8}$$

where the external diagram is commutative because of (1.7) and the internal square is cartesian in (**Sets**) by definition of K. So we get that *there exists a unique set map* ψ_A from $L(A)$ to $K(A)$, such that:

$$\gamma_A = pr_1^A \circ \psi_A \quad \text{and} \quad \delta_A = pr_2^A \circ \psi_A. \tag{1.9}$$

Now we want to prove that $\psi := \{\psi_A : L(A) \to K(A)\}_{A \in \mathscr{C}_0}$ is a natural transformation from L to K; so let us fix any morphism $f : A \to B$ in $\mathscr{C}$ and let us consider the following diagram:

$$\begin{tikzcd}
L(A) \arrow[r, "\psi_A"'] \arrow[d, "L(f)"'] \arrow[rr, bend left=30, "\gamma_A"] \arrow[rr, phantom, bend left=15, "\curvearrowright"] & K(A) \arrow[r, "pr_1^A"'] \arrow[d, "K(f)"] \arrow[dr, phantom, "\curvearrowright"] & F(A) \arrow[d, "F(f)"] \\
L(B) \arrow[r, "\psi_B"] \arrow[rr, bend right=30, "\gamma_B"'] \arrow[rr, phantom, bend right=15, "\curvearrowright"] & K(B) \arrow[r, "pr_1^B"] & F(B)
\end{tikzcd}$$

where also the external diagram is commutative because by hypothesis γ is a natural trasformation from L to F. So we get that:

$$pr_1^B \circ (K(f) \circ \psi_A) = F(f) \circ pr_1^A \circ \psi_A = F(f) \circ \gamma_A = \gamma_B \circ L(f) = pr_1^B \circ (\psi_B \circ L(f)). \tag{1.10}$$

In the same way, using the diagram:

$$\begin{tikzcd}
L(A) \arrow[r, "\psi_A"'] \arrow[d, "L(f)"'] \arrow[rr, bend left=30, "\delta_A"] \arrow[rr, phantom, bend left=15, "\curvearrowright"] & K(A) \arrow[r, "pr_2^A"'] \arrow[d, "K(f)"] \arrow[dr, phantom, "\curvearrowright"] & H(A) \arrow[d, "H(f)"] \\
L(B) \arrow[r, "\psi_B"] \arrow[rr, bend right=30, "\delta_B"'] \arrow[rr, phantom, bend right=15, "\curvearrowright"] & K(B) \arrow[r, "pr_2^B"] & H(B)
\end{tikzcd}$$

we get that:

$$pr_2^B \circ (K(f) \circ \psi_A) = pr_2^B \circ (\psi_B \circ L(f)). \tag{1.11}$$

Now if we use (1.10) and (1.11) together with the uniqueness part of the **UP** for $K(B)$, we get that:

$$\psi_B \circ L(f) = K(f) \circ \psi_A.$$

Since this holds for every morphism $f : A \to B$ in $\mathscr{C}$, we get that ψ is a natural transformation from L to K. Moreover, using (1.9) we have that ψ is such that:

$$\gamma = pr_1 \odot \psi \quad \text{and} \quad \delta = pr_2 \odot \psi;$$

in addition ψ *is unique* because for every object A of $\mathscr{C}$ we were forced in the previous construction to define ψ_A as the unique set map such that (1.8) is commutative. *So we have proved that* (1.6) *is cartesian in* $\mathscr{D}$.

In other words, *for any pair of morphisms* $\alpha : F \Rightarrow G$ *and* $\beta : H \Rightarrow G$ *in* $\mathscr{D}$, *there exists their fiber product* $K = F\,{}_{\alpha}\!\times_{\beta} H$ *in* $\mathscr{D}$.

Remark 1.8. *he previous two examples can't be generalized.* Indeed there exist categories where the fiber product never exists or exists only if we require some additional properties on the level of objects and/or morphisms. This is the case of fiber products in (**Manifolds**), as we will see in chapter 3.

> “L’Assioma della scelta è ovviamente vero,
> il principio del buon ordinamento è
> ovviamente falso, e, circa il Lemma di
> Zorn, chi è capace di capirci qualcosa?”
>
> ---
>
> Jerry Bona
> in Leonesi-Toffalori, “Matematica, miracoli e paradossi”

Chapter 2

The 2-category of complex reduced orbifolds

2.1 Uniformizing systems, embeddings and atlases

We begin with some basic definitions about complex orbifolds. Since we will work only over $\mathbb{C}$, in general we will use the word “orbifold” instead of “complex orbifold”.

Definition 2.1. Let X be a (paracompact) second countable Hausdorff topological space and let $U \subseteq X$ be open and non-empty. Then a *(complex) uniformizing system* for U is the datum of:

- a *connected* and non-empty open set $\widetilde{U} \subseteq \mathbb{C}^n$;
- a *finite* group G of holomorphic automorphisms of $\widetilde{U}$; since G is a group, it contains at least the identity on $\widetilde{U}$;
- a continuous, surjective and G-invariant map $\pi : \widetilde{U} \to U$, which induces an homeomorphism between $\widetilde{U}/G$ and U, where we give to $\widetilde{U}/G$ the quotient topology.

Sometimes we will call a set of data $(\widetilde{U}, G, \pi)$ an *orbifold chart of dimension* n for the open set U.

Remark 2.1. In all this work we will always mean that G *is a set of maps, which is also a group.* In other words, we can't have 2 different elements of G which correspond to the same holomorphic automorphism of $\widetilde{U}$; the orbifolds which have this property are usually called *reduced* or *effective.* (The precise definitions of orbifold and orbifold atlas, will be given in the following pages.)

Some authors don't use this restriction: in this case G is a priori a group and we can give a *representation* of it in terms of holomorphic automorphism of $\widetilde{U}$; so it is possible to have different elements of G which are represented by the same map. Here with "representation" of the abstract group G we mean a group homomorphism:

$$\psi : G \to \mathrm{Aut}(\widetilde{U})$$

from G to the group of holomorphic automorphisms of $\widetilde{U}$; then to say that G *acts effectively* is just equivalent to require that ψ is injective. We will say that an orbifold chart $(\widetilde{U}, G, \pi)$ is reduced if the action of G on $\widetilde{U}$ is effective.

In some of the next constructions it will be necessary to use reduced orbifolds, so from now on *we will always restrict to reduced orbifolds.*

Since we will deal always with holomorphic functions, let us recall a well known result about holomorphic functions of several variables.

Theorem 2.1.1. *(inverse mapping theorem in the complex case) Let A and B be open sets in $\mathbb{C}^n$ and let $f : A \to B$ be a holomorphic function. If f is non-singular (i.e. its jacobian matrix is non-singular) in a point $a \in A$, then there exists an open neighborhood B' of $f(a)$ in B where f is invertible; moreover $f^{-1} : B' \to f^{-1}(B') \subseteq A$ is also holomorphic.*

A proof of this fact can be found, for example, in [CG], (chapter C, theorem 6).

Lemma 2.1.2. *Let $\widetilde{U}$ be an open subset of $\mathbb{C}^n$ and let G be a* finite *group of holomporphic automorphism on $\widetilde{U}$ which* fix a point $\tilde{x} \in \widetilde{U}$*. Then for every open neighborhood $\widetilde{A}$ of $\tilde{x}$ in $\widetilde{U}$, there exists $\widetilde{B}$ such that:*

(i) $\widetilde{B}$ is an open neighborhood of $\tilde{x}$, complety contained in $\widetilde{A}$;

(ii) $\widetilde{B}$ is G-invariant;

(iii) $\widetilde{B}$ is connected.

Proof. Let us define:

$$\widetilde{B} := \bigcap_{g \in G} g(\widetilde{A});$$

by hypothesis every $g \in G$ fixes $\tilde{x}$, which belongs to $\widetilde{A}$, so also $\widetilde{B}$ contains $\tilde{x}$; moreover, every g is a holomorphic automorphism of $\widetilde{U}$ and $\widetilde{A}$ is open, so also $g(\widetilde{A})$ is so, hence $\widetilde{B}$ is a finite intersection of open subsets, so it is again open and contains $\tilde{x}$. Then (i) is proved.

Now let us fix any $h \in G$ and let us consider the set map:

$$\begin{array}{ccc} G & \rightarrow & G \\ g & \rightarrow & h \circ g =: \hat{g}; \end{array}$$

this map is bijective because h is invertible, so we have that:

$$h(\widetilde{B}) = \bigcap_{g \in G} h \circ g(\widetilde{A}) = \bigcap_{\hat{g} \in G} \hat{g}(\widetilde{A}) = \widetilde{B} \tag{2.1}$$

so $\widetilde{B}$ is stable under the action of the group G. Now if the set $\widetilde{B}$ we have found is not connected, let us take the path-connected component $\widetilde{B}'$ which contains $\tilde{x}$. Then (i) is again verified easily and so it suffices only to verify that (ii) is satisfied by $\widetilde{B}'$; so let us fix any $h \in G$ and let us first prove that $h(\widetilde{B}') \subseteq \widetilde{B}'$. In order to do this, let us take any point $\tilde{y} \in \widetilde{B}'$ and let us choose any path $\gamma : [0,1] \rightarrow \widetilde{B}'$ such that $\gamma(0) = \tilde{x}$ and $\gamma(1) = \tilde{y}$. By applying the continuos map h (which can be considered as defined from $\widetilde{B}$ to

$\widetilde{B}$ using (2.1)), we get a continuos map $\delta := h \circ \gamma$ from $[0,1]$ to $\widetilde{B}$. Now let us argue by contradiction and let us suppose that $h(\tilde{y}) \notin \widetilde{B}'$; then we obtain that $\delta : [0,1] \to \widetilde{B}$ is a continuos map such that:

$$\delta(0) = h(\gamma(0)) = h(\tilde{x}) = \tilde{x} \in \widetilde{B}' \quad \text{and} \quad \delta(1) = h(\gamma(1)) = h(\tilde{y}) \notin \widetilde{B}'$$

but this contradicts the definition of $\widetilde{B}'$ as one of the path-connected components of $\widetilde{B}$.
So we have proved that for every $h \in G$ we have that $h(\widetilde{B}') \subseteq \widetilde{B}'$. Now we want to prove that equality holds, so let us argue by contradiction and let us suppose there exists $h_0 \in G$ such that $h_0(\widetilde{B}') \subsetneq \widetilde{B}'$. Since G is a group, also $h_0^{-1} \in G$, hence we have that $h_0^{-1}(\widetilde{B}') \subseteq \widetilde{B}'$, so:

$$\widetilde{B}' = h_0 \circ h_0^{-1}(\widetilde{B}') \subseteq h_0(\widetilde{B}') \subsetneq \widetilde{B}'$$

which is absurd. Hence we have proved that $\widetilde{B}'$ is stable under the action of G, so it suffices to redefine $\widetilde{B}$ as $\widetilde{B}'$ and we are done. □

Lemma 2.1.3. *(Cartan's linearization lemma) Let $\widetilde{U}$ be a connected non-empty open set in $\mathbb{C}^n$ and let G be a finite group of holomorphic automorphisms on $\widetilde{U}$ which fix a point $\tilde{x} \in \widetilde{U}$. Then there exist:*

- *an open neighborhood $\widetilde{U}' \subset \widetilde{U}$ of $\tilde{x}$, which is G-invariant;*
- *an open neighborhood $\widetilde{V}$ of the origin in $\mathbb{C}^n$;*
- *a finite group H of* complex linear invertible maps *that act on $\widetilde{V}$;*
- *a biholomorphic map $\sigma : \widetilde{U}' \xrightarrow{\sim} \widetilde{V}$ such that $\sigma(\tilde{x}) = 0$;*
- *a group isomorphism $\bar{\sigma} : G \xrightarrow{\sim} H$ such that for every $g \in G$ we have:*

$$\sigma \circ g = \bar{\sigma}(g) \circ \sigma. \tag{2.2}$$

Proof. ([Ca], lemma 1) First of all, without loss of generality we can suppose that $\tilde{x}$ is the origin of $\mathbb{C}^n$ (at most we apply a translation, which is clearly a biholomorphic change of coordinates). Now for every $g \in G$ we define ($g' := dg_{|\tilde{x}}$ the differential of g at $\tilde{x} = 0$), which is a linear map from the tangent space of $\widetilde{U}$ at the origin to the tangent space at the point $g(\tilde{x})$, again the origin. Both the tangent spaces are isomorphic to $\mathbb{C}^n$, so from now on for us g' will be a linear map from $\mathbb{C}^n$ to itself. This map is also invertible with inverse $(g')^{-1} = (g^{-1})'$ i.e. the differential at $\tilde{x}$ of the inverse of g, which again fixes $\tilde{x}$.

Moreover, we have:

$$d(g'^{-1} \circ g)_{|\tilde{x}=0} = d(g'^{-1})_{|g(\tilde{x})} \circ dg_{|\tilde{x}} = g'^{-1} \circ g' = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}. \tag{2.3}$$

Now let us define the set map:

$$\sigma := \frac{1}{r} \sum_{g \in G} g'^{-1} \circ g$$

where r is the cardinality of the finite group G. Since every $g \in G$ fixes $\tilde{x} = 0$ and all the g' are linear maps, we have that $\sigma(0) = 0$. Moreover, using (2.3) we have that the differential at 0 of σ is the identity, so in particular σ is nonsingular in this point.

Now every term of the sum in σ is the composition of a linear complex map with an holomorphic one, so we have that σ is holomorphic. Then we can apply theorem 2.1.1 and we get two open neighborhood $\widetilde{U}', \widetilde{V}$ (with $\tilde{x} \in \widetilde{U}' \subseteq \widetilde{U}$ and $0 \in \widetilde{V}$) such that $\sigma : \widetilde{U}' \xrightarrow{\sim} \widetilde{V}$ is a biholomorphic map.

Now let us define the group $H := \{g' \text{ s.t. } g \in G\}$ and the set map $\bar{\sigma} : G \to H$ that to every g in G associates its differential g' in $\tilde{x}$. Using the properties of differential, it is easy to see that $\bar{\sigma}$ is a group homomorphism, which is also surjective by definition of H.

If $\widetilde{V}$ is not stable under the action of the group H (which fixes 0) then we can apply the previuos lemma and we restrict it to a smaller neighborhood of 0, which H-invariant, connected and contained in the previous one. Since σ was invertible with holomorphic inverse on $\widetilde{V}$, so is on this smaller set that for simplicity we will call again $\widetilde{V}$ (and consequently, we will call again $\widetilde{U}'$ the image of this set via σ^{-1}).

Now we want to prove (2.2), so let h be any element of G and let us consider the map $\phi : G \to G$ defined as $\phi(g) := g \circ h = \hat{g}$. This map is bijective, so we have:

$$\sigma \circ h = \frac{1}{r}\sum_{g\in G}(g'^{-1} \circ g \circ h) = h' \circ \left(\frac{1}{r}\sum_{g\in G} h'^{-1} \circ g'^{-1} \circ g \circ h \right) =$$

$$= h' \circ \left(\frac{1}{r}\sum_{g\in G}(g' \circ h')^{-1} \circ g \circ h \right) = h' \circ \left(\frac{1}{r}\sum_{g\in G}(g \circ h)'^{-1} \circ (g \circ h) \right) =$$

$$= h' \circ \left(\frac{1}{r}\sum_{\hat{g}\in G} \hat{g}'^{-1} \circ \hat{g} \right) = h' \circ \sigma.$$

Since this holds for every $h \in G$, then (2.2) is proved. This formula implies that $\bar{\sigma}$ is injective: indeed, it suffices to prove that if $\bar{\sigma}(g) = 1_{\widetilde{V}}$, then $g = 1_{\widetilde{U}}$, but this is obvious using (2.2) and the fact that σ is invertible by construction. Moreover, the same formula proves that $\widetilde{U}'$ is stable under the action of G using the fact that $\bar{\sigma}$ is surjective and that $\widetilde{V}$ is H-invariant. □

Definition 2.2. Let $(\widetilde{U}, G, \pi)$ be a uniformizing system and let $\tilde{x} \in \widetilde{U}$. Then we define the *isotropy subgroup* (also known as *stabilizer group*) at $\tilde{x}$ as:

$$G_{\tilde{x}} := \{g \in G \text{ s.t. } g(\tilde{x}) = \tilde{x}\}$$

which is clearly a subgroup of G.

Remark 2.2. Let $(\widetilde{U}, G, \pi)$ be a uniformizing system and let $\tilde{x} \in \widetilde{U}$ such that $G_{\tilde{x}}$ is *trivial*. Then for any point $\tilde{y} \in \widetilde{U}$ such that $\pi(\tilde{x}) = \pi(\tilde{y})$ there exists a *unique* $g \in G$ *such that* $g(\tilde{x}) = \tilde{y}$.

Indeed, the existence follows from the definition of uniformizing system, since $\pi(\widetilde{U})$ is homeomorphic to $\widetilde{U}/G$. Now let us suppose that there exists another $h \in G$ such that $h(\tilde{x}) = \tilde{y}$. Then $g^{-1} \circ h(\tilde{x}) = g^{-1}(\tilde{y}) = \tilde{x}$, i.e. $g^{-1} \circ h \in G_{\tilde{x}}$, hence $g = h$.

Lemma 2.1.4. *Let* $(\widetilde{U}, G, \pi)$ *be a uniformizing system, let* $\tilde{x} \in \widetilde{U}$ *and* $g \in G$. *If* $g(\tilde{x}) \neq \tilde{x}$, *then there exists a radius* $r = r(\tilde{x}, g) > 0$ *such that if we call* B_r *the open ball with radius* r *and centered in* $\tilde{x}$, *we have:*

$$g(B_r) \cap B_r = \varnothing. \tag{2.4}$$

Proof. Let us argue by contradiction and let us suppose that such a radius does not exist; then for every $n \in \mathbb{N}$ there exists a point $\tilde{t}_n \in g(B_{1/n}) \cap B_{1/n}$. Hence, in particular, for every n there exists $\tilde{q}_n \in B_{1/n}$ such that $g(\tilde{q}_n) = \tilde{t}_n$. Now by construction $\lim_{n\to\infty} \tilde{q}_n = \tilde{x}$ because $\tilde{q}_n \in B_{1/n}$ for every n; so by continuity of g, we have $\lim_{n\to\infty} g(\tilde{q}_n) = g(\tilde{x})$. On the other hand, $\lim_{n\to\infty} g(\tilde{q}_n) = \lim_{n\to\infty} \tilde{t}_n = \tilde{x}$. Hence $g(\tilde{x}) = \tilde{x}$, which contradicts the hypothesis. □

Remark 2.3. For every uniformizing system $(\widetilde{U}, G, \pi)$ and for every $g \in G \smallsetminus \{1_{\widetilde{U}}\}$ we define the sets:

$$\widetilde{U}_g := \{\tilde{x} \in \widetilde{U} \text{ s.t. } g(\tilde{x}) \neq \tilde{x}\} \quad \text{and} \quad \widetilde{U}^g : \{\tilde{x} \in \widetilde{U} \text{ s.t } g(\tilde{x}) = \tilde{x}\}.$$

Then for every $g \in G \smallsetminus \{1_{\widetilde{U}}\}$ *we have that* $\widetilde{U}_g$ *is dense in* $\widetilde{U}$. Indeed, if it is not dense, this implies that there exists an open subset where $g = 1_{\widetilde{U}}$; since g is holomorphic, this implies that g is the identity on all $\widetilde{U}$, contradiction.

Moreover we can prove the following very useful lemmas:

Lemma 2.1.5. *For every uniformizing system* $(\widetilde{U}, G, \pi)$ *the set:*

$$\widetilde{U}_G := \bigcap_{g \in G \smallsetminus \{1_{\widetilde{U}}\}} \widetilde{U}_g$$

of points $\tilde{x}$ of $\widetilde{U}$ with trivial stabilizer $G_{\tilde{x}}$ is dense in $\widetilde{U}$.

Proof. It suffices to prove that for every point $\tilde{x} \in \widetilde{U}$ and for every open ball B sufficently small and centered in $\tilde{x}$, the set $\widetilde{U}_G \cap B$ is non-empty. If we use lemma 2.1.4, we get that for every $g \in G \smallsetminus G_{\tilde{x}}$ there exists a positive radius $r_g = r(\tilde{x}, g)$ such that (2.4) holds. Since G is finite (and so also $G \smallsetminus G_{\tilde{x}}$), if we call $r_0 = r_0(\tilde{x})$ the minimum of this radii, then also r_0 is positive; using (2.4) we get that:

$$g(B_{r_0}) \cap B_{r_0} = \varnothing \quad \forall g \in G \smallsetminus G_{\tilde{x}}. \tag{2.5}$$

Clearly if we choose any radius r with $0 < r \leq r_0$, we have that the same relation holds also if we substitute r_0 with r. Now for every radius $0 < r \leq r_0$ we can apply lemma 2.1.2 to the set B_r and to the point $\tilde{x}$, fixed by the group $G_{\tilde{x}}$. So there exists an open neighborhood $\bar{B}_r$ of $\tilde{x}$, which is stable under the action of the group $G_{\tilde{x}}$ and such that $\bar{B}_r \subseteq B_r$.

Since $\bar{B}_r \subseteq B_r \subseteq B_{r_0}$, using (2.5) we get that:

$$g(\bar{B}_r) \cap \bar{B}_r = \varnothing \quad \forall g \in G \smallsetminus G_{\tilde{x}};$$

then *the set of points with trivial stabilizer in $\bar{B}_r$ with respect to G coincides with the set of points with trivial stabilizer with respect to $G_{\tilde{x}}$.*

Moreover, all the elements of $G_{\tilde{x}}$ fix $\tilde{x}$; then we can apply the linearization lemma and we get a holomorphic change of coordinates:

$$\sigma : \bar{B}'_r \xrightarrow{\sim} C_r$$

where $\bar{B}'_r$ is an open neighborhood of $\tilde{x}$ contained in $\bar{B}_r$, $\bar{C}_r$ is an open neighborhood of the origin in $\mathbb{C}^n$ and $\sigma(\tilde{x}) = 0$. Moreover, we get also a group H which acts *linearily* on $\bar{C}_r$, and a group isomorphism:

$$\bar{\sigma} : G_{\tilde{x}} \xrightarrow{\sim} H$$

such that for every $g \in G_{\tilde{x}}$ we have:

$$\sigma \circ g = \bar{\sigma}(g) \circ \sigma.$$

In particular, *the points with trivial stabilizer in* $\bar{B}'_r$ (*with respect to* $G_{\tilde{x}}$) *are mapped by* σ *to points with trivial stabilizer in* $\bar{C}_r$ (*with respect to* H). Indeed, let $\tilde{y}$ be a point with trivial stabilizer with respect to $G_{\tilde{x}}$ and let $h \in H \smallsetminus \{1_{\bar{C}_r}\}$; since $\bar{\sigma}$ is a group isomorphism, there exists $g \in G \smallsetminus \{1_{\widetilde{U}}\}$ such that $\bar{\sigma}(g) = h$, so:

$$\sigma(\tilde{y}) \neq \sigma(g(\tilde{y})) = \bar{\sigma}(g)(\sigma(\tilde{y})) = h(\sigma(\tilde{y})).$$

Now for every $h \in H \smallsetminus \{1_{\bar{C}_r}\}$ the set of points which are fixed by h is the eigenspace corresponding to the eigenvalue 1 for the linear function h (intersecated with $\bar{C}_r$) and this space has complex dimension at most $n-1$.

Then in the new coordinates the set of points with non trivial stabilizer is a finite union of proper linear subspaces of $\mathbb{C}^n$ intersecated with the open neighborhood of the origin $\bar{C}_r$, hence $\bar{C}_r$ contains points with trivial stabilizer.

So also $\bar{B}'_r$ contains points with trivial stabilizer with respect to $G_{\tilde{x}}$; using what we said previously we have that $\bar{B}'_r$ contains also points with trivial stabilizer with respect to the whole group G.

Now we recall that by construction $\bar{B}'_r \subseteq \bar{B}_r \subseteq B_r$, so we have proved that for every positiv radius r (less or equal than $r_0 = r_0(\tilde{x}) > 0$), there exists a point with trivial stabilizer in the open ball B_r centered in $\tilde{x}$, and this holds for every point $\tilde{x}$ in $\widetilde{U}$, so the statement is proved. □

Lemma 2.1.6. *The set of points with trivial stabilizers in* $\widetilde{U}$ *is also open in* $\widetilde{U}$.

Proof. Let $\tilde{x}$ be a point with trivial stabilizer and let us take any $g \in G \smallsetminus \{1_{\widetilde{U}}\}$. Then $g(\tilde{x}) \neq \tilde{x}$, i.e. $(g - id_{\widetilde{U}})(\tilde{x}) \neq 0$. Now g is continuous, so also $g - id_{\widetilde{U}}$ is so, hence there exists an open neighbohood A_g of $\tilde{x}$ such that $g(\tilde{y}) \neq \tilde{y}$ for all $\tilde{y} \in A_g$. Now if we consider the set:

$$A := \bigcap_{g \in G \smallsetminus \{1_{\widetilde{U}}\}} A_g$$

we get that A is an open neighborhood of $\tilde{x}$ that contains only points with trivial stabilizer. Since this holds for every point $\tilde{x}$ with trivial stabilizer, we have proved the statement. □

Definition 2.3. Let us fix two uniformizing systems $(\widetilde{U}, G, \pi)$ and $(\widetilde{V}, H, \phi)$ for open sets U, V in X with $U \subseteq V$. Then a *(complex) embedding* λ from the first to the second uniformizing system is given by an holomorphic embedding $\lambda : \widetilde{U} \to \widetilde{V}$ such that $\phi \circ \lambda = \pi$. In other words, if we call $j : U \to V$ the inclusion map, we require that the following diagram is commutative:

$$\begin{tikzcd}
\widetilde{U} \arrow[r, "\lambda"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\curvearrowright"] & \widetilde{V} \arrow[d, "\phi"] \\
U \arrow[r, hook, "j"'] & V.
\end{tikzcd}$$

The following is a very useful technical result. It was proved for the first time by I. Satake ([Sa]) with an extra assumption, and by I.Moerdijk and D.Pronk ([MP]) in the general case for orbifolds over the real numbers. The following is an analogous result proved in the case of orbifolds over the complex numbers.

Lemma 2.1.7. *Let λ and μ be two embeddings: $(\widetilde{U}, G, \pi) \to (\widetilde{V}, H, \phi)$. Then there exists a unique $h \in H$ such that $\mu = h \circ \lambda$.*

Proof. (adapted to the complex case from [Pr], proposition 4.2.2 and from [MP], appendix, proposition A.1) Lemma 2.1.5 says that the set of points with trivial stabilizer is dense in $\widetilde{V}$; moreover the set $\lambda(\widetilde{U})$ is open (since λ is an embedding and both $\widetilde{U}$ and $\widetilde{V}$ have complex dimension n) and non-empty, hence there exists $\tilde{y} \in \lambda(\widetilde{U})$ with trivial stabilizer. For symplicity, let us call $\tilde{x} := \lambda^{-1}(\tilde{y})$.

Now $\phi(\tilde{y}) = \phi \circ \lambda(\tilde{x}) = \pi(\tilde{x}) = \phi(\mu(\tilde{x}))$ and $\tilde{y}$ has trivial stabilizer, so using remark 2.2 we get that there exists a *unique* $h \in H$ such that:

$$h(\tilde{y}) = \mu(\tilde{x}) = \mu(\lambda^{-1}(\tilde{y})). \tag{2.6}$$

Now let us consider the set:

$$L := \{\tilde{z} \in \lambda(\widetilde{U}) \text{ s.t. } H_{\tilde{z}} \text{ is not trivial}\};$$

of points with non-trivial stabilizer and let us call C the path connected component of $(\lambda(\widetilde{U}) \smallsetminus L) \subseteq \widetilde{V}$ which contains the point $\tilde{y}$. Then we divide the proof of the lemma in several claims:

(a) We *claim that the element h we have just found is the same for all the points of C.*

By construction for every point $\tilde{y}' \in C$ there exists a continuos map $\alpha : [0,1] \to C$ such that $\alpha(0) = \tilde{y}$ and $\alpha(1) = \tilde{y}'$. Since α has target in $C \subseteq \lambda(\widetilde{U}) \smallsetminus L$, if we proceed as previuosly we can prove that there is a well defined map:

$$h_\alpha : [0,1] \to G$$

such that for every $t \in [0,1]$ we have that $h_\alpha(t)$ is the *unique* element in H such that:

$$h_\alpha(t)(\alpha(t)) = \mu(\lambda^{-1}(\alpha(t))). \tag{2.7}$$

(b) Now *we claim that for every $t \in [0,1[$ we have $h_\alpha(t) = h_\alpha(0)$.*

If claim (b) is proved, then we can choose any sequence $t_n \in [0,1[$ with $t_n \rightarrow 1$ and we have that:

$$h_\alpha(0)(\alpha(1)) \overset{(1)}{=} h_\alpha(0)(\lim_{t_n \rightarrow 1} \alpha(t_n)) \overset{(2)}{=} \lim_{t_n \rightarrow 1} h_\alpha(0)(\alpha(t_n)) \overset{(3)}{=}$$

$$\overset{(3)}{=} \lim_{t_n \rightarrow 1} h_\alpha(t_n)(\alpha(t_n)) \overset{(4)}{=} \lim_{t_n \rightarrow 1} \mu(\lambda^{-1}(\alpha(t_n))) \overset{(5)}{=} \mu(\lambda^{-1}(\alpha(\lim_{t_n \rightarrow 1} t_n))) \overset{(6)}{=}$$

$$\overset{(6)}{=} \mu(\lambda^{-1}(\alpha(1)))$$

hence (using uniqueness) we get that $h_\alpha(1) = h_\alpha(0)$ and this holds for every point $\tilde{y}' \in C$ (and for every path α from $\tilde{y}$ to $\tilde{y}'$). Hence h *is the same for all the points of* C, so claim (a) is proved once we have proved claim (b). Here we used the following facts:

- (1) and (6) follow from the definition of the sequence t_n and by continuity of the path α;
- (2) is continuity of $h_\alpha(0)$, which is an element of G, hence holomorphic;
- (3) *is just claim* (b), *that we have not proved yet*;
- (4) is equation (2.7);
- (5) is the continuity of the map $\mu \circ \lambda^{-1} \circ \alpha$ where the continuity of λ^{-1} follows by the fact that by hypothesis λ is an embedding, hence in particular it is an homeomorphism if restricted in target.

Hence we have only to prove claim (b): let us argue by contraddiction and let us suppose that there exists at least a point $t \in]0,1[$ such that $h_\alpha(t) \neq h_\alpha(0)$; hence we can define the point:

$$\bar{t} := \inf\{t \in]0,1[\text{ s.t. } h_\alpha(t) \neq h_\alpha(0)\}$$

and we have to distinguish between two cases:

- if $\bar{t} = 0$, then there exists a sequence $t_n \to 0$ such that $h_\alpha(t_n) \neq h_\alpha(0)$. By definition of uniformizing system the group H is *finite*, hence there exists at least an element $\bar{h} \in H \smallsetminus \{h_\alpha(0)\}$ and a subsequence t_{k_n} such that $h_\alpha(t_{k_n}) = \bar{h}$. Then we can argue as before using continuity and we can conclude that $h_\alpha(0) = \bar{h}$, but this contradicts the fact that by contruction $\bar{h} \neq h_\alpha(0)$.

- If $\bar{t} \in]0,1[$ then this means that $h_\alpha(t) = h_\alpha(0)$ for all $0 \leq t < \bar{t}$, hence using again continuity we get that $h_\alpha(\bar{t}) = h_\alpha(0)$. On the other hand, we can argue as in the previuos case finding a sequence $t_n \in]\bar{t}, 1]$, $t_n \to \bar{t}$ and an element $\bar{h} \in H \smallsetminus \{h_\alpha(0)\}$ such that $h_\alpha(t_n) = \bar{h}$ for all n; by continuity we get $h_\alpha(\bar{t}) = \bar{h}$, again a contradiction.

So claim (b) is proved, and hence also claim (a) is true, so until now we have proved that we can associate the same element $h \in H$ to all the points in C.

(c) *We claim that $\lambda(\widetilde{U}) \smallsetminus L$ has a unique path-connected component, i.e. C.*

In order to prove that, let us fix any point $\tilde{y}'$ in $\lambda(\widetilde{U}) \smallsetminus L$; now $\widetilde{U}$ is open and connected by defition of uniformizing system, hence it is also path-connected. Since λ is continuous, we have that also $\lambda(\widetilde{U})$ is path-connected, so there exists a continuous map $\gamma : [0,1] \to \lambda(\widetilde{U})$ such that $\gamma(0) = \tilde{y}$ and $\gamma(1) = \tilde{y}'$. Let us suppose that $\gamma([0,1]) \cap L \neq \varnothing$ (in the other case claim (c) is already proved). In this case we want to replace γ with another path which doesn't intersect L. First of all, we observe that (using the notation introduced in lemma 2.1.5)

$$L = (\widetilde{V} \smallsetminus \widetilde{V}^H) \cap \lambda(\widetilde{U})$$

hence, using lemma 2.1.6, L is closed in the topology of $\lambda(\widetilde{U})$; moreover, since γ is continuous, we have that $\gamma([0,1])$ is compact. Hence $\gamma([0,1]) \cap L$ is compact too. Now for every point $\tilde{z}$ in this set, we can apply the same construction made in lemma 2.1.5 in order to obtain an open neighborhood of $\tilde{z}$, completely contained in $\lambda(\widetilde{U})$ and such that we can linearize the action of the isotropy group $G_{\tilde{z}}$.

If we consider the family of all such open sets (indexed on the points of $\gamma([0,1])\cap L$), we get that this is an open cover of a compact set, hence we can extract from it a finite cover. Let us call $\{\tilde{z}_i, i = 1, \cdots, n\}$ the finite set of points we have selected and let us call B_i the corresponding open neighborhoods (those which in lemma 2.1.5 were called $\bar{B}'_{r_0}$, where $r_0 = r_0(\tilde{z}_i) > 0$). Note that without loss of generality for every $i = 1, \cdots n$ we can assume that B_i is connected (hence also path-connected) because all the elements of $G_{\tilde{z}_i}$ fix $\tilde{z}_i$, so they map the connected component which contains $\tilde{z}_i$ to itself. Moreover, without loss of generality we can assume that the B_i are chosen such that $B_i \cap B_{i+1} \neq \varnothing$ for all $i = 1, \cdots, n-1$.

Since $\gamma([0,1]) \cap L$ is compact, its inverse image via γ is a closed subset of $[0,1]$, so it is compact in it. Moreovere, it does not contain nor 0 nor 1 because by construction both $\gamma(0) = \tilde{y}$ and $\gamma(1) = \tilde{y}'$ don't belong to L; so it makes sense to define:

$$a := \min\{t \in [0,1] \text{ s.t. } \gamma(t) \in L\} \quad \text{and} \quad b := \max\{t \in [0,1] \text{ s.t. } \gamma(t) \in L\}$$

and we have that $0 < a \leq b < 1$; without loss of generality, we can assume that $\gamma(a) = \tilde{z}_1$ and $\gamma(b) = \tilde{z}_n$.

Now B_1 is an open neighborhood of $\tilde{z}_1 = \gamma(a)$, a is positive and γ is continuous, so there exists $0 \leq a' < a$ such that $q_0 := \gamma(a') \in B_1$. Moreover, by definition of a we have that $q_0 \notin L$.

By construction, we have supposed that $B_1 \cap B_2 \neq \varnothing$, moreover, it is open because both B_1 and B_2 are so, then we can use lemma 2.1.5 in order to choose a point q_1 with trivial stabilizer in $B_1 \cap B_2$. Now we can apply the linearization lemma to the set B_1, so we get a biholomorphism $\sigma : B_1 \to \sigma(B_1) =: C_1$.

Via this map, the set $B_1 \cap L$ is mapped to the set of points with non-trivial stabilizer with respect to a finite group H_1 of linear maps that act on C_1, so we have:

$$\sigma(B_1 \cap L) = \bigcup_{h \in H_1 \smallsetminus \{1_{C_1}\}} C_1^h =: L'$$

where using the notation of remark 2.3 we have that C_1^h is the set of all the points in C_1 fixed by h, i.e. the eigenspace of h corresponding to the eigenvalue 1, intersecated with C_1. Now $h \neq 1_{C_1}$, hence the complex dimension of C_1^h is at most $n-1$, so its real dimension is at most $2n-2$ if we consider $\mathbb{C}^n$ as homeomorphic to $\mathbb{R}^{2n}$. Now by construction C_1 is connected and open in $\mathbb{C}^n$, so if we choose any element $h \in H \smallsetminus \{1_{C_1}\}$ we get that $C_1 \smallsetminus C_1^h$ is again connected and open. Since $H \smallsetminus \{1_{C_1}\}$ is finite, we can apply induction and we get that *the set L' does not disconnect C_1*.

Hence $C_1 \smallsetminus L'$ is connected and open in $\mathbb{C}^n$, so it is path connected; moreover, by construction it contains the images of the points q_0 and q_1. Hence there exists a path δ_1 connecting them and which does not contain any point of L'. If we apply to δ_1 the continuous map σ^{-1} we get a path γ_1 connecting q_0 and q_1 and which doesn't intersect L.

Now if we consider a point $q_2 \in B_2 \cap B_3$ with trivial stabilizer, we can apply the same argument in B_2 and we get a path γ_2 connecting q_1 and q_2 and which doesn't intersect L, and so on. The last step is analogous to the first one and allows us to find a path γ_{n-1} which does not intersect L and which connects a point with trivial stabilizer $q_{n-1} \in B_{n-1} \cap B_n$ with a point of the form $q_n = \gamma(b')$ with $b < b' \leq 1$.

Now we can consider a new path given by the concatenation of the following paths:

$$\tilde{y} \xrightarrow{\gamma_{|[0,a']}} q_0 \xrightarrow{\gamma_1} q_1 \cdots q_{n-2} \xrightarrow{\gamma_{n-1}} q_{n-1} \xrightarrow{\gamma_{|[b',1]}} \tilde{y}'.$$

This is a path connecting $\tilde{y}$ to $\tilde{y}'$ and completely contained in $\lambda(\widetilde{U}) \smallsetminus L$; since this construction holds for every point $\tilde{y}'$ we have proved that $\lambda(\widetilde{U}) \smallsetminus L$ is path-connected, so claim (c) is proved.

Using together claim (a) and (c), we get that *there exists a unique* $h \in H$ *such that for every point* $\tilde{y}$ *in* $C = \lambda(\widetilde{U}) \smallsetminus L$ *we have* $h(\tilde{y}) = \mu(\lambda^{-1}(\tilde{y}))$.

(d) *We claim that the element* $h \in H$ *we found previously is such that* (2.6) *holds also for every point* $\tilde{z} \in L$.

In order to prove that, let us fix any point $\tilde{z} \in L$ and let B be an open neighborhood of $\tilde{z}$ where we can apply the linearization lemma. After the usual change of coordinates σ we can work in a open set B' with $\sigma(\tilde{z})$ coinciding with the origin and with $\sigma(B \cap L) =: L'$ coinciding with a finite union of linear proper subspaces of $\mathbb{C}^n$, intersecated with B'.

Now B' is an open neighborhood of the origin, so there exists an open ball $B'' \subseteq B'$ centered in the origin and we know that the set of points with trivial stabilizer is dense, so there exists $\tilde{q}$ in $B'' \smallsetminus L'$; since L' is a union of linear subspaces, then all the segment connecting $\tilde{q}$ to the origin is contained in $B'' \smallsetminus L'$. If we apply to this path the continuous map σ^{-1} we get a path connecting $\sigma^{-1}(q) \in C$ with $\tilde{z} \in L$ and which intersects L only in $\tilde{z}$. Hence, we can argue as before using continuity in order to prove that also for all the points $\tilde{z}$ in L we have:

$$h(\tilde{z}) = \mu(\lambda^{-1}(\tilde{z})). \tag{2.8}$$

Clearly, for the points of L there can be also other elements of H which make (2.8) true, but the element h that we found before is the only one that works also for all the points of C.

Hence using together (a), (c) and (d) we have proved that *there exists a unique $h \in H$ such that* (2.8) *is true for all the points of* $\lambda(\widetilde{U})$ and this is equivalent to the statement, so we are done. □

Remark 2.4. Note that claim (c) of the previuos construction is true only if we work in the complex case; in the real case (i.e. when the sets of the form $\widetilde{U}$ are open sets of $\mathbb{R}^n$) the sets C_i^h can reach the real dimension $n-1$, i.e. they can have codimension 1 (this is the case, for example, of reflexions around an hyperplane of $\mathbb{R}^n$, which are not allowed in the complex case if we want to preserve complex linearity).

In this case L can disconnect the set $\lambda(\widetilde{U})$, so it is necessary to consider what happens when we pass from one connected component to another. The basic step is the one when the two connected components we are interested in are "separated" by L; this is described in the appendix of [MP], proposition A.1.

As a consequence of lemma 2.1.7 we can prove the following:

Corollary 2.1.8. *Any embedding $\lambda : (\widetilde{U}, G, \pi) \rightarrow (\widetilde{V}, H, \phi)$ induces an injective group homomorphism $\Lambda : G \rightarrow H$ such that:*

$$\lambda \circ g = \Lambda(g) \circ \lambda \quad \forall g \in G.$$

Proof. ([ALR], section 1.1) Let us fix any $g \in G$ and let us consider the map $\mu := \lambda \circ g$. Since g is a holomorphic automorphism of $\widetilde{U}$, we get that μ is a holomorphic embedding $\widetilde{U} \rightarrow \widetilde{V}$; moreover, μ is an embedding between the uniformizing systems $(\widetilde{U}, G, \pi)$ and $(\widetilde{U}, H, \phi)$; indeed:

$$\phi \circ \mu = (\phi \circ \lambda) \circ g = \pi \circ g = \pi,$$

where the last passage follows from the fact that π is G-invariant by definition of uniformizing system. If we apply the previuos lemma to the pair (λ, μ), we find that there exists a unique $h \in H$ s.t. $\lambda \circ g = h \circ \lambda$; then we define the set map:

$$\Lambda : G \to H$$

that to every $g \in G$ associates the corresponding unique $h \in H$ just defined. Now for any pair (g_1, g_2) of automorphisms of G we get that:

$$\Lambda(g_1 \circ g_2) \circ \lambda = \lambda \circ (g_1 \circ g_2) = \Lambda(g_1) \circ (\lambda \circ g_2) = \Lambda(g_1) \circ \Lambda(g_2) \circ \lambda;$$

hence, using the uniqueness part of the previous lemma, we get that

$$\Lambda(g_1 \circ g_2) = \Lambda(g_1) \circ \Lambda(g_2),$$

i.e. Λ is a group homomorphisms. Let us prove that it is injective: since Λ is a group homomorphism, it suffices to prove that if $\Lambda(g) = id_{\widetilde{V}}$ then $g = id_{\widetilde{U}}$, but this is immediate using again the uniqueness part of the lemma. □

Remark 2.5. When we don't assume that the orbifolds are reduced, we have to define an embedding from $(\widetilde{U}, G.\pi)$ to $(\widetilde{V}, H, \phi)$ as a pair (λ, Λ) where:

- $\lambda : \widetilde{U} \to \widetilde{V}$ is an holomorphic embedding such that $\phi \circ \lambda = \pi$;
- $\Lambda : G \to H$ is an injective group homomorphism, such that for all $g \in G$ we have $\lambda \circ g = \Lambda(g) \circ \lambda$.

The first condition is just definition 2.3 and in the previuos corollary we have proved that the second condition is not necessary, but *this is true only if we use reduced orbifolds because lemma* 2.1.7 *only applies in this case.* Indeed, the proof of the lemma consists in definig a *unique* element h in H such that (2.8) is true, but this is an identity between holomorphic functions; so if the map ψ defined in remark 2.1 is not injective, the existence part of the lemma is still true, but in general we can't prove uniqueness, so also the corollary is no more true.

In this work we will only use reduced orbifolds, so we don't bother about this problem.

Corollary 2.1.9. *Let us suppose we have an embedding* $\lambda : (\widetilde{U}, G, \pi) \rightarrow (\widetilde{V}, H, \phi)$ *and we have chosen a point* $\tilde{x} \in \widetilde{U}$ *such that* $\lambda(\tilde{x}) =: \tilde{y}$ *has trivial stabilizer. Then also* $\tilde{x}$ *has trivial stabilizer.*

Proof. Let us fix $g \in G_{\tilde{x}}$: by applying the induced group homomorphism we get that:

$$\Lambda(g)(\tilde{y}) = \Lambda(g) \circ \lambda(\tilde{x}) = \lambda \circ g(\tilde{x}) = \lambda(\tilde{x}) = \tilde{y}$$

so we have that $\Lambda(g) \in H_{\tilde{y}}$, which is trivial by hypothesis, i.e. $\Lambda(g) = 1_{\widetilde{V}}$, so $g = 1_{\widetilde{U}}$ since in the previuos corollary we have also proved that Λ is injective. So the only element in the stabilizer at $\tilde{x}$ is the identity. $\square$

Lemma 2.1.10. *Let* $\lambda : (\widetilde{U}, G, \pi) \rightarrow (\widetilde{V}, H, \phi)$ *be an embedding and let* $h \in H$*. If* $h(\lambda(\widetilde{U})) \cap \lambda(\widetilde{U}) \neq \varnothing$*, then* $h(\lambda(\widetilde{U})) = \lambda(\widetilde{U})$ *and* h *belongs to the image of the induced injective group homomorphism* $\Lambda : G \rightarrow H$*.*

Proof. ([MP], appendix, lemma A.2, with some changes) Let us consider the open set $h(\lambda(\widetilde{U})) \cap \lambda(\widetilde{U})$, which is non-empty by hypothesis. Now we recall that the set of points with trivial stabilizer is dense in $\widetilde{V}$ (see lemma 2.1.5) and is an open set (see lemma 2.1.6), so there exists an open set $A \subseteq h(\lambda(\widetilde{U})) \cap \lambda(\widetilde{U})$ which contains only points with trivial stabilizer. This set is open in $\widetilde{V}$, hence it is also open in $\mathbb{C}^n$; since a basis for the topology of $\mathbb{C}^n$ is given by open balls, without loss of generality we can assume that A is an open ball centered in a point $\tilde{x}' = \lambda(\tilde{x})$.

Moreover, since this point belongs also to $h(\lambda(\widetilde{U}))$, we get that there exists a point $\tilde{y}' = \lambda(\tilde{y})$ such that $h(\tilde{y}') = \tilde{x}'$; this point is unique since h is invertible. Now by definition of embedding we have that:

$$\pi(\tilde{x}) = \phi \circ \lambda(\tilde{x}) = \phi(\tilde{x}') = \phi \circ h(\tilde{y}') = \phi(\tilde{y}') = \phi \circ \lambda(\tilde{y}) = \pi(\tilde{y}).$$

Hence there exists an element $g \in G$ such that $\tilde{x} = g(\tilde{y})$; now let us consider the injective group homomorphism $\Lambda : G \to H$ induced by λ:

$$\Lambda(g)(\tilde{y}') = \Lambda(g)(\lambda(\tilde{y})) = (\lambda \circ g)(\tilde{y}) = \lambda(\tilde{x}) = \tilde{x}';$$

so we have proved that $g \in G$ is such that:

$$\Lambda(g) \circ h^{-1}(\tilde{x}') = \tilde{x}'. \tag{2.9}$$

This element is also *unique*; indeed, let us suppose that there exists another element $g' \in G$ such that $\Lambda(g') \circ h^{-1}(\tilde{x}') = \tilde{x}'$. Then we have that:

$$(\Lambda(g'))^{-1} \circ \Lambda(g)(h^{-1}(\tilde{x}')) = h^{-1}(\tilde{x}')$$

so $(\Lambda(g'))^{-1} \circ \Lambda(g)$ belongs to the stabilizer of $h^{-1}(\tilde{x}')$ with respect to the group H. But this stabilizer is trivial since the stabilizer at $\tilde{x}'$ is so. Hence:

$$\Lambda(g'^{-1} \circ g) = \Lambda(g')^{-1} \circ \Lambda(g) = id_{\widetilde{V}}$$

Since Λ is injective by corollary 2.1.8, so we get that $g'^{-1} \circ g = id_{\widetilde{U}}$ so $g' = g$. Hence we have proved that g is the unique element of G such that (2.9) holds.

The element g depends on λ, on h and on $\tilde{x}$ (or $\tilde{x}'$ equivalently, since λ is injective). In all this proof the first two are fixed, so we can consider $g = g^{\tilde{x}}$. Clearly all this construction holds not only for the center $\tilde{x}$ of the ball A, but also for every other point in it, since $A \subseteq h(\lambda(\widetilde{U})) \cap \lambda(\widetilde{U})$ and contains only points with trivial stabilizer. So we can define a function:

$$f : A \to G$$

that to every element $\tilde{z}' = \lambda(\tilde{z})$ associates the corresponding unique element $g^{\tilde{z}} \in G$ such that:

$$\Lambda(g^{\tilde{z}}) \circ h^{-1}(\tilde{z}') = \tilde{z}'.$$

Now let us proceed as in the proof of lemma 2.1.7 in order to prove that actually the map f is constant; so for any point $\tilde{z}'$ in A let us choose any path γ connecting $\tilde{x}'$ with $\tilde{z}'$ (for example, let us choose the oriented segment between them) and let us prove that f is constant on this path.

We claim that $f(\gamma(t))$ is equal to $f(\tilde{x}') = f(\gamma(0))$ for all $t \in [0,1[$. If this is true, then let us choose any sequence $t_n \in [0,1[$ with $t_n \to 1$ and let us call $\gamma(t_n) =: \tilde{x}'_n = \lambda(\tilde{x}_n)$; by continuity we get that:

$$\begin{aligned} \tilde{z}' = \lim_{n\to\infty} \tilde{x}'_n = \lim_{n\to\infty} \Lambda(g^{\tilde{x}_n}) \circ h^{-1}(\tilde{x}'_n) = \\ = \lim_{n\to\infty} \Lambda(g^{\tilde{x}}) \circ h^{-1}(\tilde{x}'_n) = \Lambda(g^{\tilde{x}}) \circ h^{-1}(\lim_{n\to\infty} \tilde{x}'_n) = \\ = \Lambda(g^{\tilde{x}}) \circ h^{-1}(\tilde{z}'). \end{aligned}$$

Since $g^{\tilde{z}}$ is unique, we get that $g^{\tilde{z}} = g^{\tilde{x}}$, hence we have proved that f is constant on A, so let us call $g \in G$ the constant value of this function. We omit the proof of the claim, which is analogous to the proof of claim (b) in the previuos lemma.

So until now we have proved that there exists a unique $g \in G$ such that for every point $\tilde{z}' \in A$ we have:

$$\Lambda(g) \circ h^{-1}(\tilde{z}') = \tilde{z}'$$

i.e. $\Lambda(g) \circ h^{-1}$ coincides with the identity on the open set $A \subseteq \widetilde{V}$. Since we are working with holomorphic functions, this is true for the whole $\widetilde{V}$, hence *we have that there exists a unique $g \in G$ such that*:

$$\Lambda(g) = h.$$

Moreover,

$$h(\lambda(\widetilde{U})) = \Lambda(g) \circ \lambda(\widetilde{U}) = \lambda \circ g(\widetilde{U}) = \lambda(\widetilde{U})$$

where the last passage follows from the fact that $g \in G$ is an automorphism of $\widetilde{U}$ by definition of G. □

Definition 2.4. A *(reduced) orbifold atlas of dimension* n on a paracompact and second countable Hausdorff topological space X is a family $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i \in I}$ of (reduced) uniformizing systems (with all the $\widetilde{U}_i$'s open in the same $\mathbb{C}^n$) such that:

(i)

$$\bigcup_{(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}} \pi_i(\widetilde{U}_i) = X;$$

(ii) if $(\widetilde{U}_i, G_i, \pi_i)$, $(\widetilde{U}_j, G_j, \pi_j) \in \mathcal{U}$ are uniformizing systems for U_i and U_j respectively, then for every point $x \in U_i \cap U_j$ there exists an open neighborhood $U_k \subseteq U_i \cap U_j$ of x in X, a uniformizing system $(\widetilde{U}_k, G_k, \pi_k) \in \mathcal{U}$ for U_k and embeddings:

$$(\widetilde{U}_i, G_i, \pi_i) \xleftarrow{\lambda_{ki}} (\widetilde{U}_k, G_k, \pi_k) \xrightarrow{\lambda_{kj}} (\widetilde{U}_j, G_j, \pi_j).$$

To be more precise, an orbifold atlas is the datum of a family $\mathcal{U}$ of uniformizing systems that satisfies (i) and (ii), *together with the family of all possible embeddings between charts of* $\mathcal{U}$, but with a little abuse of notation we will always write $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i \in I}$ to denote both the family of uniformizing systems and the family of embeddings.

In the following pages, for every uniformizing system $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we will denote with U_i the open set $\pi_i(\widetilde{U}_i) \subseteq X$.

Remark 2.6. The most important reason for such a definition will be clear in the proof of proposition 2.1.13, where we will show that we can give to every quotient of a manifold (by a finite group of holomorphic automorphisms) a natural structure of orbifold, while in general we can't give to it a structure of manifold. This is also one of the most important reasons for the study of orbifold theory in differentiable geometry.

Remark 2.7. Let $x \in U_i \cap U_j$ and let $(\widetilde{U}_k, G_k, \pi_k)$ be as in the previuos definition. Since $\pi_i : \widetilde{U}_i \to U_i$ is surjective, there exists at least a point $\tilde{x}_i \in \widetilde{U}_i$ such that $x = \pi_i(\tilde{x}_i)$. In the same way, there exists a point $\tilde{x}_k \in \widetilde{U}_k$ such that $x = \pi_k(\tilde{x}_k) = \pi_i(\lambda_{ki}(\tilde{x}_k))$. Hence we get that $\pi_i(\tilde{x}_i) = \pi_i(\lambda_{ki}(\tilde{x}_k))$, so by definition of uniformizing system we get that there exists a (not necessarily unique) $g \in G_i$ such that $(g \circ \lambda_{ki})(\tilde{x}_k) = \tilde{x}_i$.

Now $g \circ \lambda_{ki}$ is again an embedding from $(\widetilde{U}_k, G_k, \pi_k)$ to $(\widetilde{U}_i, G_i, \pi_i)$, so whenever we apply the previous definition and we have fixed $\tilde{x}_i$ such that $\pi_i(\tilde{x}_i) = x$, by substituting $g \circ \lambda_{ki}$ to λ_{ki} there is no loss of generality in assuming that we have chosen a point $\tilde{x}_k \in \widetilde{U}_k$ such that $\lambda_{ki}(\tilde{x}_k) = \tilde{x}_i$. Using the same argument, we can also assume that if we have chosen also a point $\tilde{x}_j \in \widetilde{U}_j$, we have $\lambda_{kj}(\tilde{x}_k) = \tilde{x}_j$.

In other words, *every time we have two uniformizing systems* $(\widetilde{U}_i, G_i, \pi_i)$ *and* $(\widetilde{U}_j, G_j, \pi_j)$ *for 2 open neighborhoods* U_i *and* U_j *for* x *without loss of generality we can assume we are in the following situation:*

$$\begin{tikzcd}
\tilde{x}_i & \tilde{x}_k & \tilde{x}_j \\
\rotatebox[origin=c]{-90}{$\in$} & \rotatebox[origin=c]{-90}{$\in$} & \rotatebox[origin=c]{-90}{$\in$} \\
\widetilde{U}_i \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\curvearrowright"] & \widetilde{U}_k \arrow[l, "\lambda_{ki}"'] \arrow[r, "\lambda_{kj}"] \arrow[d, "\pi_k"] \arrow[dr, phantom, "\curvearrowright"] & \widetilde{U}_j \arrow[d, "\pi_j"] \\
U_i & U_k \arrow[l, hook'] \arrow[r, hook] & U_j . \\
 & \rotatebox[origin=c]{90}{$\in$} & \\
 & x &
\end{tikzcd}$$

Remark 2.8. Let us fix any point $x \in X$, let $(\widetilde{U}_i, G_i, \pi_i)$ be a uniformizing system for an open neighborhood U_i for x and let us fix any point $\tilde{x}_i \in \widetilde{U}_i$

such that $\pi_i(\tilde{x}_i) = x$. Let us suppose we have chosen also another uniformizing system $(\widetilde{U}_j, G_j, \pi_j)$ and a point $\tilde{x}_j$ with the same properties.
Then by definition of atlas there exists a third uniformizing system $(\widetilde{U}_k, G_k, \pi_k)$ in $\mathcal{U}$ together with embeddings $\lambda_{ki}, \lambda_{kj}$ in the previuos two. Moreover, using remark 2.7, we can assume that there exists a point $\tilde{x}_k \in \widetilde{U}_k$ such that $\lambda_{ki}(\tilde{x}_k) = \tilde{x}_i$ and $\lambda_{kj}(\tilde{x}_k) = \tilde{x}_j$.

Now using lemma 2.1.8 we get injective group homomorphisms:

$$\Lambda_{ki} : G_k \to G_i \quad \text{and} \quad \Lambda_{kj} : G_k \to G_j.$$

By construction, for every $g \in (G_k)_{\tilde{x}_k}$ we have:

$$\Lambda_{ki}(g)(\tilde{x}_i) = (\Lambda_{ki}(g) \circ \lambda_{ki})(\tilde{x}_k) = (\lambda_{ki} \circ g)(\tilde{x}_k) = \lambda_{ki}(\tilde{x}_k) = \tilde{x}_i;$$

hence $\Lambda_{ki}(g) \in (G_i)_{\tilde{x}_i}$; in the same way we get that $\Lambda_{kj}(g) \in (G_j)_{\tilde{x}_j}$ for any $g \in (G_k)_{\tilde{x}_k}$. In other words, we can induce injective group homomorphisms, again called with the same notations:

$$\Lambda_{ki} : (G_k)_{\tilde{x}_k} \to (G_i)_{\tilde{x}_i} \quad \text{and} \quad \Lambda_{kj} : (G_k)_{\tilde{x}_k} \to (G_j)_{\tilde{x}_j}.$$

Moreover, using lemma 2.1.10 we get that these group homomorphisms are also onto, so we get the group isomorphisms:

$$(G_i)_{\tilde{x}_i} \cong (G_k)_{\tilde{x}_k} \cong (G_j)_{\tilde{x}_j}.$$

Hence we can give the following definition:

Definition 2.5. Whenever we fix an orbifold atlas $\mathcal{U}$ on a space X, the *local group* of a point $x \in X$ is any of the isotropy subgroups for some preimage $\tilde{x}$ of x in any uniformizing system for an open neighborhood of x. The previuos discussion proves that *the local group at x is well defined up to isomorphisms.*

In particular, it makes sense to check if a local group is trivial or not, since this notion is invariant under group isomorphisms.

Note that in particular this holds in the case when we have chosen the same uniformizing system (i.e. in the case when $i = j$), but different preimages for x.

We will see in remark 2.14 that the notion of local group depends only on the orbifold structure, i.e. on the equivalence class of orbifold atlases that we will describe in section 4.

Our aim now is to prove that *every manifold can be considered as an orbifold.* For us a manifold will be a class of *compatible* manifold atlases on a paracompact and second countable Hausdorff topological space M. The standard definition of manifold atlas says that this is a collection of pairs (U_i, ϕ_i) where the U_i's are open sets of M and the ϕ_i's are the coordinates functions from them to open sets of $\mathbb{C}^n$ such that the transition maps on the intersections of two open sets U_i and U_j are biholomorphic.

In general, it is not possible to associate to a manifold atlas an orbifold atlas on the same topological space: the most important problem that arises is the fact that in the definition of orbifold we require that every chart is *connected*, while this condition does not appear in the standard definition of manifold. In order to solve this problem, let us give the following definition, which is not standard in literature, but will be very useful in the next pages:

Definition 2.6. A manifold altas $\mathcal{M} = \{(U_i, \phi_i)\}_{i\in I}$ is said to be *admissible* if the following two conditions hold:

(i) every domain U_i is connected;

(ii) for every pair of indexes $i, j \in I$ such that $U_i \cap U_j \neq \varnothing$ and for every point x in this intersection, there exists an index $k \in I$ such that $x \in U_k$ and $U_k \subseteq U_i \cap U_j$.

Lemma 2.1.11. *For every manifold atlas $\mathcal{M}$ on a space M there exists an admissible atlas $\widetilde{\mathcal{M}}$ which refines $\mathcal{M}$ (and so is compatible with it).*

Proof. Let us consider the atlas $\mathcal{M}'$ defined in the following way: for every index $i \in I$ we consider the set U_i as the disjoint union of its connected components $\{U_{ij}\}_{j \in J}$ (where the set of indexes J depends on i); then for every $j \in J$ we define the chart (U_{ij}, ϕ_{ij}) where ϕ_{ij} is just the restriction of ϕ_i to U_{ij} and we consider the family:

$$\mathcal{M}' = \{(U_{ij}, \phi_{ij})\}_{i \in I, j \in J}.$$

By construction every chart of this family is compatible with all the charts of $\mathcal{M}$; moreover, since for every $i \in I$ we have that $U_i = \cup_{j \in J} U_{ij}$, the union of the domains of the charts of $\mathcal{M}'$ covers M. So $\mathcal{M}'$ is an atlas compatible with $\mathcal{M}$. Moreover, this new atlas refines the previous one because the domain of every chart of it is contained in the domain of at least one chart of the previuos one and the transition maps between the domains of the form U_{ij} and the domains of the form U_i are just restrictions of the transition maps of the atlas $\mathcal{M}$, so they are again biholomorphic.

Now let us define the class:

$$\mathcal{F} := \{\text{all the orbifold atlases for the manifold } M$$
$$\text{which refine } \mathcal{M}' \text{and such that they satisfy condition (i)}\}$$

i.e. all the atlases which refine $\mathcal{M}'$ and such that the domain of all their charts are connected. This class is non-empty; indeed it contains at least the atlas $\mathcal{M}'$.

Now on $\mathcal{F}$ we can define the following relation: $\mathcal{M}_1 \preceq \mathcal{M}_2$ iff the following condition holds:

(A) $\mathcal{M}_2$ contains all the charts of the form $(C_{U \cap U'}, \phi|_{C_{U \cap U'}})$ for all the choices of pairs of charts (U, ϕ), (U', ϕ') in $\mathcal{M}_1$ and for all the connected components $C_{U \cap U'}$ of $U \cap U'$ (if any).

In particular, if in (A) we choose $(U,\phi) = (U',\phi')$, we get that $U \cap U' = U$, which is connected (since $\mathcal{M}_1$ belongs to $\mathcal{F}$), so the chart $(C_{U\cap U'}, \phi|_{C_{U\cap U'}})$ is just equal to the chart (U,ϕ), so condition (A) implies:

(B) $\mathcal{M}_2$ contains $\mathcal{M}_1$.

Now we claim that $\preceq$ is actually a relation of order on $\mathcal{F}$; indeed:

- *reflexivity* follows again from the fact that the charts (U,ϕ) of any manifold atlas on $\mathcal{F}$ have all connected domain;
- let $\mathcal{M}_1 \preceq \mathcal{M}_2$ and also $\mathcal{M}_\in \preceq \mathcal{M}_1$. Then by applying (B) we get that $\mathcal{M}_1 \subseteq \mathcal{M}_1$ and $\mathcal{M}_2 \subseteq \mathcal{M}_1$, so the two coincide. Hence *anti-simmetry* is proved;
- let $\mathcal{M}_1 \preceq \mathcal{M}_2 \preceq \mathcal{M}_3$. Then using (B) for the first relation and (A) for the second one, we get that $\mathcal{M}_1 \preceq \mathcal{M}_3$, hence *transitivity* holds.

Now let us prove that every chain in $\mathcal{F}$ has an upper bound, so let us fix any chain $\{\mathcal{M}_i\}_{i\in I} \subseteq \mathcal{F}$ where I is an ordered set such that if $i \leq j$, then $\mathcal{M}_i \preceq \mathcal{M}_j$. In order to find an upper bound, let us define: $\mathcal{A} = \{(V_a,\psi_a)\}_{a\in A}$ as the family of all the charts in all the families $\mathcal{M}_i$ and let $\mathcal{B}$ be the family of all connected components of intersections of pairs of charts in $\mathcal{A}$. To be more precise,

$$\mathcal{B} := \{(C_{V_a,V_b}, \psi_{ab})\}_{a,b\in A}$$

where for every pair of charts (V_a,ψ_a), (V_b,ψ_b) in $\mathcal{A}$ we denote with C_{V_a,V_b} any of the connected components of $V_a \cap V_b$ and ψ_{ab} is the restriction of ψ_a to this set.

By construction, we have that $\mathcal{B}$ belongs to $\mathcal{F}$. Indeed all the domains of its charts are connected by construction; moreover let us fix any chart (C_{V_a,V_b}, ψ_{ab}) in $\mathcal{B}$ and let us suppose that $(V_a,\psi_a) \in \mathcal{M}_i$ and $(V_b,\psi_b) \in \mathcal{M}_j$. By hypothesis $\mathcal{M}_i \in \mathcal{F}$, hence the domain V_a is contained in the domain of a chart (V, ξ) of $\mathcal{M}'$ and the transition map $\xi \circ \psi_a^{-1}$ is biholomorphic. Hence also the transition map:

$$\xi \circ \psi_{ab} : \psi_a(V_a \cap V_b) \to \xi(V_a \cap V_b)$$

is holomorphic. Since this holds for every chart $(V_{ab}, \psi_{ab}) \in \mathcal{B}$, we have proved that $\mathcal{B}$ refines $\mathcal{M}'$. Hence $\mathcal{B}$ belongs to the class $\mathcal{F}$; moreover, by construction it is clear that$\mathcal{M}_i \preceq \mathcal{B}$ for every $i \in I$.

Hence we can apply Zorn's lemma in order to prove that there exists in $\mathcal{F}$ a maximal element $\widetilde{\mathcal{M}}$ with respect to $\preceq$. In particular, (A) implies that for every pair of charts of $\widetilde{\mathcal{M}}$ their "intersection" is again a chart of the atlas, so request (ii) is satisfied. Request (i) is a direct consequence of the fact that $\widetilde{\mathcal{M}}$ belongs to $\mathcal{F}$; hence $\widetilde{\mathcal{M}}$ is an admissible atlas. Moreover, it belongs to $\mathcal{F}$, so it refines $\mathcal{M}'$, which refines $\mathcal{M}$, so the statement is proved. □

Now we are ready to state and prove the following proposition:

Proposition 2.1.12. *Let us fix a second countable paracompact Hausdorff topological space M. Then to every* admissible *manifold atlas $\mathcal{M}$ on M we can associate an orbifold atlas $\overline{\mathcal{M}}$ on the same topological space.*

Proof. Let us suppose that $\mathcal{M} := \{(U_i, \phi_i)\}_{i \in I}$; then for every index $i \in I$ we define the orbifold chart:

$$(\widetilde{U}_i, G_i, \pi_i) := \left(\phi_i(U_i), \{id_{\widetilde{U}_i}\}, \phi_i^{-1}\right) \tag{2.10}$$

By hypothesis $\mathcal{M}$ is admissible, so property (i) of uniformizing systems holds, i.e. for every $i \in I$, U_i is open and connected. Moreover, ϕ_i is an homeomorphism by definition of manifold atlas, so $\widetilde{U}_i$ is an open and non-empty connected set of $\mathbb{C}^n$; moreover, $G_i = \{id_{\widetilde{U}_i}\}$ is a group of holomorphic automorphisms on $\widetilde{U}_i$ and $\pi_i = \phi_i^{-1}$ is continuous and G_i-invariant. In addition, we have that $\widetilde{U}_i/G_i$ is homeomorphic to $\widetilde{U}_i$, which is homeomorphic to U_i via ϕ_i^{-1}.

So also the axiom (ii) of definition 2.1 is satisfied; hence for every $i \in I$ the triple (2.10) defines a reduced uniformizing system for the open set U_i of M. Now we want to prove that:

$$\overline{\mathcal{M}} := \{(\widetilde{U}_i, G_i, \pi_i)\}_{i \in I}$$

is an orbifold atlas on M. By definition of manifold, the union of the U_i covers M and by definition of $\widetilde{U}_i$ we have that $\pi_i(\widetilde{U}_i) = \phi_i^{-1} \circ \phi_i(U_i) = U_i$, so axiom (i) of definition 2.4 is satisfied.

Let us prove that also axiom (ii) holds, so let us suppose that for some pair of indexes $i, j \in I$ we have that $U_i \cap U_j \neq \varnothing$ in M and let us fix a point x in the intersection of them. Now by property (ii) of admissible atlases, there exists an index $k \in I$ and a chart $(U_k, \phi_k) \in \mathcal{M}$ such that $x \in U_k$ and $U_k \subseteq U_i \cap U_j$. Since $\mathcal{M}$ is a manifold altas, we have that this last chart is compatible with (U_i, ϕ_i), so the transition map:

$$\phi_i \circ \phi_k^{-1} : \phi_k(U_k) \to \phi_i(U_k)$$

is a biholormorphism. Hence, if we define:

$$\lambda_{ki} := \phi_i \circ \phi_k^{-1} : \widetilde{U}_k \to \widetilde{U}_i$$

we have that this map is a complex embedding between open sets of $\mathbb{C}^n$. Moreover, since the groups G_k and G_i are both trivial, this map is equivariant, so it is an embedding from $(\widetilde{U}_k, G_k, \pi_k)$ to $(\widetilde{U}_i, G_i, \pi_i)$ in the sense of definition 2.3. In the same way we can define an embedding λ_{kj} from $(\widetilde{U}_k, G_k, \pi_k)$ to $(\widetilde{U}_j, G_j, \pi_j)$, so axiom (ii) of definition 2.4 is satisfied. Hence $\overline{\mathcal{M}}$ is an orbifold atlas. □

Note that the previous construction does not hold for a general atlas, but only for admissible ones.

Remark 2.9. Note that the previuos lemma does not give the uniqueness of $\widetilde{\mathcal{M}}$ because it uses Zorn's lemma. So we are interested in what happens when we choose different admissible atlases (either because we fix different atlases and then we apply proposition 2.1.12, either because we choose a (non admissible) atlas and lemma 2.1.11 gives us more than one admissible atlas which refines it. Then using the previous proposition, we associate to every such atlas an orbifold atlas for M, so the question is: *what is the relationship between the orbifold atlases associated to different (but compatible) admissible manifold atlases?* In order to solve this problem (see proposition 2.5.7 below) we will have to give in orbifold theory a definition analogous to the one of compatibility between manifold atlases. This new definition, called *equivalence of orbifold atlases*, will be the main argument of the last section of this chapter.

Example 2.1. The non-uniqueness of the admissible atlas can be found easily in dimension 1. Let us consider the real line $\mathbb{R}$ with the euclidean topology: it has clearly an admissible atlas given by the only chart $(\mathbb{R}, id)$; however, this manifold has also the following admissible atlas:

$$\mathcal{M} := \{(]2n, 2n+2[, id)\}_{n \in \mathbb{Z}} \cup \{(]2n-1, 2n+1[, id)\}_{n \in \mathbb{Z}} \cup \{(]m, m+1[, id)\}_{m \in \mathbb{Z}}$$

Note that the union of the first two families is already an atlas on $\mathbb{R}$, but it does not contain the charts on the intersections of two adjacent domains, which are exactly the charts of the third family. Adding ths family gives us an admissible manifold atlas because the last charts added don't intersect with each other and are completely contained in the charts of the first two families.

In the introduction of this work we said that orbifolds arise often in literature as global quotients of manifolds under the action of holomorphic (or smooth) actions of finite groups. The following proposition proves this fact.

Proposition 2.1.13. *Let $\mathcal{M}$ be a manifold atlas on a topological space M and let G be a finite groups that acts* effectively *on M as a group of holomor-*

phic automorphisms. Then we can associate to these data an orbifold atlas $\mathcal{A}$ *for the global quotient* M/G.

Proof. Let $\mathcal{M} = \{(U_i, \phi_i)\}_{i\in I}$; for simplicity, we will denote with capital letters (i.e: $P, Q, ..$) the points of the quotient space M/G and with $\bar{P}, \bar{P}', \cdots$, their pre-images in M. For every index $i \in I$ and for every point $\bar{P}$ in U_i we set $\tilde{P} := \phi_i(\bar{P}) \in \phi_i(U_i) =: \widetilde{U}_i \subseteq \mathbb{C}^n$.

Moreover, for every $\bar{P} \in U_i$ we define the isotropy subgroup $G_{\bar{P}}$ to be the subgroup of the automorphisms of G which fix the point $\bar{P}$. As in some constructions of the previuos propositions, if we consider the finite intersection:

$$U_{i,G_{\bar{P}}} := \bigcap_{g\in G_{\bar{P}}} g(U_i)$$

we get that this set is open in M and contains the point $\bar{P}$. Moreover, it is invariant under the action of the isotropy group $G_{\bar{P}}$. Now we consider its image via the coordinate function ϕ_i. This is an open set in $\mathbb{C}^n$ and contains the point $\tilde{P} = \phi_i(\bar{P})$. We introduce also the group:

$$\tilde{G}_{\tilde{P}} := \{\phi_i \circ g \circ \phi_i^{-1}\}_{g\in G_{\bar{P}}};$$

by hypothesis the set $G_{\bar{P}}$ consists of holomorphic automorphisms of M, i.e. holomorphic automorphisms if composed with the coordinates functions. Then the group just defined is made of holomorphic automorphisms of $\widetilde{U}_i$. Moreover, by construction the set $\phi_i(U_{i,G_{\bar{P}}})$ is invariant under the action of this group, which in particular fixes the point $\tilde{P}$. Lastly, for every open neighborhood $\widetilde{V} \subseteq \widetilde{U}_i$ of a point $\tilde{P}$ we define:

$$\widetilde{V}_{i,\tilde{P}} := \text{ connected component of } \bigcap_{\tilde{g}\in\tilde{G}_{\tilde{P}}} \tilde{g}(\widetilde{V}) \text{ which contains } \tilde{P}.$$

By construction this set is open, non-empty (it contains $\tilde{P}$) and invariant under the action of the group $\tilde{G}_{\tilde{P}}$. Then we can define the triple:

$$(\widetilde{V}_{i,\tilde{P}}, \tilde{G}_{\tilde{P}}, \pi_i) \tag{2.11}$$

where $\pi_i := \pi \circ \phi_i^{-1}|_{\widetilde{V}_{i,\tilde{P}}}$ and $\pi : M \to M/G$ is the quotient map. By definition of $\tilde{G}_{\tilde{P}}$ we have that $\widetilde{V}_{i,\tilde{P}}/\tilde{G}_{\tilde{P}}$ is homeomorphic to $\pi_i(\widetilde{V}_{i,\tilde{P}})$. Moreover, one can prove as in the previuous constructions that this last set is open in M/G, so (2.11) is a uniformizing system for an open neighborhood of P in M/G. Then we are ready to define the orbifold atlas:

$$\mathcal{A} := \{(\widetilde{V}_{i,\tilde{P}}, \tilde{G}_{\tilde{P}}, \pi_i)\}_{P \in M/G} \tag{2.12}$$

where for every point $P \in M/G$ the family $\mathcal{A}$ is indexed over the following variables:

- $\bar{P}$ is chosen as *one* of the preimages of P and the index $i \in I$ is chosen as *one* of the indexes such that $\bar{P} \in U_i$. Using the previous notation, here $\tilde{P} = \phi_i(\bar{P})$;
- V varies over *all* the open neighborhoods of $\tilde{P}$ contained in $\widetilde{U}_i$.

By construction, the sets $\pi_i(\widetilde{V}_{i,\tilde{P}}) =: U_{i,P}$ are an open cover of M/G, so axiom (i) of definition 2.4 is satisfied. Now let us prove also axiom (ii), so let us fix two uniformizing systems:

$$(\widetilde{V}_{i,\tilde{P}}, \tilde{G}_{\tilde{P}}, \pi_i) \quad \text{and} \quad (\widetilde{V}_{j,\tilde{Q}}, \tilde{G}_{\tilde{Q}}, \pi_j)$$

in $\mathcal{A}$ such that $U_{i,P} \cap U_{j,Q} \neq \varnothing$ and let R be a point in the intersection. Then by definition of π there exists a pair of points (not necessarily coinciding) $\bar{R}, \bar{R}'$ such that $\pi(\bar{R}) = R = \pi(\bar{R}')$ and such that $\tilde{R} := \phi_i(\bar{R}) \in \widetilde{V}_{i,\tilde{P}}$ and $\tilde{R}' := \phi_j(\bar{R}') \in \widetilde{W}_{j,\tilde{Q}}$. Moreover, by definition of π there exists a (not necessarily unique) $g_0 \in G$ such that $g_0(\bar{R}') = \bar{R}$, hence the set:

$$A := g_0(\phi_j^{-1}(\widetilde{V}_{j,\tilde{Q}})) \cap (\phi_i^{-1}(\widetilde{V}_{i,\tilde{P}}))$$

is open and non-empty (it contains $\bar{R}$). Hence its image in $\widetilde{V}_{i,\tilde{P}}$ via the homeomorphism ϕ_i is an open neighborhood of the point $\tilde{R}$. If we denote this set with A, it makes sense to consider the orbifold chart:

$$(\widetilde{A}_{i,\tilde{R}}, \tilde{G}_{\tilde{R}}, \pi_i) \in \mathcal{A}.$$

This chart comes with a natural embedding λ from it into $(\widetilde{V}_{i,\tilde{P}}, \tilde{G}_{\tilde{P}}, \pi_i)$ given by the inclusion of $\widetilde{A}_{i,\tilde{R}}$ in $\widetilde{V}_{i,\tilde{P}}$. Let us also define the map:

$$\mu := \phi_j \circ g_0^{-1} \circ \phi_i^{-1} : \widetilde{A}_{i,\tilde{R}} \rightarrow \widetilde{W}_{j,\tilde{Q}};$$

since g_0^{-1} is an holomorphic automorphism of M, then μ (which is just g_0^{-1} in coordinates) is an holomorphic embedding. Moreover, it commutes with the projection maps π_i and π_j, so μ represents an embedding of orbifolds:

$$\mu : (\widetilde{A}_{i,\tilde{R}}, \tilde{G}_{\tilde{R}}, \pi_i) \rightarrow (\widetilde{V}_{j,\tilde{Q}}, \tilde{G}_{\tilde{Q}}, \pi_j).$$

Hence we have proved that axiom (ii) of definition 2.4 is satisfied, so $\mathcal{A}$ is an orbifold atlas for the topological space M/G. The only thing we haven't yet proved is the fact that the topological space satisfies the definition of orbifold atlases. In particular, M/G is paracompact and second countable because it is the quotient of the paracompact and second countable space M, but it is not obviuos that it is also Hausdorff, so this is the last thing we have to prove.

In order to do that, let us fix any pair of points $P \neq Q$ in M/G and let us call $\{\widetilde{P}_1, \cdots, \widetilde{P}_k\}$ and $\{\widetilde{Q}_1, \cdots, \widetilde{Q}_l\}$ their preimages in M via the quotient map $\pi : M \rightarrow M/G$. These two sets are both finite (even if in general not equal), because their cardinality is at most the cardinality of G. Moreover, since $P \neq Q$, they have empty intersection. Hence, since M is a manifold (hence Hausdorff), for every pair of points $\widetilde{P}_i, \widetilde{Q}_j$ there exists open disjoint neighborhoods V_{ij} of $\widetilde{P}_i$ and W_{ij} of $\widetilde{Q}_j$. So we can define the *finite* intersections:

$$V_i := \bigcap_{j=1,\cdots,l} V_{ij} \quad \text{and} \quad W_j := \bigcap_{i=1,\cdots,l} W_{ij};$$

and we get that every V_i is an open neighborhood of $\widetilde{P}_i$ which is disjoint by every W_{ij}, and hence by every W_i. In the same way, W_j is an open neighborhood of $\widetilde{Q}_j$ which is disjoint by every V_i. Hence we have that the sets:

$$V := \bigcup_{i=1,\cdots,k} V_i \quad \text{and} \quad W := \bigcup_{j=1,\cdots,l} W_j$$

are open disjoint sets; the first one contains all the preimages of P, while the second one contains every preimage of Q. Now we can define the sets:

$$\widetilde{V} := \bigcap_{g\in G} g(V) \quad \text{and} \quad \widetilde{W} := \bigcap_{g\in G} g(W)$$

which are both finite intersections of open sets, hence again open. Moreover, a direct check shows that they are both saturated to respect to the action of the group G and that the first one contains again all the points $\{\widetilde{P}_1, \cdots, \widetilde{P}_k\}$ and the second one contains again the set $\{\widetilde{Q}_1, \cdots, \widetilde{Q}_l\}$. Moreover, $\widetilde{V} \subseteq V$ and $\widetilde{W} \subseteq W$, hence $\widetilde{V} \cap \widetilde{W} = \varnothing$.

So the sets $\pi(\widetilde{V})$ and $\pi(\widetilde{W})$ in M/G are open disjoint neighborhoods of P and Q respectively; hence the topological space M/G is Hausdorff. $\square$

Remark 2.10. Note that in the special case when $G = \{id_M\}$ we have obtained an alternative proof of the fact that every manifold can be considered as an orbifold. Actually also in this proof we implicitly used the notion of admissible atlas.

Remark 2.11. The construction of the orbifold atlas $\mathcal{A}$ depends strictly on the choice of the manifold atlas $\mathcal{M}$, so *what happens if we obtain an orbifold atlas $\mathcal{A}'$ from the previuos proposition applied to another altas $\mathcal{M}'$ compatible with $\mathcal{M}$?*

This problem will be solved in the last section of this chapter, proposition 2.5.8.

Example 2.2. Let us fix an integer n and positive integer numbers $a_0, \cdots, a_n$; then we define the action of $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$ on $\mathbb{C}^{n+1} \smallsetminus \{0\}$ given for every $\lambda \in \mathbb{C}^*$ by:

$$\begin{array}{rccc} \lambda : & \mathbb{C}^{n+1} \smallsetminus \{0\} & \rightarrow & \mathbb{C}^{n+1} \smallsetminus \{0\} \\ & (z_0, \cdots, z_n) & \rightarrow & (\lambda^{a_0} z_0, \cdots, \lambda^{a_n} z_n) \end{array}$$

and we denote with $W\mathbb{P}(a_0, \cdots, a_n)$ the quotient space $\mathbb{C}^{n+1} \smallsetminus \{0\}/\mathbb{C}^*$, called the *weighted projective space* with weights $(a_0, \cdots, a_n)$; its points will be usually denoted with $[z_0 : \cdots : z_n]$.

Note that this is not a special case of the previous proposition because here $\mathbb{C}^*$ is not finite. In the special case when all the weights are equal, we obtain the usual complex projective space $\mathbb{P}^n$ (in particular, this is obvious in the case when alle the weights are equal to one); in the general case, we can't obtain a manifold structure,but we can adapt the construction of the charts for the projective space in order to describe an orbifold structure on this topological space.

Let us call $\pi : \mathbb{C}^{n+1} \smallsetminus \{0\} \rightarrow W\mathbb{P}(a_0, \cdots, a_n)$ the quotient map and let $X_i := \{z_i \neq 0\} \subset \mathbb{C}^{n+1} \smallsetminus \{0\}$; this set is clearly open and saturated with respect to the action of $\mathbb{C}^*$, so if we call $U_i := \pi(X_i)$ we get that this set is open in the weighted projective space.

Now for every $i = 0, \cdots, n$, let $\widetilde{U}_i$ be equal to $\mathbb{C}^n$ with coordinates $(z_0, \cdots, \hat{z}_i, \cdots, z_n)$ and let us define the group $G_i := \{\mu \in \mathbb{C}^n \text{ s.t. } \mu^{a_i} = 1\}$, i.e. the group of a_i-th roots of unity. This is a *finite* group that acts on $\widetilde{U}_i$ as follows: for every $\mu \in G_i$ and for every point $(z_0, \cdots, \hat{z}_i, \cdots, z_n)$ in X_i we define the action:

$$\mu(z_0, \cdots, \hat{z}_i, \cdots, z_n) := (\mu^{a_0} z_0, \cdots, \mu^{a_n} z_n). \tag{2.13}$$

Now let us define the inclusion map:

$$\begin{array}{cccc} \phi_i : & \widetilde{U}_i & \rightarrow & \mathbb{C}^{n+1} \smallsetminus \{0\} \\ & (z_0, \cdots, \hat{z}_i, \cdots, z_n) & \rightarrow & (z_0, \cdots, 1, \cdots z_n) \end{array}$$

and let us set $\pi_i := \pi \circ \phi_i : \widetilde{U}_i \rightarrow U_i$. This map is continuous because it is the composition of two continuous maps (π is continuous by definition of quotient topology); moreover, it is surjective, indeed if we consider any point of the form $[z_0 : \cdots : z_n] \in U_i$, then $z_i \neq 0$, so if we choose $\lambda \in \mathbb{C}^*$ such that $\lambda^{a_i} = 1/z_i$ we get that this point is also equal to $[\lambda^{a_0} z_0 : \cdots : 1 : \cdots : \lambda^{a_n} z_n]$, which is equal to $\pi_i(\lambda^{a_0} z_0, \cdots, \hat{z}_i, \cdots, \lambda^{a_n} z_n)$.

Moreover, two points $(z_0, \cdots, \hat{z}_i, \cdots, z_n)$ and $(w_0, \cdots, \hat{w}_i, \cdots, w_n)$ in $\widetilde{U}_i$ are identified in U_i by π_i if and only if there exists $\lambda \in \mathbb{C}^*$ such that:

$$\lambda(z_0, \cdots, 1, \cdots, z_n) = (w_0, \cdots, 1, \cdots, w_n)$$

i.e. if and only if:

$$\begin{cases} \lambda^{a_i} = 1 \Leftrightarrow \lambda \in G_i \\ \lambda(z_0, \cdots, \hat{z}_i, \cdots, z_n) = (w_0, \cdots, \hat{w}_i, \cdots, w_n) \end{cases}$$

where the last equation is just the action of G_i on $\widetilde{U}_i$ as in (2.13). Hence $\widetilde{U}_i / G_i$ is homeomorphic to U_i via π_i, so we have proved that $(\widetilde{U}_i, G_i, \pi_i)$ is a uniformizing system for the open set $U_i \subseteq W\mathbb{P}(a_0, \cdots, a_n)$. Clearly the family of all these charts (indexed on $i = 0, \cdots, n$) covers the weighted projective space, but it is not an orbifold atlas because it does not satisfies condition (ii) of definition 2.4.

In order to satisfy also this condition, we have to proceed in this way: first of all, for every pair of indexes $i \neq j \in \{0, \cdots, n\}$ we set:

$$\widetilde{U}_{ij} := \{\tilde{z} = (z_0, \cdots \hat{z}_i, \cdots, z_n) \in \widetilde{U}_i \text{ s.t. } z_j \neq 0\}$$

and we define the function $\phi_{ij} : \widetilde{U}_{ij} \to \widetilde{U}_{ji}$ as:

$$\phi_{ij}(z_0, \cdots, \hat{z}_i, \cdots, z_j, \cdots, z_n) := \left(\frac{z_0}{z_j^{a_0/a_j}}, \cdots, \frac{1}{z_j^{a_i/a_j}} \cdots, \hat{z}_j, \cdots, \frac{z_n}{z_j^{a_n/a_j}} \right)$$

where here z_j^{1/a_j} is chosen to be any of the a_j-th roots of z_j (here we assume that $i < j$, if not we have to permute the coordinate expression of ϕ_{ij}). This map is clearly holomorphic because on $\widetilde{U}_{ij}$ the coordinate z_j is everywhere different from zero. Moreover, this function is invertible with holomorphic inverse given by ϕ_{ji}, which has the same formal expression of ϕ_{ij} except for the order of the coordinates:

$$\phi_{ji}(z_o, \cdots, z_i, \cdots, \hat{z}_j, \cdots, z_n) = \left(\frac{z_0}{z_i^{a_0/a_i}}, \cdots, \hat{z}_i, \cdots, \frac{1}{z_i^{a_j/a_i}}, \cdots, \frac{z_n}{z_i^{a_n/a_i}} \right).$$

This function is again holomorphic and it is a direct check to prove that it is the inverse of ϕ_{ij}. Hence we have proved that for every $i \neq j$ (and not only $i < j$) the function ϕ_{ij} is biholomorphic and that $\phi_{ij}^{-1} = \phi_{ji}$.

Now let us fix any point $z := [z_o : \cdots : z_n]$ and let us *choose* an index $i = 0, \cdots, n$ such that $z \in U_i$ (in general, this index is not unique). In other words, we choose i such that $z_i \neq 0$, so without loss of generality $z_i = 1$ and let us *choose* a fixed "representant" $\tilde{z} := (z_0, \cdots, \hat{z}_i, \cdots, z_n) \in \widetilde{U}_i$ for it. Then let $G_{i,\tilde{z}}$ be its stabilizer subgroup with respect to the action of the group G_i. Then *for every open neighborhood* V of this point in $\widetilde{U}_i$ we define:

$$\widetilde{V}_{i,\tilde{z}} := \text{connected component of } \bigcap_{g \in G_{i,\tilde{z}}} g(V) \text{ which contains } \tilde{z}.$$

As usual, this set is an open neighborhood of $\tilde{z}$ and it is invariant under the action of the group $G_{i,\tilde{z}}$. It is easy to prove that the triple $(\widetilde{V}_{i,\tilde{z}}, G_{i,\tilde{z}}, \pi_i)$

(where here π_i is the restriction of π_i to $\widetilde{V}_{i,\tilde{z}}$) is a uniformizing system for the open set $\pi_i(\widetilde{V}_{i,\tilde{z}}) \subseteq W\mathbb{P}(a_0, \cdots, a_n)$.

Now we define the familiy:

$$\mathcal{A} := \{(\widetilde{V}_{i,\tilde{z}}, G_{i,\tilde{z}}, \pi_i)\}$$

indexed over:

- all the points $z \in W\mathbb{P}(a_0, \cdots, a_n)$; for any such point, we choose *one* index i and *one* point $\tilde{z} \in \widetilde{U}_i$ as described before;

- all the open neighborhoods V of $\tilde{z}$ in $\widetilde{U}_i$; for every such neighborhood $\widetilde{V}_{i,\tilde{z}}$ is constructed as before.

This family clearly satisfies axiom (i), so let us prove only axiom (ii) of definition 2.4; so let us fix two orbifold charts:

$$(\widetilde{V}_{i,\tilde{z}}, G_{i,\tilde{z}}, \pi_i) \quad \text{and} \quad (\widetilde{V}_{j,\tilde{w}}, G_{j,\tilde{w}}, \pi_j)$$

such that $A := \pi_i(\widetilde{V}_{i,\tilde{z}}) \cap \pi_j(\widetilde{V}_{j,\tilde{w}}) \neq \varnothing$ in the weighted projective space and let us fix a point $x = [x_0 : \cdots : x_n]$ in this intersection. By definition of the family $\mathcal{A}$ for the point x we have chosen an index $k \in \{0, \cdots, n\}$ and a point $\tilde{x} \in \widetilde{U}_k$ such that $\pi_k(\tilde{x}) = x$. Then the set $B := \pi_k^{-1}(A)$ is an open neighborhood of $\tilde{x}$ in $\widetilde{U}_k$, so the family $\mathcal{A}$ contains an orbifold chart of the form $(\widetilde{B}_{k,\tilde{x}}, G_{k,\tilde{x}}, \pi_k)$. By construction $\widetilde{B}_{k,\tilde{x}}$ is an open connected neighborhood of $\tilde{x}$ completely contained in $\widetilde{U}_k$, so it makes sense to define the set maps:

$$\lambda := \phi_{ki}|_{\widetilde{B}_{k,\tilde{x}}} : \widetilde{B}_{k,\tilde{x}} \to \widetilde{V}_{i.\tilde{z}} \quad \text{and} \quad \mu := \phi_{kj}|_{\widetilde{B}_{k,\tilde{x}}} : \widetilde{B}_{k,\tilde{x}} \to \widetilde{V}'_{j.\tilde{w}}$$

which are holomorphic embeddings because restrictions (in domain) of biholomorphic maps. Moreover, these maps commute with the projection maps, i.e. the following diagrams are commutative:

$$\begin{tikzcd}
\widetilde{B}_{k,\tilde{x}} \arrow[r, "\lambda"] \arrow[d, "\pi_k"'] \arrow[dr, phantom, "\curvearrowright"] & \widetilde{V}_{i,\tilde{z}} \arrow[d, "\pi_i"] \\
\pi_k(\widetilde{B}_{k,\tilde{x}}) \arrow[r, hook] & \pi_i(\widetilde{V}_{i,\tilde{z}})
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{B}_{k,\tilde{x}} \arrow[r, "\mu"] \arrow[d, "\pi_k"'] \arrow[dr, phantom, "\curvearrowright"] & \widetilde{V}_{j,\tilde{w}} \arrow[d, "\pi_j"] \\
\pi_k(\widetilde{B}_{k,\tilde{x}}) \arrow[r, hook] & \pi_j(\widetilde{V}_{j,\tilde{w}}).
\end{tikzcd}$$

Let us prove this fact only for the first diagram, the second one is analogous; without loss of generality, let us suppose that $k < i$ and let us fix a point $(z_0, \cdots, \hat{z}_k, \cdots, z_i, \cdots, z_n)$ in $\widetilde{B}_{k,\tilde{x}}$. Then:

$$\pi_k(z_0, \cdots, \hat{z}_k, \cdots, z_i, \cdots, z_n) = [z_0 : \cdots : 1 \cdots : z_i : \cdots : z_n] \tag{2.14}$$

and

$$\phi_{ki}(z_0, \cdots, \hat{z}_k, \cdots, z_i, \cdots, z_n) = \left(\frac{z_0}{z_i^{a_0/a_i}}, \cdots, \frac{1}{z_i^{a_k/a_i}}, \cdots, \hat{z}_i, \cdots, \frac{z_n}{z_i^{a_n/a_i}} \right)$$

so:

$$\pi_i \circ \phi_{ki}(z_0, \cdots, \hat{z}_k, \cdots, z_i, \cdots, z_n) = \left[\frac{z_0}{z_i^{a_0/a_i}} : \cdots : \frac{1}{z_i^{a_k/a_i}} : \cdots : 1 : \cdots : \frac{z_n}{z_i^{a_n/a_i}} \right] \tag{2.15}$$

Now if we choose $\lambda = z_i^{1/a_i} \in \mathbb{C}^*$ we get that (2.14) is equal to (2.15), so the first diagram commutes. In the same way one can prove that also the second diagram commutes, hence λ and μ are embeddings in the sense of orbifolds:

$$(\widetilde{V}_{j,\tilde{w}}, G_{j,\tilde{w}}, \pi_j) \xleftarrow{\mu} (\widetilde{B}_{k,\tilde{x}}, G_{k,\tilde{x}}, \pi_k) \xrightarrow{\lambda} (\widetilde{V}_{i,\tilde{z}}, G_{i,\tilde{z}}, \pi_i).$$

So we have proved that the weighted projective space $W\mathbb{P}(a_0, \cdots, a_n)$ has a natural structure of orbifold atlas. One can easily see that this structure just reduces to the manifold structure on the complex projective space $\mathbb{CP}^n$

in the case when all the weights are equal to 1.

A special case of weighted projective space is the *teardrop orbifold* $W\mathbb{P}(1,2)$. This is the most simple case of a manifold (in this case the 2-sphere) with only a "labelled" point; for the details, see for example [LU2], example 2.2.

Example 2.3. *Elliptic curves* are probably the first known example of orbifold. For a discussion about these objects, see for example [K] or [Si]; for our purposes here we can use the following standard description of these objects: the classification of elliptic curves can be completely reduced to the classification of the set of "parameters" $\lambda \in \mathbb{H}$ where $\mathbb{H}$ is the Poincaré half complex plane:

$$\mathbb{H} := \{\lambda \in \mathbb{C} \text{ s.t. } \operatorname{Im}(\lambda) > 0\}.$$

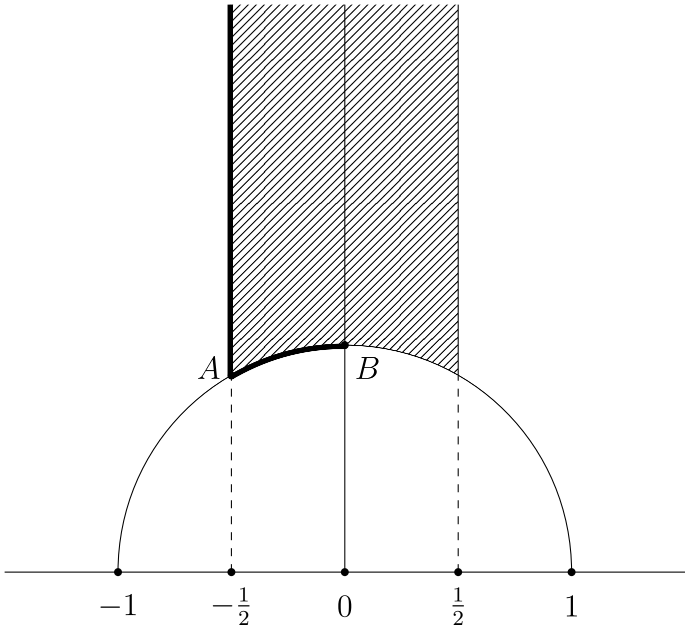
Two different parameters λ, λ' describe the same curve if and only if there exists a matrix $A \in SL_2(\mathbb{Z})$ that takes λ to λ', where if we fix a matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

we define its action on $\lambda \in \mathbb{H}$ as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \lambda := \frac{a\lambda + b}{c\lambda + d}.$$

So it is natural to consider the quotient space $\mathbb{H}/SL_2(\mathbb{Z})$. Here the group of holomorphic automorphisms $SL_2(\mathbb{Z})$ on $\mathbb{H}$ is not finite, so again we can't apply directly proposition 2.5.8. However, one can easily see that a fundamental domain of this action is the following:



In other words, this set is defined by the following two conditions:

- $-\frac{1}{2} \leq \lambda < \frac{1}{2}$;
- $\begin{cases} |\lambda| \geq 1 & \text{if } \mathrm{Re}(\lambda) \leq 0 \\ |\lambda| > 1 & \text{if } \mathrm{Re}(\lambda) > 0 \end{cases}$

One can easily see that this set contains no equivalent points. Moreover, the stabilizers of every point are trivial except for the points labeled with A and B. The first one is fixed by a cyclic group of order 4 (this point corresponds to the square lattice), while the second one is fixed by a cyclic group of order 6 (this corresponds to the hexagonal lattice). For more details, see for example [HC], [K] and [Si].

2.2 Local liftings and compatible systems

Now our aim is to make orbifolds into a category, i.e. we want to define what a morphism between orbifolds is. In order to have such a morphism we have first of all to define a continuous map between the underlying topological spaces, but differently from the case of morphisms between manifolds, this will not be sufficent in general. The idea to keep in mind in the following definitions is that a morphism between orbifolds is essentially a continuous function which can be *locally lifted to a holomorphic* function between uniformizing systems in source and target.

First of all, let us consider any fixed orbifold atlas $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i \in I}$ on X. *Such an atlas can be considered as a category as follows*:

$$\mathcal{U} = \begin{cases} \text{objects:} & \text{uniformizing systems } (\widetilde{U}_i, G_i, \pi_i) \\ \text{morphisms:} & \text{embeddings } \lambda_{ij} : (\widetilde{U}_i, G_i, \pi_i) \to (\widetilde{U}_j, G_j, \pi_j) \end{cases}$$

with identity morphisms $1_{\widetilde{U}_i}$ for any uniformizing system $(\widetilde{U}_i, G_i, \pi_i)$ and composition defined in the obvious way. Note that in order to have an atlas we have to add some additional properties on the category $\mathcal{U}$ as in definition (2.1). Then we are ready to give the following:

Definition 2.7. Let $\mathcal{U}$ and $\mathcal{V}$ be atlases for X and Y respectively and let $U \subseteq X$ and $V \subseteq Y$ be open sets with uniformizing systems $(\widetilde{U}, G, \pi) \in \mathcal{U}$ and $(\widetilde{V}, H, \phi) \in \mathcal{V}$ respectively. Let $f : U \to V$ be a continuous function; then a *lifting of* f from $(\widetilde{U}, G, \pi)$ *to* $(\widetilde{V}, H, \phi)$ is a holomorphic function $\tilde{f}_{\widetilde{U},\widetilde{V}} : \widetilde{U} \to \widetilde{V}$ such that:

$$\phi \circ \tilde{f}_{\widetilde{U},\widetilde{V}} = f \circ \pi. \tag{2.16}$$

Definition 2.8. Let $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i \in I}$ and $\mathcal{V} = \{(\widetilde{V}_j, H_j, \phi_j)\}_{j \in J}$ be atlases (not necessarily of the same dimension) for X and Y respectively and let $f : X \to Y$ be a continuous function between topological spaces. Then a *compatible system* for f is the datum of:

(1) a functor $\tilde{f} : \mathcal{U} \to \mathcal{V}$ between the associated categories such that if we call $(\widetilde{V}_i, H_i, \phi_i) \in \mathcal{V}$ the image of any element $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ via $\tilde{f}$, we have $f(\pi_i(\widetilde{U}_i)) \subseteq \phi_i(\widetilde{V}_i)$;

(2) a collection $\{\tilde{f}_{\widetilde{U}_i,\widetilde{V}_i}\}_{(\widetilde{U}_i,G_i,\pi_i)\in\mathcal{U}}$ where for any $(\widetilde{U}_i,G_i,\pi_i)\in\mathcal{U}$ we have that $\tilde{f}_{\widetilde{U}_i,\widetilde{V}_i}$ is a lifting for the continuos function $f|_{U_i}: U_i \to f(U_i) \subseteq V_i$ from $(\widetilde{U}_i,G_i,\pi_i)\in\mathcal{U}$ to $(\widetilde{V}_i,H_i,\phi_i)\in\mathcal{V}$;

such that the following condition holds: for every embedding $\lambda_{ij}:(\widetilde{U}_i,G_i,\pi_i)\to(\widetilde{U}_j,G_j,\pi_j)$ in $\mathcal{U}$, we have:

$$\tilde{f}_{\widetilde{U}_j,\widetilde{V}_j}\circ\lambda_{ij} = \tilde{f}(\lambda_{ij})\circ\tilde{f}_{\widetilde{U}_i,\widetilde{V}_i} \tag{2.17}$$

i.e. we are in the following situation:

$$\begin{tikzcd}
\widetilde{U}_i \arrow[rr, "\tilde{f}_{\widetilde{U}_i,\widetilde{V}_i}"] \arrow[dr, "\lambda_{ij}"'] \arrow[dd, "\pi_i"'] & & \widetilde{V}_i \arrow[dr, "\tilde{f}(\lambda_{ij})"] \arrow[dd, "\phi_i" near end] & \\
& \widetilde{U}_j \arrow[rr, "\tilde{f}_{\widetilde{U}_j,\widetilde{V}_j}"] \arrow[dd, "\pi_j"' near start] & & \widetilde{V}_j \arrow[dd, "\phi_j"] \\
U_i \arrow[rr, "f_{|U_i}"' near end] \arrow[dr, hook] & & V_i \arrow[dr, hook] & \\
& U_j \arrow[rr, "f_{|U_j}"'] & & V_j
\end{tikzcd}$$

where *all the faces of the cube are commutative*; indeed:

- the lower face is commutative because we are just restricting the function f from U_j to U_i;
- the left and right sides are commutative by definition of embedding in $\mathcal{U}$ and $\mathcal{V}$ respectively;
- the front and back sides are both commutatives because of (2.16);
- the top side is commutative because of (2.17).

With a little abuse of notation, we will always write $\tilde{f} : \mathcal{U} \to \mathcal{V}$ to denote a compatible system for f, i.e. with $\tilde{f}$ we will usually mean not only the functor which satisfies (1), but also the collection of local liftings described in (2).

Definition 2.9. (*composition of compatible systems*) Now let us consider 3 fixed orbifolds atlases $\mathcal{U}, \mathcal{V}, \mathcal{W}$ for X, Y and Z respectively, together with 2 continuous functions $f : X \to Y$, $g : Y \to Z$ and compatible systems $\tilde{f} : \mathcal{U} \to \mathcal{V}$ and $\tilde{g} : \mathcal{V} \to \mathcal{W}$. For every uniformizing system $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$, let us call:

$$(\widetilde{V}_i, H_i, \phi_i) := \tilde{f}(\widetilde{U}_i, G_i, \pi_i) \quad \text{and} \quad (\widetilde{W}_i, K_i, \xi_i) := \tilde{g}(\widetilde{V}_i, H_i, \phi_i).$$

Then we define the compatible system $\tilde{g} \circ \tilde{f}$ as the functor $\tilde{g} \circ \tilde{f} : \mathcal{U} \to \mathcal{W}$ together with the collection of liftings:

$$\left\{ (\tilde{g} \circ \tilde{f})_{\widetilde{U}_i, \widetilde{W}_i} := \tilde{g}_{\widetilde{V}_i, \widetilde{W}_i} \circ \tilde{f}_{\widetilde{U}_i, \widetilde{V}_i} \right\}_{(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}}$$

for the continuous map $g \circ f$.

Let us prove that actually this is a compatible system for $g \circ f$: for any embedding $\lambda_{ij} : (\widetilde{U}_i, G_i, \pi_i) \to (\widetilde{U}_j, G_j, \pi_j)$ we have the following commutative diagram:

$$\begin{tikzcd}
(\widetilde{U}_i, G_i, \pi_i) \arrow[r, "\tilde{f}_{\widetilde{U}_i, \widetilde{V}_i}"] \arrow[d, "\lambda_{ij}"'] \arrow[dr, phantom, "\curvearrowright"] & (\widetilde{V}_i, H_i, \phi_i) \arrow[r, "\tilde{g}_{\widetilde{V}_i, \widetilde{W}_i}"] \arrow[d, "\tilde{f}(\lambda_{ij})"] \arrow[dr, phantom, "\curvearrowright"] & (\widetilde{W}_i, K_i, \xi_i) \arrow[d, "\tilde{g}(\tilde{f}(\lambda_{ij}))"] \\
(\widetilde{U}_j, G_j, \pi_j) \arrow[r, "\tilde{f}_{\widetilde{U}_j, \widetilde{V}_j}"'] & (\widetilde{V}_j, H_j, \phi_j) \arrow[r, "\tilde{g}_{\widetilde{V}_j, \widetilde{W}_j}"'] & (\widetilde{W}_j, K_j, \xi_j);
\end{tikzcd}$$

so condition (2.17) holds. Hence we have actually defined a compatible system for $g \circ f$.

2.3 Natural transformations between compatible systems

The following definition is a slight change of definition 1.3.6 in [Pe], which I think was too much restrictive for our purposes.

Definition 2.10. Let us fix atlases $\mathcal{U}$ and $\mathcal{V}$ for X and Y respectively and let $\tilde{f}_1, \tilde{f}_2 : \mathcal{U} \to \mathcal{V}$ be compatible systems for *the same* continuous map $f : X \to Y$. For simplicity, for every uniformizing system $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ and for every embedding λ_{ij}, let us call:

$$(\widetilde{V}_i^m, H_i^m, \phi_i^m) := \tilde{f}_m(\widetilde{U}_i, G_i, \pi_i) \quad \text{for} \quad m = 1, 2$$

$$\text{and} \quad \lambda_{ij}^m := \tilde{f}_m(\lambda_{ij}) \quad \text{for} \quad m = 1, 2.$$

Then a *natural transformation of compatible systems from* $\tilde{f}_1$ *to* $\tilde{f}_2$ is the datum of a family:

$$\left\{\delta_{\widetilde{U}_i} = \delta_{(\widetilde{U}_i, G_i, \pi_i)} : (\widetilde{V}_i^1, H_i^1, \phi_i^1) \to (\widetilde{V}_i^2, H_i^2, \phi_i^2)\right\}_{(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}}$$

of embeddings in $\mathcal{V}$, such that:

(i) for every $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we have:

$$(\tilde{f}_2)_{\widetilde{U}_i, \widetilde{V}_i^2} = \delta_{\widetilde{U}_i} \circ (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1};$$

(ii) for every embedding λ_{ij} in $\mathcal{U}$ the following diagram of embeddings in $\mathcal{V}$ is commutative:

$$\begin{tikzcd}
(\widetilde{V}_i^1, H_i^1, \phi_i^1) \arrow[r, "\delta_{\widetilde{U}_i}"] \arrow[d, "\lambda_{ij}^1"'] \arrow[dr, phantom, "\curvearrowright"] & (\widetilde{V}_i^2, H_i^2, \phi_i^2) \arrow[d, "\lambda_{ij}^2"] \\
(\widetilde{V}_j^1, H_j^1, \phi_j^1) \arrow[r, "\delta_{\widetilde{U}_j}"'] & (\widetilde{V}_j^2, H_j^2, \phi_j^2).
\end{tikzcd} \tag{2.18}$$

Whenever we have a natural transformation δ between compatible systems from $\tilde{f}_1$ to $\tilde{f}_2$, we will denote it as $\delta : \tilde{f}_1 \Rightarrow \tilde{f}_2$.

Remark 2.12. If we ignore the additional properties of the compatible systems $\tilde{f}_1$ and $\tilde{f}_2$ and we consider them just as functors, we get that condition (ii) is just the description of a natural transformation from the functor $\tilde{f}_1$ to the functor $\tilde{f}_2$ as already given in definition 1.7.

Remark 2.13. Let us consider the following diagram:

$$\begin{tikzcd}
(\widetilde{V}_i^1, H_i^1, \phi_i^1) \arrow[d, "\lambda_{ij}^1"'] \arrow[rr, bend left=40, "\delta_{\widetilde{U}_i}"] \arrow[rr, phantom, bend left=20, "\curvearrowright"] \arrow[dr, phantom, "\curvearrowright"] & (\widetilde{U}_i, G_i, \pi_i) \arrow[l, "(\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1}"] \arrow[r, "(\tilde{f}_2)_{\widetilde{U}_i, \widetilde{V}_i^2}"'] \arrow[d, "\lambda_{ij}"] \arrow[dr, phantom, "\curvearrowright"] & (\widetilde{V}_i^2, H_i^2, \phi_i^2) \arrow[d, "\lambda_{ij}^2"] \\
(\widetilde{V}_j^1, H_j^1, \phi_j^1) \arrow[rr, bend right=40, "\delta_{\widetilde{U}_j}"'] \arrow[rr, phantom, bend right=20, "\curvearrowright"] & (\widetilde{U}_j, G_j, \pi_j) \arrow[l, "(\tilde{f}_1)_{\widetilde{U}_j, \widetilde{V}_j^1}"'] \arrow[r, "(\tilde{f}_2)_{\widetilde{U}_j, \widetilde{V}_j^2}"] & (\widetilde{V}_j^2, H_j^2, \phi_j^2)
\end{tikzcd}$$

where the two squares are commutative because of (2.17) applied to $\tilde{f}_1$ and $\tilde{f}_2$ respectively, and the upper and lower parts are commutative because of part (i) of the previuos definition (applied to $\widetilde{U}_i$ and $\widetilde{U}_j$ respectively). We would like to deduce from this diagram that also the external one is commutative, i.e. property (ii) of the previous definition. However, we can just say that:

$$\lambda_{ij}^2 \circ \delta_{\widetilde{U}_i} \circ (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1} = \lambda_{ij}^2 \circ (\tilde{f}_2)_{\widetilde{U}_i, \widetilde{V}_i^2} =$$

$$= (\tilde{f}_2)_{\widetilde{U}_j, \widetilde{V}_j^2} \circ \lambda_{ij} = \delta_{\widetilde{U}_j} \circ (\tilde{f}_1)_{\widetilde{U}_j, \widetilde{V}_j^1} \circ \lambda_{ij} =$$

$$= \delta_{\widetilde{U}_j} \circ \lambda_{ij}^1 \circ (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1};$$

hence we can only deduce that $\lambda^2_{ij} \circ \delta_{\widetilde{U}_i} = \delta_{\widetilde{U}_j} \circ \lambda^1_{ij}$ on the set $(\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}^1_i}(\widetilde{U}_i)$. So if this set contains an open ball, we get that (ii) is verified using the fact that both $\lambda^2_{ij} \circ \delta_{\widetilde{U}_i}$ and $\delta_{\widetilde{U}_j} \circ \lambda^1_{ij}$ are holomorphic. When this condition is not satisfied (for example, when the underlying continuous map f identifies all the space to a single point), we have that *condition* (ii) *is not overabundant.*

Definition 2.11. Let us take orbifold atlases $\mathcal{U}$ for X and $\mathcal{V}$ for Y, a continuous map $f : X \rightarrow Y$, 3 compatible systems $\tilde{f}_m : \mathcal{U} \rightarrow \mathcal{V}$ for $m = 1, 2, 3$ and natural transformations $\delta : \tilde{f}_1 \Rightarrow \tilde{f}_2$ and $\sigma : \tilde{f}_2 \Rightarrow \tilde{f}_3$. Then let us define the *vertical composition* $\sigma \odot \delta : \tilde{f}_1 \Rightarrow \tilde{f}_3$ as follows: for any $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we set:

$$(\sigma \odot \delta)_{\widetilde{U}_i} := \sigma_{\widetilde{U}_i} \circ \delta_{\widetilde{U}_i}$$

which is clearly an embedding in $\mathcal{V}$ between the images of $(\widetilde{U}_i, G_i, \pi_i)$ via $\tilde{f}_1$ and $\tilde{f}_3$ respectively. Moreover, we want to prove that $\sigma \odot \delta$ is again a natural transformation, hence we have to prove conditions (i) and (ii) of definition 2.10; if we use the same notations of definition 2.10 (for $m = 1, 2, 3$), we have that:

(i) for any $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we get:

$$(\tilde{f}_3)_{\widetilde{U}_i, \widetilde{V}^3_i} = \sigma_{\widetilde{U}_i} \circ (\tilde{f}_2)_{\widetilde{U}_i, \widetilde{V}^2_i} = \sigma_{\widetilde{U}_i} \circ \delta_{\widetilde{U}_i} \circ (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}^1_i}$$

(ii) for any embedding λ_{ij} in $\mathcal{U}$ we have the commutative diagram:

$$\begin{tikzcd}
(\widetilde{V}^1_i, H^1_i, \phi^1_i) \arrow[r, "\delta_{\widetilde{U}_i}"] \arrow[d, "\lambda^1_{ij}"'] \arrow[dr, phantom, "\curvearrowright"] & (\widetilde{V}^2_i, H^2_i, \phi^2_i) \arrow[r, "\sigma_{\widetilde{U}_i}"] \arrow[d, "\lambda^2_{ij}"'] \arrow[dr, phantom, "\curvearrowright"] & (\widetilde{V}^3_i, H^3_i, \phi^3_i) \arrow[d, "\lambda^3_{ij}"] \\
(\widetilde{V}^2_j, H^2_j, \phi^2_j) \arrow[r, "\delta_{\widetilde{U}_j}"'] & (\widetilde{V}^2_j, H^2_j, \phi^2_j) \arrow[r, "\sigma_{\widetilde{U}_j}"'] & (\widetilde{V}^3_j, H^3_j, \phi^3_j).
\end{tikzcd}$$

Hence $\sigma \odot \delta$ is a natural transformation from $\tilde{f}_1$ to $\tilde{f}_3$.

Note that, using remark 2.12, (ii) actually comes from the fact that the vertical composition of natural tranformations between functors is again a natural tranformation, as described in §1.2.

Definition 2.12. Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be ordifold atlases for X, Y and Z respectively; let $\tilde{f}_m$ and $\tilde{g}_m$ be compatible systems for $f : X \to Y$ and $g : Y \to Z$ respectively, for $m = 1, 2$. Moreover, assume that we have natural transformations $\delta : \tilde{f}_1 \Rightarrow \tilde{f}_2$ and $\eta : \tilde{g}_1 \Rightarrow \tilde{g}_2$. Then we define a *horizontal composition*:

$$\eta * \delta : (\tilde{g}_1 \circ \tilde{f}_1) \Rightarrow (\tilde{g}_2 \circ \tilde{f}_2)$$

as follows: for any $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we set:

$$(\eta * \delta)_{\widetilde{U}_i} := \eta_{\widetilde{V}_i^2} \circ \tilde{g}_1(\delta_{\widetilde{U}_i}).$$

By construction, we have that actually this is an embedding from $\tilde{g}_1 \circ \tilde{f}_1(\widetilde{U}_i, G_i, \pi_i)$ to $\tilde{g}_2 \circ \tilde{f}_2(\widetilde{U}_i, G_i, \pi_i)$. Then we want to prove that actually $\eta * \delta$ is a natural transformation from $\tilde{g}_1 \circ \tilde{f}_1$ to $\tilde{g}_2 \circ \tilde{f}_2$. In order to do that, let us define:

$$(\widetilde{W}_i^{mn}, K_i^{mn}, \xi_i^{mn}) := \tilde{g}_n \circ \tilde{f}_m(\widetilde{U}_i, G_i, \pi_i) \quad \text{for} \quad m, n = 1, 2.$$

With this notation, we get that:

(i) for any uniformizing system $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we have:

$$\begin{aligned}(\tilde{g}_2 \circ \tilde{f}_2)_{\widetilde{U}_i, \widetilde{W}_i^{22}} &= (\tilde{g}_2)_{\widetilde{V}_i^2, \widetilde{W}_i^{22}} \circ (\tilde{f}_2)_{\widetilde{U}_i, \widetilde{V}_i^2} = \\ &= \eta_{\widetilde{V}_i^2} \circ (\tilde{g}_1)_{\widetilde{V}_i^2, \widetilde{W}_i^{21}} \circ \delta_{\widetilde{U}_i} \circ (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1} = \\ &= \eta_{\widetilde{V}_i^2} \circ \tilde{g}_1(\delta_{\widetilde{U}_i}) \circ (\tilde{g}_1)_{\widetilde{V}_i^1, \widetilde{W}_i^{11}} \circ (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1} = \\ &= (\eta * \delta)_{\widetilde{U}_i} \circ (\tilde{g}_1 \circ \tilde{f}_1)_{\widetilde{U}_i, \widetilde{W}_i^{11}};\end{aligned}$$

(ii) moreover, for every embedding λ_{ij} in $\mathcal{U}$ we get the commutative diagram:

$$\begin{tikzcd}
(\widetilde{W}_i^{11}, K_i^{11}, \xi_i^{11}) \arrow[r, "\tilde{g}_1(\delta_{\widetilde{U}_i})"] \arrow[d, "\tilde{g}_1\circ\tilde{f}_1(\lambda_{ij})"'] \arrow[dr, phantom, "\curvearrowright"] & (\widetilde{W}_i^{21}, K_i^{21}, \xi_i^{21}) \arrow[r, "\eta_{\widetilde{V}_i^2}"] \arrow[d, "\tilde{g}_1\circ\tilde{f}_2(\lambda_{ij})"] \arrow[dr, phantom, "\curvearrowright"] & (\widetilde{W}_i^{22}, K_i^{22}, \xi_i^{22}) \arrow[d, "\tilde{g}_2\circ\tilde{f}_2(\lambda_{ij})"] \\
(\widetilde{W}_j^{11}, K_j^{11}, \xi_j^{11}) \arrow[r, "\tilde{g}_1(\delta_{\widetilde{U}_j})"'] & (\widetilde{W}_j^{21}, K_j^{21}, \xi_j^{21}) \arrow[r, "\eta_{\widetilde{V}_j^2}"'] & (\widetilde{W}_j^{22}, K_j^{22}, \xi_j^{22})
\end{tikzcd}$$

where the first square is just obtained by applying the functor $\tilde{g}_1$ to (2.18) and the second one comes directly from the fact that η is a natural tranformation from $\tilde{g}_1$ to $\tilde{g}_2$ and $\tilde{f}_2(\lambda_{ij})$ is an embedding in $\mathcal{V}$.

Hence we have proved that $\eta * \delta$ is a natural transformation from $\tilde{g}_1 \circ \tilde{f}_1$ to $\tilde{g}_2 \circ \tilde{f}_2$. As before, part (ii) corresponds to the fact that a horizontal composition of natural transformations is again a natural transformation if we work in (**Cat**).

2.4 The 2-category (Pre-Orb)

As described in the introduction, the first aim of this thesis is to construct a 2-category, where the objects are orbifold atlases, the morphisms are compatible systems and the 2-morphisms are natural transformations between compatible systems. We have already defined the composition $\circ$ on the level of morphisms and the vertical and horizontal compositions $\odot$ and $*$ on the level of 2-morphisms, hence it suffices only to verify the axioms given in chapter 1.

Proposition 2.4.1. *The definitions of orbifold atlases, compatible systems, natural transformations and compositions* $\circ, \odot, *$ *give rise to a 2-category, that we will denote with (**Pre-Orb**).*

Proof. First of all, let us fix any pair of objects (i.e. orbifold atlases) $\mathcal{U}$ and $\mathcal{V}$ and let us define a category $(\mathbf{Pre\text{-}Orb})(\mathcal{U},\mathcal{V})$ with $\odot$ as composition. We define the space of objects $(\mathbf{Pre\text{-}Orb})(\mathcal{U},\mathcal{V})_0$ to be the set of all compatible systems $\tilde{f}:\mathcal{U}\rightarrow\mathcal{V}$ for all continuos maps $f:X\rightarrow Y$; for any pair of compatible systems $\tilde{f}$ and $\tilde{g}$ for f and g respectively, we define:

$$(\mathbf{Pre\text{-}Orb})(\mathcal{U},\mathcal{V})(\tilde{f},\tilde{g}):=\begin{cases}\text{natural transformations from } \tilde{f} \text{ to } \tilde{g} & \text{if } f=g\\ \varnothing & \text{else.}\end{cases}$$

The vertical composition $\odot$ is clearly associative; moreover, for every object $\tilde{f}$ (i.e. for every compatible system: $\mathcal{U}\rightarrow\mathcal{V}$), we can define the natural transformation $i_{\tilde{f}}$ as follows: for any uniformizing system $(\widetilde{U}_i,G_i,\pi_i)\in\mathcal{U}$ we set as usual $(\widetilde{V}_i,H_i,\phi_i):=\tilde{f}(\widetilde{U}_i,G_i,\pi_i)$ and we define:

$$(i_{\tilde{f}})_{\widetilde{U}_i}:=1_{\widetilde{V}_i};$$

clearly $i_{\tilde{f}}$ is a natural transformation from $\tilde{f}$ to itself. Moreover, for any $\alpha:\tilde{f}\Rightarrow\tilde{g}$ and for any $\beta:\tilde{h}\Rightarrow\tilde{f}$ we have:

$$\alpha\odot i_{\tilde{f}}=\alpha \quad \text{and} \quad i_{\tilde{f}}\odot\beta=\beta.$$

Hence $(Pre{-}Orb)(\mathcal{U},\mathcal{V})$ is a small category, so we have defined the datum (1) and (2) of definition 1.9. Now for any triple $\mathcal{U},\mathcal{V},\mathcal{W}$ of orbifold atlases, let us define the functor “composition”:

$$c_{\mathcal{U},\mathcal{V},\mathcal{W}}:(\mathbf{Pre\text{-}Orb})(\mathcal{U},\mathcal{V})\times(\mathbf{Pre\text{-}Orb})(\mathcal{V},\mathcal{W})\rightarrow(\mathbf{Pre\text{-}Orb})(\mathcal{U},\mathcal{W})$$

as follows:

- for every $\tilde{f}:\mathcal{U}\rightarrow\mathcal{V}$ and $\tilde{g}:\mathcal{V}\rightarrow\mathcal{V}$ we set:

$$c_{\mathcal{U},\mathcal{V},\mathcal{W}}(\tilde{f},\tilde{g}):=\tilde{g}\circ\tilde{f};$$

- for every $\delta : \tilde{f}_1 \Rightarrow \tilde{f}_2$ in $(\mathbf{Pre\text{-}Orb})(\mathcal{U}, \mathcal{V})$ and for every $\eta : \tilde{g}_1 \Rightarrow \tilde{g}_2$ in $(\mathbf{Pre\text{-}Orb})(\mathcal{V}, \mathcal{W})$ we set:

$$c_{\mathcal{U},\mathcal{V},\mathcal{W}}(\delta, \eta) := \eta * \delta : (\tilde{g}_1 \circ \tilde{f}_1) \Rightarrow (\tilde{g}_2 \circ \tilde{f}_2). \tag{2.19}$$

We want to prove that this is actually a functor, so let us consider any diagram of the form:

$$\begin{tikzcd}[column sep=huge]
\mathcal{U} \arrow[r, bend left=60, "\tilde{f}_1"] \arrow[r, "\tilde{f}_2"'] \arrow[r, bend right=60, "\tilde{f}_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \delta"] \arrow[r, phantom, shift right=4ex, "\Downarrow \sigma"] & \mathcal{V} \arrow[r, bend left=60, "\tilde{g}_1"] \arrow[r, "\tilde{g}_2"'] \arrow[r, bend right=60, "\tilde{g}_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \eta"] \arrow[r, phantom, shift right=4ex, "\Downarrow \mu"] & \mathcal{W}.
\end{tikzcd}$$

We want to prove that $c_{\mathcal{U},\mathcal{V},\mathcal{W}}$ preserves compositions, i.e. that:

$$c_{\mathcal{U},\mathcal{V},\mathcal{W}}\left((\sigma, \mu) \odot (\delta, \eta)\right) \overset{?}{=} c_{\mathcal{U},\mathcal{V},\mathcal{W}}(\sigma, \mu) \odot c_{\mathcal{U},\mathcal{V},\mathcal{W}}(\delta, \eta). \tag{2.20}$$

We recall that in chapter 1 we have defined the composition between morphisms in the product of categories as made "component by component"; in particular in this case we have that:

$$(\sigma, \mu) \odot (\delta, \eta) = (\sigma \odot \delta, \mu \odot \eta);$$

then using (2.19) we get that to prove (2.20) is equivalent to prove that:

$$(\mu \odot \eta) * (\sigma \odot \delta) \overset{?}{=} (\mu * \sigma) \odot (\eta * \delta)$$

In other words, we have to prove that the *interchange law* (see §1.3) is satisfied. So let us verify that this last identity is true: for any uniformizing system $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we have:

$$((\mu \odot \eta) * (\sigma \odot \delta))_{\widetilde{U}_i} = (\mu \odot \eta)_{\widetilde{V}_i^3} \circ \tilde{g}_1((\sigma \odot \delta)_{\widetilde{U}_i}) =$$

$$= \mu_{\widetilde{V}_i^3} \circ \eta_{\widetilde{V}_i^3} \circ \tilde{g}_1(\sigma_{\widetilde{U}_i}) \circ \tilde{g}_1(\delta_{\widetilde{U}_i}) \overset{*}{=} \mu_{\widetilde{V}_i^3} \circ \tilde{g}_2(\sigma_{\widetilde{U}_i}) \circ \eta_{\widetilde{V}_i^2} \circ \tilde{g}_1(\delta_{\widetilde{U}_i}) =$$

$$= (\mu * \sigma)_{\widetilde{U}_i} \odot (\eta * \delta)_{\widetilde{U}_i} = ((\mu * \sigma) \odot (\eta * \delta))_{\widetilde{U}_i}$$

where the passage denoted with $\overset{*}{=}$ is just the naturality of $\eta : \tilde{g}_1 \Rightarrow \tilde{g}_2$ applied to the embedding $\lambda := \sigma_{\widetilde{U}_i}$. Hence (2.20) is proved, so $c_{\mathcal{U},\mathcal{V},\mathcal{W}}$ preserves compositions.

Moreover, for any object $(\tilde{f}, \tilde{g})$ in $(\textbf{Pre-Orb})(\mathcal{U}, \mathcal{V}) \times (\textbf{Pre-Orb})(\mathcal{V}, \mathcal{W})$, and for any $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we have:

$$c_{\mathcal{U},\mathcal{V},\mathcal{W}}(i_{(\tilde{f},\tilde{g})})_{\widetilde{U}_i} = c_{\mathcal{U},\mathcal{V},\mathcal{W}}(i_{\tilde{f}}, i_{\tilde{g}})_{\widetilde{U}_i} = (i_{\tilde{g}})_{\widetilde{V}_i} \circ \tilde{g}((i_{\tilde{f}})_{\widetilde{U}_i}) =$$

$$= 1_{\widetilde{W}_i} \circ \tilde{g}(1_{\widetilde{V}_i}) = 1_{\widetilde{W}_i} \circ 1_{\widetilde{W}_i} = 1_{\widetilde{W}_i} = (i_{\tilde{g} \circ \tilde{f}})_{\widetilde{U}_i}$$

Hence $c_{\mathcal{U},\mathcal{V},\mathcal{W}}$ preserves also the identities, so it is a functor. Hence point (3) of definition 1.9 is well defined. It remains only to define the “identities” of point (4). So for every atlas $\mathcal{U}$:

- we define $1_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{U}$ to be a compatible system over the identity on X, described as the identity functor from the category associated to $\mathcal{U}$ to itself, together with the collection of liftings for the identity map on X:

$$\left\{(1_{\mathcal{U}})_{\widetilde{U}_i,\widetilde{U}_i} := 1_{\widetilde{U}_i}\right\}_{(\widetilde{U}_i,G_i,\pi_i) \in \mathcal{U}};$$

- we define $i_{\mathcal{U}}$ as a natural transformation from $1_{\mathcal{U}}$ to itself, described for every $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ as:

$$(i_{\mathcal{U}})_{\widetilde{U}_i} := 1_{\widetilde{U}_i}.$$

The last two definitions are equivalent to give a functor as in (4) of definition 1.9, so *we have defined all the data of a 2-category; now we have only to verify the axioms of remark* 1.3.

(a) For every triple of compatible systems:

$$\mathcal{U} \xrightarrow{\tilde{f}} \mathcal{V} \xrightarrow{\tilde{g}} \mathcal{W} \xrightarrow{\tilde{h}} \mathcal{Z}$$

we have that $(\tilde{h} \circ \tilde{g}) \circ \tilde{f} = \tilde{h} \circ (\tilde{g} \circ \tilde{f})$ as functors; moreover, for every uniformizing system $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$, if we call $(\widetilde{V}_i, H_i, \phi_i)$, $(\widetilde{W}_i, K_i, \xi_i)$ and $(\widetilde{Z}_i, L_i, \psi_i)$ the images of $(\widetilde{U}_i, G_i, \pi_i)$ via $\tilde{f}$, $\tilde{g} \circ \tilde{f}$ and $\tilde{h} \circ \tilde{g} \circ \tilde{f}$ respectively, we get:

$$((\tilde{h} \circ \tilde{g}) \circ \tilde{f})_{\widetilde{U}_i, \widetilde{Z}_i} = (\tilde{h} \circ \tilde{g})_{\widetilde{V}_i, \widetilde{Z}_i} \circ \tilde{f}_{\widetilde{U}_i, \widetilde{V}_i} =$$

$$= \tilde{h}_{\widetilde{W}_i, \widetilde{Z}_i} \circ \tilde{g}_{\widetilde{V}_i, \widetilde{W}_i} \circ \tilde{f}_{\widetilde{U}_i, \widetilde{V}_i} = (\tilde{h} \circ (\tilde{g} \circ \tilde{f}))_{\widetilde{U}_i, \widetilde{Z}_i}$$

hence $(\tilde{h} \circ \tilde{g}) \circ \tilde{f} = \tilde{h} \circ (\tilde{g} \circ \tilde{f})$ as compatible systems.

(b) For every diagram of compatible systems and natural transformations of compatible systems:

$$\begin{tikzcd}
\mathcal{U} \arrow[r, bend left, "\tilde{f}_1"] \arrow[r, bend right, "\tilde{f}_2"'] \arrow[r, phantom, "\Downarrow \delta"] & \mathcal{V} \arrow[r, bend left, "\tilde{g}_1"] \arrow[r, bend right, "\tilde{g}_2"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathcal{W} \arrow[r, bend left, "\tilde{h}_1"] \arrow[r, bend right, "\tilde{h}_2"'] \arrow[r, phantom, "\Downarrow \omega"] & \mathcal{Z}
\end{tikzcd}$$

and for every uniformizing system $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$, if we use the notations of definition 2.12 we have:

$$((\omega * \eta) * \delta)_{\widetilde{U}_i} = (\omega * \eta)_{\widetilde{V}_i^2} \circ ((\tilde{h}_1 \circ \tilde{g}_1)(\delta_{\widetilde{U}_i})) = \omega_{\widetilde{W}_i^{22}} \circ \tilde{h}_1(\eta_{\widetilde{V}_i^2}) \circ (\tilde{h}_1 \circ \tilde{g}_1(\delta_{\widetilde{U}_i})) =$$

$$= \omega_{\widetilde{W}_i^{22}} \circ \tilde{h}_1(\eta_{\widetilde{V}_i^2} \circ \tilde{g}_1(\delta_{\widetilde{U}_i})) = \omega_{\widetilde{W}_i^{22}} \circ \tilde{h}_1((\eta * \delta)_{\widetilde{U}_i}) = (\omega * (\eta * \delta))_{\widetilde{U}_i}.$$

Hence we have proved that $(\omega * \eta) * \delta = \omega * (\eta * \delta)$.

(c) For every compatible system $\tilde{f} : \mathcal{U} \to \mathcal{V}$ we have $\tilde{f} \circ 1_{\mathcal{U}} = \tilde{f}$ as functors. Moreover, for every $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we have $(\tilde{f} \circ 1_{\mathcal{U}})_{\widetilde{U}_i, \widetilde{V}_i} = \tilde{f}_{\widetilde{U}_i, \widetilde{V}_i} \circ 1_{\widetilde{U}_i} = \tilde{f}_{\widetilde{U}_i, \widetilde{V}_i}$. Hence $\tilde{f} \circ 1_{\mathcal{U}} = \tilde{f}$ in the sense of compatible systems. In the same way one can check that $1_{\mathcal{V}} \circ \tilde{f} = \tilde{f}$.

(d) For each natural transformation $\delta : \tilde{f}_1 \Rightarrow \tilde{f}_2$ in (**Pre-Orb**)$(\mathcal{U}, \mathcal{V})$ and for every $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$, we have:

$$(\delta * i_{\mathcal{U}})_{\widetilde{U}_i} = \delta_{\widetilde{U}_i} \circ 1_{\widetilde{U}_i} = \delta_{\widetilde{U}_i};$$

so $\delta * i_{\mathcal{U}} = \delta$. In the same way we get that $i_{\mathcal{V}} * \delta = \delta$.

This concludes the proof that (**Pre-Orb**) is a 2-category. □

2.5 Equivalent orbifold atlases

We said in remark 2.9 that as in the case of manifolds, actualy we are not interested in atlases, but in equivalence classes of atlases or, equivalently, in "maximal" atlases. In literature it is well known the definition of equivalent atlases, but I found nowhere the proof that this is actually a relation of equivalence, so this section is devoted to prove this fact in details. The other important results of this section are proposition 2.5.7 and proposition 2.5.8; the first one allows us to think to every manifold (i.e: equivalence class of compatible manifold atlases) as an orbifold (i.e: equivalence class of equivalent orbifold atlases, in the sense described below). The second one proves that the structure of orbifold for a global quotient of a manifold (via a finite group of automorphisms) is well defined, i.e. it does not depend on the manifold atlas we choose for our calculation. This was a problem remained unsolved in the previous sections.

Definition 2.13. An atlas $\mathcal{U}$ on X is said to *refine* another atlas $\mathcal{V}$ on the same topological space if for every uniformizing system in $\mathcal{U}$ there exists an embedding of it into some uniformizing system of $\mathcal{V}$. Equivalenty, $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i \in I}$ is a refinement of $\mathcal{V} = \{(\widetilde{V}_j, H_j, \phi_j)\}_{j \in J}$ iff there exists a set map $\gamma : I \longrightarrow J$ and embeddings $\lambda_i : (\widetilde{U}_i, G_i, \pi_i) \rightarrow (\widetilde{V}_{\gamma(i)}, H_{\gamma(i)}, \phi_{\gamma(i)})$ for every $i \in I$.

In a some sense, we can consider this as a compatible system $\mathcal{U} \rightarrow \mathcal{V}$ for the identity on X, except for the fact that in general this will not be a functor (actually, it is not even defined on embeddings). However, we will use the same abuse of notation we used for compatible system, i.e. we will write $(\widetilde{V}_i, H_i, \phi_i)$ instead of $(\widetilde{V}_{\gamma(i)}, H_{\gamma(i)}, \phi_{\gamma(i)})$.

Definition 2.14. Two atlases are said to be *equivalent* if they have a common refinement.

Now we want to prove that *this gives rise to an equivalence relation on the set of atlases on a fixed space.* In order to do that, we first state and prove some lemmas and useful remarks.

Lemma 2.5.1. *Let us fix a uniformizing system* $(\widetilde{U}, G, \pi)$ *and a point* $\tilde{x} \in \widetilde{U}$. *Then for every open neighborhood* $\widetilde{A} \subseteq \widetilde{U}$ *of* $\tilde{x}$, *there exists a uniformizing system of the form* $(\widetilde{B}, G_{\tilde{x}}, \pi_{|\widetilde{B}})$ *such that:*

- $\widetilde{B}$ *is an open connected neighborhood of* $\tilde{x}$;
- $\widetilde{B} \subseteq \widetilde{A}$;
- *up to a linear change of coordinates the group* $G_{\tilde{x}}$ *acts linearily on* $\widetilde{B}$;
- *the inclusion* $i : \widetilde{B} \rightarrow \widetilde{U}$ *gives rise to an embedding of orbifold charts:*

$$i : (\widetilde{B}, G_{\tilde{x}}, \pi_{|\widetilde{B})}) \rightarrow (\widetilde{U}, G, \pi).$$

Note that we don't require that $\widetilde{A}$ is stable under the action of the group G or $G_{\tilde{x}}$.

Proof. If we apply lemma 2.1.4 for all $g \in G \smallsetminus G_{\tilde{x}}$, we get a finite collection of positive radii $r_g = r(\tilde{x}, g)$ that satisfy (2.4); so if we call r_0 their minimum, we get that also r_0 is positive. Now $r_0 \leq r_g$ for every $g \in G \smallsetminus G_{\tilde{x}}$, so lemma 2.1.4 says that:

$$g(B_{r_0}) \cap B_{r_0} = \varnothing \quad \forall g \in G \smallsetminus G_{\tilde{x}}. \tag{2.21}$$

Now $\widetilde{A} \cap B_{r_0}$ is an open neighborhood of $\tilde{x}$ in $\widetilde{U}$, so we can apply lemma 2.1.2 to this set with respect to the point $\tilde{x}$ and to the group $G_{\tilde{x}}$. Hence we get an open connected neighborhood $\widetilde{A}'$ of $\tilde{x}$, completely contained in $\widetilde{A} \cap B_{r_0}$ and stable under the action of the group $G_{\tilde{x}}$. So we can apply Cartan's linearization lemma to the the pair $(\widetilde{A}', G_{\tilde{x}})$ in the point $\tilde{x}$, so there exists:

- a connected open neighborhood $\widetilde{B} \subseteq \widetilde{A}' \subseteq \widetilde{A}$ around $\tilde{x}$, stable under the action of the group $G_{\tilde{x}}$;
- a *connected* open neighborhood $\widetilde{C}$ of $0 \in \mathbb{C}^n$;
- a finite group H of *linear invertible* maps that act on $\widetilde{C}$;
- a biholomorphic map $\sigma : \widetilde{B} \xrightarrow{\sim} \widetilde{C}$ such that: $\sigma(\tilde{x}) = 0$;
- a group isomorphism $\bar{\sigma} : G_{\tilde{x}} \xrightarrow{\sim} H$ such that:

$$\sigma \circ g = \bar{\sigma}(g) \circ \sigma \quad \forall g \in G_{\tilde{x}}. \tag{2.22}$$

Now let us take any $g \in G_{\tilde{x}}$; if compose equation (2.22) with σ^{-1} both on the left and on the right side, we get that:

$$g \circ \sigma^{-1} = \sigma^{-1} \circ \bar{\sigma}(g)$$

so, recalling that $\widetilde{C} = \sigma(\widetilde{B})$,

$$g(\widetilde{B}) = g \circ \sigma^{-1}(\widetilde{C}) = \sigma^{-1} \circ \bar{\sigma}(g)(\widetilde{C}) \overset{*}{=} \sigma^{-1}(\widetilde{C}) = \widetilde{B}$$

where $\overset{*}{=}$ comes from the fact that by construction $\widetilde{C}$ is H-invariant and $\bar{\sigma}(g) \in H$. This holds for every $g \in G_{\tilde{x}}$, so $\widetilde{B}$ is $G_{\tilde{x}}$-invariant. Moreover, after the change of coordinates σ this group acts linearily on $\widetilde{B}$.

In addition, by construction $\widetilde{B}=\widetilde{A}'\subseteq\widetilde{A}\cap B_{r_0}$, so we have that $\widetilde{B}\subseteq\widetilde{A}$ and (using (2.21)):

$$g(\widetilde{B})\cap\widetilde{B}=\varnothing\quad\forall g\in G\smallsetminus G_{\tilde{x}}.\tag{2.23}$$

Now we want to prove that the triple $(\widetilde{B},G_{\tilde{x}},\pi_{|\widetilde{B}})$ is a uniformizing system. By construction $\widetilde{B}$ is an open connected neighborhood of $\tilde{x}$ and we have already proved that it is $G_{\tilde{x}}$-invariant, so the first two properties of definition 2.1 are satisfied.

Let us define $\{\tilde{x}_1,\cdots,\tilde{x}_k\}$ to be the set of all the preimages in $\widetilde{U}$ of $x:=\pi(\tilde{x})$ in X via π and let us suppose that $\tilde{x}_1=\tilde{x}$. Then for every $i=1,\cdots,n$ let us choose an element $g_i\in G$ (not necessarily unique) such that $g_i(\tilde{x}_1)=\tilde{x}_i$ (without loss of generality, $g_1=1_{\widetilde{U}}$). Now let us define the set:

$$\widetilde{D}:=\bigcup_{i=1,\cdots,k}g_i(\widetilde{B})$$

and let us prove that it is a disjoint union of open sets, all homeomorphic to $\widetilde{B}$. By contraddiction, let us suppose that there exist $i\neq j$ such that $g_i(\widetilde{B})\cap g_j(\widetilde{B})\neq\varnothing$. Then by applying g_j^{-1} we would get that:

$$g_j^{-1}\circ g_i(\widetilde{B})\cap\widetilde{B}\neq\varnothing.\tag{2.24}$$

On the other hand, by construction we have that:

$$g_j^{-1}\circ g_i(\tilde{x})=g_j^{-1}\circ g_i(\tilde{x}_1)=g_j^{-1}(\tilde{x}_i)\neq\tilde{x}$$

because we supposed that $g_j(\tilde{x})=\tilde{x}_j\neq\tilde{x}_i$; so $g_j^{-1}\circ g_i\notin G_{\tilde{x}}$, hence we have obtained a contradiction between (2.23) and (2.24). So every set $g_i(\widetilde{B})$ is disjoint from any other one with a different index; moreover, all these sets are homeomorphic to $\widetilde{B}$ because G is a set of holomorphic automorphisms. So there exists an homeomorphism:

$$\psi : \widetilde{D} \xrightarrow{\sim} \coprod_{i=1,\cdots,k} \widetilde{B}.$$

Now we want also to prove that $\widetilde{D}$ is saturated with respect to the action of the whole group G. Indeed, let us fix any point $\tilde{y} \in \widetilde{D}$ and any $g \in G$ and let us prove that $g(\tilde{y}) \in \widetilde{D}$. Since $\widetilde{D}$ is the disjoint union of the sets of the form $g_i(\widetilde{B})$, there exists a unique $i \in \{1, \cdots, k\}$ such that $\tilde{y} \in g_i(\widetilde{B})$, so let us call $\tilde{y}'$ the unique point in $\widetilde{B}$ such that $g_i(\tilde{y}') = \tilde{y}$. So our claim is equivalent to say that $g \circ g_i(\tilde{y}') \in \widetilde{D}$. So in order to prove our claim, it suffices to prove that for every h in G we have $h(\widetilde{B}) \subseteq \widetilde{D}$.

So let us consider the point $h(\tilde{x})$: by construction we have that there exists a unique $i \in \{1, \cdots, k\}$ such that $h(\tilde{x}) = x_i = g_i(\tilde{x})$. Hence $g_i^{-1} \circ h \in G_{\tilde{x}}$; using the fact that $\widetilde{B}$ is $G_{\tilde{x}}$-invariant, we get that:

$$g_i^{-1} \circ h(\widetilde{B}) = \widetilde{B}$$

hence:

$$h(\widetilde{B}) = g_i(\widetilde{B}) \subseteq \widetilde{D}.$$

So we have proved that $\widetilde{F}$ is saturated with respect to the action of the group G; moreover, it is open in $\widetilde{U}$, so by definition of quotient topology, if we define:

$$B := \pi(\widetilde{D}) \subseteq X$$

we have that B is an open neighborhood of $x = \pi(\tilde{x})$ in $U := \pi(\widetilde{U})$. Since U is open in X (by definition of uniformizing system), we have that B is an open neighborhood of x in X. Now a direct check proves that $B = \pi(\widetilde{B})$; so our aim now is to prove that the triple $(\widetilde{B}, G_{\tilde{x}}, \pi_{|\widetilde{B}})$ is a uniformizing system for the open neighborhood B for x.

So we want to prove that $\pi_{|\widetilde{B}}$ induces an homeomorphism from $\widetilde{B}/G_{\tilde{x}}$ to B. By hypothesis $(\widetilde{U}, G, \pi)$ is a uniformizing system, so the map π induces an homeomorphism φ from $\widetilde{U}/G$ to $U = \pi(\widetilde{U})$, so (by definition of quotient topology) it induces also an homeomorphism:

$$\varphi_{|\widetilde{D}} : \widetilde{D}/G \xrightarrow{\sim} B = \pi(\widetilde{D}).$$

By contruction $\widetilde{D}$ is (up to homeomorphism) equal to the disjoint union of some copies of $\widetilde{B}$. Moreover, since $B = \pi(\widetilde{B})$, to every point in B we can associate a point in $\widetilde{B}$ (also if not necessarily unique), so in order to describe the topology on B, it suffices to consider how the group G (restricted if necessary) acts on $\widetilde{B}$. Now using (2.23) we have that all the elements in $G \smallsetminus G_{\tilde{x}}$ map $\widetilde{B}$ to another copy of it, disjoint from it; so in order to study the topology of B, it suffices to consider the action of $G_{\tilde{x}}$ on $\widetilde{B}$. To be more precise, one can induce an homeomorphism:

$$\varphi' : \widetilde{B}/G_{\tilde{x}} \xrightarrow{\sim} B = \pi(\widetilde{B}).$$

so we have proved the third condiction of definition 2.1, i.e. $(\bar{B}, G_{\tilde{x}}, \pi_{|\bar{B}})$ is a uniformizing system for the open neighborhood B for x. Moreover, if we define $i : \widetilde{B} \to \widetilde{U}$ to be the inclusion map, we get that this is clearly an holomorphic embedding. Moreover, it is clear that $\pi_{|\widetilde{B}} = \pi \circ i$, so we have an embedding of uniformizing systems:

$$i : (\widetilde{B}, G_{\tilde{x}}, \pi_{|\widetilde{B}}) \to (\widetilde{U}, G, \pi).$$

$\square$

In the previuous proposition we have also implicitly proved the following result:

Corollary 2.5.2. *For every orbifold atlas $\mathcal{U}$ on X and for every point $x \in X$ there exists an open neighborhood U of it in X and a uniformizing system $(\widetilde{U}, G, \pi)$ for U such that the group G is just the local group at x as defined in remark 2.8. Moreover, this uniformizing system comes with an embedding* *into a uniformizing system of $\mathcal{U}$ (even if, in general, it does not belong to this atlas).*

In addition, the topology around every point $x \in X$ is given by an open connected neighborhood of the origin in $\mathbb{C}^n$ (which corresponds to $\tilde{x}$) modulo a finite group of linear invertible maps that act on this set. In particular, if the local group at x is trivial, then the topology around x is the euclidean topology in $\mathbb{C}^n$.

Lemma 2.5.3. *Let us fix a paracompact Hausdorff topological space X and 3 atlases $\mathcal{U}, \mathcal{V}, \mathcal{V}'$ for it. Suppose that both $\mathcal{V}$ and $\mathcal{V}'$ refine $\mathcal{U}$; then there exists a common refinement $\mathcal{W}$ of $\mathcal{V}$ and $\mathcal{V}'$ (i.e. $\mathcal{V}$ and $\mathcal{V}'$ are equivalent in the sense of definition 2.13).*

Proof. Let us define a family $\mathcal{W}$ whose elements are *all the uniformizing systems $(\widetilde{W}, K, \xi)$ for open connected sets $W \subseteq X$, such that there exist two uniformizing systems $(\widetilde{V}, H, \pi) \in \mathcal{V}$ and $(\widetilde{V}', H', \pi') \in \mathcal{V}'$ together with embeddings*:

$$(\widetilde{V}, H, \phi) \xleftarrow{\gamma} (\widetilde{W}, K, \xi) \xrightarrow{\delta} (\widetilde{V}', H', \phi').$$

First of all, we want to prove that for every point $x \in X$ there exists an open neighborhood W of it in X and a uniformizing system $(\widetilde{W}, K, \xi) \in \mathcal{W}$ for W. Let us fix such a point x. Since both $\mathcal{V}$ and $\mathcal{V}'$ are orbifold atlases for X, there exists open neighborhoods V and V' for x and uniformizing systems $(\widetilde{V}, H, \phi) \in \mathcal{V}$ and $(\widetilde{V}', H', \phi') \in \mathcal{V}'$ for V and V' respectively.

Now we know that $\mathcal{V}$ and $\mathcal{V}'$ are both refinements of $\mathcal{U}$, so there exists two uniformizing systems $(\widetilde{U}, G, \pi), (\widetilde{U}', G', \pi') \in \mathcal{U}$ and embeddings:

$$\lambda : (\widetilde{V}, H, \phi) \to (\widetilde{U}, G, \pi) \quad \text{and} \quad \mu : (\widetilde{V}', H', \phi') \to (\widetilde{U}', G', \pi').$$

We recall that $x \in V \cap V'$ and that $V = \phi(\widetilde{V}) = \pi \circ \lambda(\widetilde{V}) \subseteq \pi(\widetilde{U}) = U$ and in the same way $V' \subseteq U'$, hence $x \in U \cap U'$; since $\mathcal{U}$ is an atlas, there exists an open neighborhood $U_1 \subseteq (U \cap U')$ of x in X and a uniformizing system $(\widetilde{U}_1, G_1, \pi_1)$ of U_1 in $\mathcal{U}$, together with two embeddings:

$$(\widetilde{U}, G, \pi) \xleftarrow{\alpha} (\widetilde{U}_1, G_1, \pi_1) \xrightarrow{\beta} (\widetilde{U}', G', \pi').$$

Now $(\widetilde{U}_1, G_1, \pi_1)$ is a uniformizing system for x, so there exists a (not necessarily unique) $\tilde{x}_1 \in \widetilde{U}_1$ such that $\pi_1(\tilde{x}_1) = x$. Using remark 2.7 we can assume without loss of generality that

$$\alpha(\tilde{x}_1) \in \lambda(\widetilde{V}) \subseteq \widetilde{U} \quad \text{and} \quad \beta(\tilde{x}_1) \in \mu(\widetilde{V}') \subseteq \widetilde{U}'. \tag{2.25}$$

Hence if we define:

$$\widetilde{A} := \alpha^{-1}\left(\lambda(\widetilde{V})\right) \cap \beta^{-1}\left(\mu(\widetilde{V}')\right)$$

we get that this set is not empty because it contains $\tilde{x}_1$; moreover, it is also open since α and β are continuos and λ and η are embeddings between open sets of the same dimension. So $\widetilde{A}$ is an open neighborhood of $\tilde{x}_1$ in $\widetilde{U}_1 \subseteq \mathbb{C}^n$, hence we can apply lemma 2.5.1 and we get that there exists a uniformizing system $(\widetilde{U}_2, G_2, \pi_2)$ (in general not belonging to $\mathcal{U}$) for an open neighborhood $\pi_2(\widetilde{U}_2)$ for x such that $\widetilde{U}_2 \subseteq \widetilde{A} \subseteq \widetilde{U}_1$, together with an embedding (which coincides with the inclusion) $i : (\widetilde{U}_2, G_2, \pi_2) \rightarrow (\widetilde{U}_1, G_1, \pi_1)$.

In other words, we have obtained the following diagram:

$$\begin{tikzcd}
 & & & (\widetilde{V}, H, \phi) \arrow[dl, "\lambda"'] \\
 & & (\widetilde{U}, G, \pi) & \\
(\widetilde{U}_2, G_2, \pi_2) \arrow[r, "i"] \arrow[uurrr, dashed, bend left, "\gamma"] \arrow[ddrrr, dashed, bend right, "\delta"'] & (\widetilde{U}_1, G_1, \pi_1) \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \\
 & & (\widetilde{U}', G', \pi') & \\
 & & & (\widetilde{V}', H', \phi') \arrow[ul, "\mu"]
\end{tikzcd}$$

Now we have that:

$$\alpha \circ i(\widetilde{U}_2) \subseteq \alpha(\widetilde{A}) \subseteq \alpha(\alpha^{-1}(\lambda(\widetilde{V}))) = \lambda(\widetilde{V})$$

so it makes sense to define the set map:

$$\gamma := \lambda^{-1}_{|\alpha \circ i(\widetilde{U}_2)} \circ \alpha \circ i : \widetilde{U}_2 \to \widetilde{V}$$

defined between two open sets in $\mathbb{C}^n$. Now λ is holomorphic and injective, hence if we restrict its codomain to $\alpha \circ i(\widetilde{U}_2) \subseteq \lambda(\widetilde{V})$ we have that λ is invertible; moreover, by hypothesis λ is an embedding, so it is nonsingular in every point of $\widetilde{V}$, so if we apply theorem 2.1.1 we get that its inverse is again a holomorphic function. Hence γ is the composition of 3 holomorphic injective and non singular maps, so it is an holomorphic embedding. Now we want to prove that it is also an embedding in the sense of definition 2.3 from $(\widetilde{U}_2, G_2, \pi_2)$ to $(\widetilde{V}, H, \phi)$:

$$\pi_2 = \pi_1 \circ i = \pi \circ \alpha \circ i = \pi \circ \lambda \circ \lambda^{-1} \circ \alpha \circ i = \phi \circ \lambda^{-1} \circ \alpha \circ i = \phi \circ \gamma \quad (2.26)$$

so γ is actually an embedding between compatible systems. In the same way, we can define an embedding $\delta : (\widetilde{U}_2, G_2, \pi_2) \to (\widetilde{V}', H', \phi')$

Hence by definition of the family $\mathcal{W}$, we have that $(\widetilde{U}_2, G_2, \pi_2) \in \mathcal{W}$. This can be done for every point $x \in X$, so we have proved that the union of the images of the uniformizing systems in $\mathcal{W}$ covers X, i.e. property (i) of definition 2.4 of orbifold atlases is satified by $\mathcal{W}$.

Now we want to prove also property (ii), so let us take any pair of uniformizing systems $(\widetilde{W}_i, K_i, \xi_i) \in \mathcal{W}$ for $i = 1, 2$ and let us fix any point $x \in \xi_1(\widetilde{W}_1) \cap \xi_2(\widetilde{W}_2)$ (if any). Recalling the definition of $\mathcal{W}$ we get that there exists uniformizing systems $(\widetilde{V}_i, H_i, \phi_i) \in \mathcal{V}$ and $(\widetilde{V}_i', H_i', \phi'_i) \in \mathcal{V}'$ for $i = 1, 2$, and embeddings:

$$(\widetilde{V}_1, H_1, \phi_1) \xleftarrow{\gamma_1} (\widetilde{W}_1, K_1, \xi_1) \xrightarrow{\delta_1} (\widetilde{V}_1', H_1', \phi_1');$$
$$(\widetilde{V}_2, H_2, \phi_2) \xleftarrow{\gamma_2} (\widetilde{W}_2, K_2, \xi_2) \xrightarrow{\delta_2} (\widetilde{V}_2', H_2', \phi_2').$$

Now by construction $\phi(\widetilde{V}_1) \cap \phi(\widetilde{V}_2) \ni x$ and both $(\widetilde{V}_1, H_1, \phi_1)$ and $(\widetilde{V}_2, H_2, \phi_2)$ are uniformizing systems in $\mathcal{V}$, so there exists an open neighborhood V_3 of x in X, completely contained in $\phi(\widetilde{V}_1) \cap \phi(\widetilde{V}_2)$, a uniformizing system $(\widetilde{V}_3, H_3, \phi_3) \in \mathcal{V}$ for it and embeddings ν_1, ν_2 as follows:

$$(\widetilde{V}_1, H_1, \phi_1) \xleftarrow{\nu_1} (\widetilde{V}_3, H_3, \phi_3) \xrightarrow{\nu_2} (\widetilde{V}_2, H_2, \phi_2).$$

As in the previous construction of $\widetilde{A}$, we can assume that the set:

$$\widetilde{A}' := \nu_1^{-1}\left(\gamma_1(\widetilde{W}_1)\right) \cap \nu_2^{-1}\left(\gamma_2(\widetilde{W}_2)\right)$$

is an open neighborhood of some point $\tilde{x}_3 \in \widetilde{V}_3$ such that $\phi_3(\tilde{x}_3) = x$, so using again lemma 2.5.1 for $\widetilde{A}'$, we get that there exists a uniformizing system $(\widetilde{V}_4, H_4, \phi_4)$ (not necessary in the atlas $\mathcal{V}$) such that $\phi_4(\widetilde{V}_4) \ni x$; moreover, the lemma gives us also an embedding j of $(\widetilde{V}_4, H_4, \phi_4)$ into $(\widetilde{V}_3, H_3, \phi_3)$. So let us consider the diagram:

$$\begin{tikzcd}
 & & & (\widetilde{W}_1, K_1, \xi_1) \arrow[dl, "\gamma_1"'] \\
 & & (\widetilde{V}_1, H_1, \phi_1) & \\
(\widetilde{V}_4, H_4, \phi_4) \arrow[r, "j"] \arrow[uurrr, dashed, bend left, "\theta_1"] \arrow[ddrrr, dashed, bend right, "\theta_2"'] & (\widetilde{V}_3, H_3, \phi_3) \arrow[ur, "\nu_1"] \arrow[dr, "\nu_2"'] & & \\
 & & (\widetilde{V}_2, H_2, \phi_2) & \\
 & & & (\widetilde{W}_2, K_2, \xi_2) \arrow[ul, "\gamma_2"]
\end{tikzcd}$$

If we apply the same construction described before, we can define two embeddings:

$$\theta_1 : (\widetilde{V}_4, H_4, \phi_4) \to (\widetilde{W}_1, K_1, \xi_1) \quad \text{and} \quad \theta_2 : (\widetilde{V}_4, H_4, \phi_4) \to (\widetilde{W}_2, K_2, \xi_2)$$

making the diagram commute. Now if we consider the embeddings:

$$\mathcal{V} \ni (\widetilde{V}_1, H_1, \phi_1) \xleftarrow{\nu_1 \circ j} (\widetilde{V}_4, H_4, \phi_4) \xrightarrow{\delta_1 \circ \theta_1} (\widetilde{V}_1', H_1, \phi_1') \in \mathcal{V}'$$

we have proved that $(\widetilde{V}_4, H_4, \phi_4)$ is an element of the family $\mathcal{W}$. Moreover, property (ii) of definition 2.4 is satisfied if we consider the open neighborhood $\phi_4(\widetilde{V}_4)$ of x in X and the pair of embeddings θ_1, θ_2. □

Proposition 2.5.4. *The relation of "being equivalent" is actually an equivalence relation on the set of orbifold atlases.*

Proof. It is easy to show that this relation is symmetric and reflexive, so let us consider only transitivity and let us fix a triple of atlases $\mathcal{U}_i$ for $i = 1, 2, 3$, a refinement $\mathcal{U}_4$ of the first two and a refinement $\mathcal{U}_5$ of the last two. Then in particular $\mathcal{U}_4$ and $\mathcal{U}_5$ are both refinements of $\mathcal{U}_2$.

Hence using lemma 2.5.3 we get that there exists a common refinement $\mathcal{U}_6$ of $\mathcal{U}_4$ and $\mathcal{U}_5$. Now composing with the given refinements we get that $\mathcal{U}_6$ is a common refinement of $\mathcal{U}_1$ and $\mathcal{U}_3$, so $\mathcal{U}_1$ and $\mathcal{U}_3$ are equivalent, hence transitivity is proved. □

So it makes sense to give the following definition.

Definition 2.15. ([ALR], def. 1.2) A *complex orbifold structure* on a paracompact Hausdorff topological space X is an equivalence class of orbifold atlases on X. We will denote such an object by $\mathcal{X}$ or $[X]$. We will call *orbifold* the pair $(X, \mathcal{X})$, or, by abuse of notation, just the orbifold structure $\mathcal{X}$.

Remark 2.14. In remark 2.8 we defined the local group at a point x in X for a *fixed* orbifold atlas $\mathcal{X}$ and we proved that this notion is well defined up to group isomorphisms. Now what happens if we choose two equivalent orbifold atlases $\mathcal{X}_i$ for $i = 1, 2$ and we call $G_{i,x}$ the corresponding local groups? We recall that by definition $G_{i,x}$ was defined to be equal to the isotropy group $G_{i,\tilde{x}^i}$ at *any* of the preimages $\tilde{x}^i$ of x in some chart of $\mathcal{X}_i$ (and this notion was well defined up to isomorphisms).

Since $\mathcal{X}_1$ and $\mathcal{X}_2$ are equivalent, there exists an atlas $\mathcal{X}_3$ which refines both. So let $(\widetilde{U}^3, G^3, \pi^3)$ be a uniformizing system in it such that it contains a point $\tilde{x}^3$ such that $\pi^3(\tilde{x}^3) = x$. By hypothesis, $\mathcal{X}_3$ refines $\mathcal{X}_1$, so there exists an orbifold chart $(\widetilde{U}^1, G^1, \pi^1) \in \mathcal{X}_1$ and an embedding:

$$\lambda : (\widetilde{U}^3, G^3, \pi^3) \to (\widetilde{U}^1, G^1, \pi^1)$$

so using the same proof of remark 2.8 we get that the group $G_{3,\tilde{x}^3}$ is isomorphic to $G_{1,\tilde{x}^1}$. In the same way, using the fact that $\mathcal{X}_3$ refines $\mathcal{X}_2$, we get that $G_{3,\tilde{x}^3}$ is isomorphic to $G_{2,\tilde{x}^2}$.

Hence *up to isomorphisms the local group at any point $x \in X$ depends only on the orbifold structure and not on the orbifold atlas chosen.*

Definition 2.16. We say that $\mathcal{X}$ has *dimension n* if there is an atlas $\mathcal{U}$ of dimension n in the class $\mathcal{X}$.

Lemma 2.5.5. *This is equivalent to say that* every *atlas of the class has the same dimension n.*

Proof. Indeed, let us consider another atlas $\mathcal{V}$ in the class $\mathcal{X}$. Now let us take any chart $(\widetilde{V}, H, \phi) \in \mathcal{V}$; using lemma 2.1.5 we have that the set of points with trivial stabilizer in $\widetilde{V}$ is dense, so in particular it is non-empty. So let us fix any point $\tilde{y} \in \widetilde{V}$ with trivial stabilizer. Using corollary 2.5.2 we have that the topology around $\phi(\tilde{y})$ in X is the same of $\mathbb{C}^m$, where m is the dimension of the atlas $\mathcal{V}$. Using remark 2.14 we have that this point has trivial stabilizer also with respect to $\mathcal{U}$, so using again corollary 2.5.2 we have that the topology around this point in x is also the same of $\mathbb{C}^n$. Since an open set of $\mathbb{C}^n$ is homeorphic to an open set of $\mathbb{C}^m$ iff $n = m$, we get that also the atlas $\mathcal{V}$ has dimension n. □

Given any orbifold $\mathcal{X}$ sometimes it is convenient to work with a fixed atlas of the class, chosen in a "canonical" way. In order to do this, we give the following definition:

Definition 2.17. We define the *maximal atlas associated to an orbifold structure* $\mathcal{X}$ to be the family:

$$\mathcal{U} := \bigcup_{\mathcal{U}_m \in \mathcal{X}} \mathcal{U}_m.$$

i.e. the family of all the uniformizing systems of all the atlases of the class $\mathcal{X}$.

Proposition 2.5.6. *$\mathcal{U}$ is actually an orbifold atlas for X and it belongs to the orbifold structure $\mathcal{X}$.*

Proof. Since every family $\mathcal{U}_m \in \mathcal{X}$ satisfies property (i) of orbifold atlases, so is $\mathcal{U}$, hence it suffices only to verify property (ii) of definition 2.4. So let us fix two uniformizing systems $(\widetilde{U}_1, G_1, \pi_1) \in \mathcal{U}_1$ and $(\widetilde{U}_2, G_2, \pi_2) \in \mathcal{U}_2$ where $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{X}$, let us suppose that $\pi_1(\widetilde{U}_1) \cap \pi_2(\widetilde{U}_2)$ is non-empty in X and let us fix a point x in the intersection.

Since $\mathcal{U}_1$ and $\mathcal{U}_2$ are equivalent, there exists an atlas $\mathcal{U}_3$ which refines both them. Let $(\widetilde{U}_3, G_3, \pi_3) \in \mathcal{U}_3$ be a uniformizing system such that there exists $\tilde{x}_3 \in \widetilde{U}_3$ with $\pi_3(\tilde{x}_3) = x$ (this chart exists because of property (i) of orbifold atlases for $\mathcal{U}_3$). By definition of refinement, there exists uniformizing systems $(\widetilde{U}_1', G_1', \pi_1') \in \mathcal{U}_1$ and $(\widetilde{U}_2', G_2', \pi_2') \in \mathcal{U}_2$ together with embeddings:

$$(\widetilde{U}_1', G_1', \pi_1') \xleftarrow{\lambda_1} (\widetilde{U}_3, G_3, \pi_3) \xrightarrow{\lambda_2} (\widetilde{U}_2', G_2', \pi_2');$$

for simplicity, let us call $\tilde{x}_1' := \lambda_1(\tilde{x}_3)$ and $\tilde{x}_2' := \lambda_2(\tilde{x}_3)$.

Now $(\widetilde{U}_1, G_1, \pi_1)$ and $(\widetilde{U}_1', G_1', \pi_1')$ are both uniformizing systems of $\mathcal{U}_1$ and their images in X both contain x. So by property (ii) for the orbifold atlas $\mathcal{U}_1$, there exists a uniformizing system $(\widetilde{U}_1'', G_1'', \pi_1'') \in \mathcal{U}_1$ for an open neighborhood of x, together with embeddings:

$$(\widetilde{U}_1, G_1, \pi_1) \xleftarrow{\alpha_1} (\widetilde{U}''_1, G''_1, \pi''_1) \xrightarrow{\beta_1} (\widetilde{U}'_1, G'_1, \pi'_1).$$

Let us call $\tilde{x}''_1$ a point in $\widetilde{U}''_1$ such that $\pi''_1(\tilde{x}''_1) = x$; using remark 2.7 we can assume that $\beta_1(\tilde{x}''_1) = \tilde{x}'_1$. In the same way, we can choose a uniformizing system $(\widetilde{V}''_2, G''_2, \pi''_2) \in \mathcal{U}_2$, embeddings:

$$(\widetilde{U}_2, G_2, \pi_2) \xleftarrow{\alpha_2} (\widetilde{U}''_2, G''_2, \pi''_2) \xrightarrow{\beta_2} (\widetilde{U}'_2, G'_2, \pi'_2)$$

and point $\tilde{x}''_2 \in \widetilde{U}''_2$ such that $\beta_2(\tilde{x}''_2) = \tilde{x}'_2$. Then we can define the set:

$$\widetilde{A} := \lambda_1^{-1}(\beta_1(\widetilde{U}''_1)) \cap \lambda_2^{-1}(\beta_2(\widetilde{U}''_2));$$

as in lemma 2.5.3 this set is open and non-empty in $\widetilde{U}_3$ because it contains $\tilde{x}_3$, so we can apply lemma 2.5.1 to this set with respect to the point $\tilde{x}_3$ and we get that there exists a uniformizing system $(\widetilde{U}_4, G_4, \pi_4)$ and an embedding $\eta : (\widetilde{U}_4, G_4, \pi_4) \to (\widetilde{U}_3, G_3, \pi_3)$ such that $\tilde{x}_3 \in \eta(\widetilde{U}_4)$ and $\eta(\widetilde{U}_4) \subseteq \widetilde{A}$. In other words, we are using a diagram of embeddings of the form:

$$\begin{tikzcd}
 & & & (\widetilde{U}_1, G_1, \pi_1) \\
 & & (\widetilde{U}''_1, G''_1, \pi''_1) \arrow[ur, "\alpha_1"] \arrow[dr, "\beta_1"'] & \\
 & & & (\widetilde{U}'_1, G'_1, \pi'_1) \\
(\widetilde{U}_4, G_4, \pi_4) \arrow[r, "\eta"] \arrow[uuurrr, dashed, bend left, "\gamma_1"] \arrow[dddrrr, dashed, bend right, "\gamma_2"'] & (\widetilde{U}_3, G_3, \pi_3) \arrow[urr, "\lambda_1"] \arrow[drr, "\lambda_2"'] & & \\
 & & & (\widetilde{U}'_2, G'_2, \pi'_2) \\
 & & (\widetilde{U}''_2, G''_2, \pi''_2) \arrow[ur, "\beta_2"] \arrow[dr, "\alpha_2"'] & \\
 & & & (\widetilde{U}_2, G_2, \pi_2)
\end{tikzcd}$$

Now the set maps:

$$\gamma_1 := \alpha_1 \circ \beta_1^{-1} \circ \lambda_1 \circ \eta$$

$$\gamma_2 := \alpha_2 \circ \beta_2^{-1} \circ \lambda_2 \circ \eta$$

are both well defined (because $\lambda_i \circ \eta(\widetilde{U}_4) \subseteq \lambda_i(\widetilde{A}) \subseteq \beta_i(\widetilde{U}_i'')$ for $i = 1, 2$); moreover, one can use a calculation similar to (2.26) in order to prove that these maps give rise to embeddings:

$$(\widetilde{U}_1, G_1, \pi_1) \xleftarrow{\gamma_1} (\widetilde{U}_4, G_4, \pi_4) \xrightarrow{\gamma_2} .$$

So we have described a uniformizing system $(\widetilde{U}_4, G_4, \pi_4)$ such that $\pi_4(\widetilde{U}_4)$ contains x, together with embeddings in the fixed uniformizing systems chosen at the beginning. Hence in order to prove that property (ii) of definition 2.4 is satisfied, it suffices to prove that $(\widetilde{U}_4, G_4, \pi_4)$ belongs to the family $\mathcal{U}$.

In order to do that, we define $\mathcal{U}_4$ to be *the family of all uniformizing systems on X which have an embedding in some uniformizing system of $\mathcal{U}_3$*. We recall that in lemma 2.5.3 we have proved that if we fix a pair of equivalent orbifold atlases, then the family of all uniformizing systems which admits embeddings in some charts of both atlases is again an orbifold atlas which refines both them. In particular, if we choose these two atlases to be the same (i.e. $\mathcal{U}_3$ in this case), we have immediately that $\mathcal{U}_4$ is an atlas for X and it refines $\mathcal{U}_3$, so it is equivalent to it; hence it is an atlas of the class $\mathcal{X}$. Moreover, since $(\widetilde{U}_4, G_4, \pi_4)$ comes with an embedding in $(\widetilde{U}_3, G_3, \pi_3) \in \widetilde{U}_3$, then $(\widetilde{U}_4, G_4, \pi_4) \in \mathcal{U}_4$, so it belongs also to the family $\mathcal{U}$ defined at the beginning of the proof.

So property (ii) of orbifold atlases is proved for the family $\mathcal{U}$. Moreover, given the atlas $\mathcal{U}_1$, we have that $\mathcal{U}_1$ refines $\mathcal{U}$, so it is equivalent to it. Hence $\mathcal{U} \in \mathcal{X}$. □

We recall that in remark 2.9 we consider the case when we fix different (but equivalent) admissible manifold atlases and associate to them the corresponding orbifold atlases. The question was: what is the relationship between these orbifold atlases? The answer is given by the following proposition:

Proposition 2.5.7. *Let us fix a pair of* admissible *manifold atlases* $\mathcal{M}_m$ *for* $m = 1, 2$ *on* M *and let us call* $\overline{\mathcal{M}_i}$ *the associated orbifold atlases, as described in proposition 2.1.12. If* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are* compatible*, then* $\overline{\mathcal{M}_1}$ *and* $\overline{\mathcal{M}_2}$ *are* equivalent *in the sense of definition 2.14.*

Proof. Let us suppose that $\mathcal{M}_m = \{(U_i^m, \phi_i^m)\}_{i \in I^m}$ for $m = 1, 2$ and let us consider the family $\mathcal{A}$:

$$\mathcal{A} := \{\text{all the charts of the form } (U_i^1 \cap U_j^2, \phi_i^1|_{U_i^1 \cap U_j^2})\}_{i \in I^1, j \in I^2}$$

where the indexes i and j are such that $U_i^1 \cap U_j^2 \neq \varnothing$. It is easy to see that this family is again a manifold atlas on M, which refines both the previos two. Now for this atlas we can apply lemma 2.1.11 in order to get a new manifold atlas $\mathcal{M}_3 = \{(U_i^3, \phi_i^3)\}_{i \in I^3}$, which is *admissible* and which refines $\mathcal{A}$. Then $\mathcal{M}_3$ refines also both $\mathcal{M}_1$ and $\mathcal{M}_2$.

Let us call $\overline{\mathcal{M}_m} = \{(\widetilde{U}_i^m, G_i^m, \pi_i^m)\}_{i \in I^m}$ the orbifold atlases associated to $\mathcal{M}_m$ using proposition 2.1.12 for $m = 1, 2, 3$. Then we want to prove that $\overline{\mathcal{M}_3}$ refines both $\overline{\mathcal{M}_1}$ and $\overline{\mathcal{M}_2}$ in the sense of orbifolds; let us prove this only for the first case, the second one is analogous.

By construction we have that $\mathcal{M}_3$ refines $\mathcal{M}_1$, so there exists a set map $f : I^3 \to I^1$ such that for every $i \in I^3$ we have that $U_i^3 \subseteq U_{f(i)}^1$. Then we can define the function $\lambda_{i,f(i)} := \phi_{f(i)}^1 \circ (\phi_i^3)^{-1}$ defined from $\widetilde{U}_i^3$ to $\widetilde{U}_{f(i)}^1$.

By construction every chart of $\mathcal{M}_3$ is compatible with every chart of $\mathcal{M}_1$ and $U_i^3 \subseteq U_{f(i)}^1$, so the function $\lambda_{i,f(i)}$ is a biholomorphism if restricted in codomain, hence $\lambda_{i,f(i)}$ is a complex embedding. Moreover, we recall that by construction the groups G_i^3 and $G_{f(i)}^1$ (defined in lemma 2.1.12) are both trivial, so $\lambda_{i,f(i)}$ is an embedding between orbifold charts:

$$\lambda_{i,f(i)} : (\widetilde{U}_i^3, G_i^3, \pi_i^3) \to (\widetilde{U}_{f(i)}^1, G_{f(i)}^1, \pi_{f(i)}^1).$$

Since this holds for every $i \in I^3$, we have proved that $\overline{\mathcal{M}_3}$ refines $\overline{\mathcal{M}_1}$. In the same way we can also prove that $\overline{\mathcal{M}_3}$ refines $\overline{\mathcal{M}_2}$; hence $\overline{\mathcal{M}_1}$ and $\overline{\mathcal{M}_2}$ are equivalent orbifold atlases. □

We recall also that in remark 2.11 we wondered whether the notion of orbifold associated to a global quotient of a manifold was well defined or not. Indeed in proposition 2.1.13 we associated to every *fixed* atlas on the manifold M an orbifold atlas for its quotient (under the global action of a finite group G of automorphisms), but we said nothing about what happens if we choose different manifold atlases. The following proposition fills this gap.

Proposition 2.5.8. *Let us suppose we have chosen a pair of* compatible *manifold atlases $\mathcal{M}_1, \mathcal{M}_2$ on the same space M. Let G be a finite group of holomorphic automorphisms that act on M and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the corresponding orbifold atlases associated to $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively using proposition 2.1.13. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are* equivalent *in the sense of orbifolds.*

Proof. Let us suppose that $\mathcal{M}_1 = \{(U_i^1, \phi_i^1)\}_{i \in I^1}$ and $\mathcal{M}_2 = \{(U_j^2, \phi_j^2)\}_{j \in I^2}$; then we define the manifold atlas $\mathcal{M}_3 := \{(U_i^1 \cap U_j^2, \phi_i^1|_{U_i^1 \cap U_j^2}\}_{i \in I^1, j \in I^2}$. Since the previous two are compatible manifold atlases, then also $\mathcal{M}_3$ is a manifold atlas (and refines the previous two). Let us call $\mathcal{A}_3$ the orbifold atlas on M/G associated to $\mathcal{M}_3$ by proposition 2.1.13 and let us fix a generic orbifold chart $(\widetilde{W}, G, \xi) \in \mathcal{A}_3$. Using the explicit description of $\mathcal{A}_3$ in the above mentioned proposition, we get that there exists indexes $i \in I$, $j \in J$, a point $\tilde{P} \in \phi_i(U_i^1 \cap U_j^2)$ and an open *connected* neighborhood $\widetilde{V}_{i,\tilde{P}}$ of $\tilde{P}$ completely contained in $\phi_i(U_i^1 \cap U_j^2)$, such that:

$$(\widetilde{W}, G, \xi) = (\widetilde{V}_{i,\tilde{P}}, G_{\tilde{P}}, \pi_i). \tag{2.27}$$

So we can define an embedding λ from this chart to the same chart of $\mathcal{A}_1$ given by the identity. Since this works for every uniformizing system

$(\widetilde{W}, G, \xi)$ of $\mathcal{A}_3$, we have proved that $\mathcal{A}_3$ refines $\mathcal{A}_1$ in the sense of orbifolds.

Now we want to prove that the chart (2.27) has an embedding also in a chart of $\mathcal{A}_2$. In order to do that, let us consider the open set $B := \phi_j^2 \circ \phi_1^{-1}(\widetilde{V}_{i,\tilde{P}})$. We know that $\widetilde{V}_{i,\tilde{P}}$ is open and contains the point $\tilde{P}$; since both the maps involved are homeomorphisms, we get that B is an open connected neighborhood of $\tilde{P}' := \phi_j^2 \circ (\phi_i^1)^{-1}(\tilde{P})$ completely contained in $\phi_j(\widetilde{U}_j^2)$, so it concides with a set of the form $\widetilde{B}_{j,\tilde{P}'}$ as defined in proposition 2.1.13. Hence it makes sense to consider the orbifold chart $(\widetilde{B}_{j,\tilde{P}'}, \tilde{G}_{\tilde{P}'}, \pi_j)$, which belongs to $\mathcal{A}_2$ because of the explicit construction of this orbifold atlas in the mentioned above proposition. Moreove, we can consider:

$$\mu := \phi_j^2 \circ (\phi_j^1)^{-1} : (\widetilde{V}_{i,\tilde{P}}, G_{\tilde{P}}, \pi_i) \to (\widetilde{B}_{j,\tilde{P}'}, \tilde{G}_{\tilde{P}'}, \pi_j).$$

The map μ is an holomorphic embedding because it is just the inclusion map written in coordinates in a complex manifold; moreover, it commutes with the projection maps, so it is an embedding in the sense of orbifolds. Since this works for every chart $(\widetilde{W}, G, \xi)$ in $\mathcal{A}_3$, we have proved that $\mathcal{A}_3$ refines $\mathcal{A}_2$.

Hence $\mathcal{A}_3$ is a common refinement of $\mathcal{A}_1$ and $\mathcal{A}_2$, so these 2 atlases are equivalent orbifold atlases for the topological space M/G. □

Definition 2.18. The orbifold structure for the quotient space M/G described by propositions 2.1.13 and 2.5.8 is usually denoted with $[M/G]$.

> "**Proposition 4-12** [...] suppose that the smooth factors maps $(f_1, \cdots, f_n)$ are in fact transverse. The set-theoretic fiber product $P := A_1[f_1] \times_S \cdots \times_S [f_n]A_n$ is then a smooth manifold [...].
> **Proof** This is a standard result, for which we appeal to standard texts, such as Lang ([10]) - but beware that the proof given in Lang consists of the single word "Obvious"."
>
> Lyle Ramshaw-Julieb Bash
> [RB]

Chapter 3

The 2-category of internal groupoids in a category $\mathscr{C}$

3.1 Groupoid objects in a fixed category

Let us fix a category $\mathscr{C}$ and two objects R and U in it. In the case where both R and U are sets (or sets with additional properties), we would like to think of R as a set of "arrows" between points of U. Hence we have to define two morphisms "source" and "target" from R to U which "associate" to every point in R a pair of points in U (their source and target). Moreover, we would like to think of R as a "groupoid" in the sense of a category where all the arrows are invertible. Hence we have to define a binary operation of "multiplication" between composable arrows of R and an operation of "inversion" from R to itself. Moreover, in order to have a category we have to associate to every point x of U an arrow "identity" in R with source and target coinciding with x. Furthermore, we would like that all these five maps (source, target, multiplication, inverse and identity) are morphisms in the category $\mathscr{C}$ we are working in.

Finally, such five morphisms will have to satisfy some compatibility axioms. So we get the following definition:

Definition 3.1. ([BEC+], section 3.1) A *groupoid object* or *internal groupoid* in a fixed category $\mathscr{C}$ is the datum of two objects R, U and five morphisms of $\mathscr{C}$:

- two morphisms $s, t : R \rightrightarrows U$ such that the fiber product $R_t \times_s R$ *exists* in $\mathscr{C}$; these two maps are usually called *source* and *target* of the groupoid object;
- a morphism $m : R_t \times_s R \to R$ called *multiplication*;
- $i : R \to R$, known as *inverse* of the groupoid object;
- a morphism $e : U \to R$, called *identity*;

which satisfy the following axioms:

(i) the compositions $U \xrightarrow{e} R \xrightarrow{s} U$ and $U \xrightarrow{e} R \xrightarrow{t} U$ are both the identity map on U:

$$\begin{tikzcd} U \arrow[r, "e"] \arrow[dr, "1_U"'] \arrow[dr, phantom, shift left=3ex, "\curvearrowright"] & R \arrow[d, "s"] \\ & U \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "e"] \arrow[dr, "1_U"'] \arrow[dr, phantom, shift left=3ex, "\curvearrowright"] & R \arrow[d, "t"] \\ & U; \end{tikzcd}$$

(ii) if we call pr_1 and pr_2 the two projections from the fibered product $R_t \times_s R$ to R, then we get $s \circ m = s \circ pr_1$ and $t \circ m = t \circ pr_2$, i.e:

$$\begin{tikzcd} R_t \times_s R \arrow[r, "m"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\curvearrowright"] & R \arrow[d, "s"] \\ R \arrow[r, "s"'] & U \end{tikzcd} \qquad \begin{tikzcd} R_t \times_s R \arrow[r, "m"] \arrow[d, "pr_2"'] \arrow[dr, phantom, "\curvearrowright"] & R \arrow[d, "t"] \\ R \arrow[r, "t"'] & U; \end{tikzcd}$$

(iii) (associativity) the two morphisms $m \circ (1_R \times m)$ and $m \circ (m \times 1_R)$ are equal:

$$\begin{tikzcd}
R_t \times_s R_t \times_s R \arrow[r, "1_R \times m"] \arrow[d, "m \times 1_R"'] \arrow[dr, phantom, "\curvearrowright"] & R_t \times_s R \arrow[d, "m"] \\
R_t \times_s R \arrow[r, "m"'] & R;
\end{tikzcd}$$

(iv) (unit) the two morphisms $m \circ (e \circ s, 1_R)$ and $m \circ (1_R, e \circ t)$ from R to R are both equal to the identity of R:

$$\begin{tikzcd}
R \arrow[r, "(e \circ s, 1_R)"] \arrow[dr, "1_R"'] & R_t \times_s R \arrow[d, "m"] \arrow[dl, phantom, "\curvearrowright"] \\
 & R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "(1_R, e \circ t)"] \arrow[dr, "1_R"'] & R_t \times_s R \arrow[d, "m"] \arrow[dl, phantom, "\curvearrowright"] \\
 & R;
\end{tikzcd}$$

;

(v) (inverse) $i \circ i = 1_R$, $s \circ i = t$ and therefore $t \circ i = s$. Moreover, we require that $m \circ (1_R, i) = e \circ s$ and $m \circ (i, 1_R) = e \circ t$:

$$\begin{tikzcd}
R \arrow[r, "i"] \arrow[dr, "1_R"'] & R \arrow[d, "i"] \arrow[dl, phantom, "\curvearrowright"] \\
 & R
\end{tikzcd}
\quad
\begin{tikzcd}
R \arrow[r, "i"] \arrow[dr, "s"'] & R \arrow[d, "t"] \arrow[dl, phantom, "\curvearrowright"] \\
 & R
\end{tikzcd}
\quad
\begin{tikzcd}
R \arrow[r, "(1_R, i)"] \arrow[d, "s"'] \arrow[dr, phantom, "\curvearrowright"] & R_t \times_s R \arrow[d, "m"] \\
U \arrow[r, "e"'] & R
\end{tikzcd}
\quad
\begin{tikzcd}
R \arrow[r, "(i, 1_R)"] \arrow[d, "t"'] \arrow[dr, phantom, "\curvearrowright"] & R_t \times_s R \arrow[d, "m"] \\
U \arrow[r, "e"'] & R.
\end{tikzcd}$$

Using this list of axioms one can deduce the following useful property:

Lemma 3.1.1. $m \circ (i \circ pr_2, i \circ pr_1) = i \circ m$

This is stated (but not proved) in [BEC+], exercise 3.1. Actually, we will use this lemma only in the concrete category (**Manifolds**), hence the statement can be checked set-theoretically, and this is very simple using the previous axioms.

If we assume that the category $\mathscr{C}$ has been fixed, we will denote a groupoid object by any of the following 3 notations:

- (U, R, s, t, m, e, i);
- $R\ {}_s\times_t\ R \xrightarrow{m} R \xrightarrow{i} R \underset{t}{\overset{s}{\rightrightarrows}} U \xrightarrow{e} R$;
- $R \underset{t}{\overset{s}{\rightrightarrows}} U$.

Some articles (see, for example, [M], [MP], [MP2]) refer to a groupoid object with the notation $G = (G_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0)$. This is used whenever it is simpler to think to a groupoid as a category G where all morphisms are invertible.

Remark 3.1. Given any pair of morphisms $R \underset{t}{\overset{s}{\rightrightarrows}} U$ in a fixed category $\mathscr{C}$, we must take care if their fibered product $R\ {}_s\times_t\ R$ exists or not. This is always ensured e.g. when we work in the categories (**Sets**), (**Groups**) or (**Schemes**), but in general it is no more true in the category (**Manifolds**) (complex manifolds and holomorphic maps between them).

In this last category it is known from category theory that if the fibered product of any pair of morphisms $A \xrightarrow{f} B$ and $C \xrightarrow{g} B$ exists, it is obtained adding a natural structure of manifold on the set-theoretical fibered product. Now the first step can always be done (in (**Sets**)), while the second one requires additional conditions on the maps used, so when in section 5 we will deal with this category we will have to take care about the hypothesis we put on s and t.

We are mainly interested in the category (**Manifolds**), so the only examples about groupoid objects will be given in section 5. For the same reason, we will not give in the following sections any example of morphisms and 2-morphisms of groupoids and will fill this gap only in the next chapter.

Lemma 3.1.2. *To every groupoid object $R \underset{t}{\overset{s}{\rightrightarrows}} U$ in a fixed* concrete *category $\mathscr{C}$ we can associate a* small *category $\mathscr{R}$ where all the morphisms are invertible.*

This is why some authors refers to a groupoid object as a category with only isomorphisms; we will see in the following proof that in order to associate a category to the groupoid object we are forced to loose some informations about this object, so *the two concepts are not equivalent.*

Proof. We recall that in §1.1 we defined a concrete category $\mathscr{C}$ as a category where the objects are sets (with additional properties) and the morphisms are set maps (with additional properties); so in this case we have in particular that both R and U are sets. Hence we can define a new category $\mathscr{R}$ as follows:

- the class of objects $\mathscr{R}_0$ is just the *set* U (so the category will be *small*);
- for every pair of objects $x, y \in U$ we define:

$$\mathscr{R}(x,y) := \{f \in R \text{ s.t. } s(f) = x \text{ and } t(f) = y\}.$$

Now for every triple of objects x, y, z we define also the composition law:

$$\mathscr{R}(x,y) \times \mathscr{R}(y,z) \to \mathscr{R}(x,z).$$

as follows: let us fix any $f \in \mathscr{R}(x,y)$ and any $g \in \mathscr{R}(y,z)$; then by definition of $\mathscr{R}$ we get that $t(f) = y = s(g)$, hence $(f,g) \in R\,{}_t\!\times_s R$, so it makes sense to define:

$$g \circ f := m(f,g).$$

Now we have also to define the identity for every object in $U = \mathscr{R}_0$; so for every object x we set:

$$1_x := e(x).$$

This is well defined, indeed $s(1_x) = s \circ e(x) = id(x) = x$ and $t(1_x) = t \circ e(x) = id(x) = x$, so $1_x \in \mathscr{R}(x,x)$.

Then the axioms of a category are easily verified using the properties of groupoid objects. Actually axioms (i) to (v) of groupoid objects are obtained by the properties of categories, by "substracting" anything that refers to set and set maps. This is also the main reason of the names of the structure maps s, t, i, e of any groupoid object $R \overset{s}{\underset{t}{\rightrightarrows}} U$, also in the case when $\mathscr{C}$ is not a concrete category. For an explanation of the word "multiplication", see examples 3.3 and 3.4 at the end of this chapter.

□

Note that in this construction we loose completely the informations about the inversion map i.

3.2 Morfisms between groupoid objects

Now we are ready to define morphisms between groupoid objects in a fixed category $\mathscr{C}$.

Definition 3.2. Given two groupoid objects $R \overset{s}{\underset{t}{\rightrightarrows}} U$ and $R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'$ in a fixed category $\mathscr{C}$, a *morphism* between them is a pair (ψ, Ψ), where $\psi : U \to U'$ and $\Psi : R \to R'$ are both morphisms in $\mathscr{C}$, which together commute with all structure morphisms of the two groupoid objects. In other words, we ask that all the following diagrams are commutative:

$$\begin{tikzcd}
R \arrow[r, "\Psi"] \arrow[d, "s"'] \arrow[dr, phantom, "\curvearrowright"] & R' \arrow[d, "s'"] \\
U \arrow[r, "\psi"'] & U'
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\Psi"] \arrow[d, "t"'] \arrow[dr, phantom, "\curvearrowright"] & R' \arrow[d, "t'"] \\
U \arrow[r, "\psi"'] & U'
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "\psi"] \arrow[d, "e"'] \arrow[dr, phantom, "\curvearrowright"] & U' \arrow[d, "e'"] \\
R \arrow[r, "\Psi"'] & R'
\end{tikzcd}$$

$$\begin{tikzcd}
R\,{}_t\!\times_s R \arrow[r, "\Psi\times\Psi"] \arrow[d, "m"'] \arrow[dr, phantom, "\curvearrowright"] & R'\,{}_{t'}\!\times_{s'} R' \arrow[d, "m'"] \\
R \arrow[r, "\Psi"'] & R'
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\Psi"] \arrow[d, "i"'] \arrow[dr, phantom, "\curvearrowright"] & R' \arrow[d, "i'"] \\
R \arrow[r, "\Psi"'] & R'.
\end{tikzcd}$$

Remark 3.2. Note that if we fix the morphism Ψ, then ψ is uniquely determined by these properties: indeed using axiom (i) for the second groupoid object and the previous diagrams, we get that:

$$\begin{tikzcd}
U \arrow[r, "\psi"'] \arrow[d, "e"'] \arrow[dr, phantom, "\curvearrowright"] & U' \arrow[d, "e'\curvearrowright"] \arrow[dr, "1_{U'}"] & \\
R \arrow[r, "\Psi"] & R' \arrow[r, "s'"] & U'
\end{tikzcd}$$

i.e. $\psi = s' \circ \Psi \circ e$. Note also that given this formula, ψ is certainly a morphism in $\mathscr{C}$, since it is a composition of morphisms in this category.

Definition 3.3. Let us consider 3 groupoid objects in $\mathscr{C}$ together with 2 morphisms:

$$(R \underset{t}{\overset{s}{\rightrightarrows}} U) \xrightarrow{(\psi,\Psi)} (R' \underset{t'}{\overset{s'}{\rightrightarrows}} U') \xrightarrow{(\phi,\Phi)} (R'' \underset{t''}{\overset{s''}{\rightrightarrows}} U'').$$

Then we define the *composition*:

$$(\phi, \Phi) \circ (\psi, \Psi) := (\phi \circ \psi, \Phi \circ \Psi).$$

It is easy to prove that this is actually a morphism of groupoid objects: $(R \underset{t}{\overset{s}{\rightrightarrows}} U) \to (R'' \underset{t''}{\overset{s''}{\rightrightarrows}} U'')$; for example, in order to prove that it preserves the source it suffices to compose the following 2 commutative squares:

$$\begin{tikzcd}
R \arrow[r, "\Psi"] \arrow[d, "s"'] \arrow[dr, phantom, "\curvearrowright"] & R' \arrow[r, "\Phi"] \arrow[d, "s'"] \arrow[dr, phantom, "\curvearrowright"] & R'' \arrow[d, "s''"] \\
U \arrow[r, "\psi"'] & U' \arrow[r, "\phi"'] & U'';
\end{tikzcd}$$

in the same way one can prove that $(\phi \circ \psi, \Phi \circ \Psi)$ preserves also targets, multiplications, identities and inverses. Moreover, it is clear that both $\phi \circ \psi$ and $\Phi \circ \Psi$ are morphisms in $\mathscr{C}$, hence we have defined a new morphism of groupoid objects in $\mathscr{C}$.

Lemma 3.2.1. *Let us fix a morphism* $(\psi, \Psi) : (R \overset{s}{\underset{t}{\rightrightarrows}} U) \to (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U')$ *and let us suppose that* $\mathscr{C}$ *is a* concrete *category. Then we can associate to* (ψ, Ψ) *a functor:*

$$\widetilde{\Psi} : \mathscr{R} \to \mathscr{R}'$$

between the categories associated to the two groupoid objects in lemma 3.1.2.

Proof. The functor $\widetilde{\Psi}$ is defined on the level of objects as ψ and on the level of morphisms as Ψ. Then the axioms for a covariant functor are just the first 4 squares of the previous definition. ☐

Note that using this lemma the composition of morphisms just defined induces a composition of the corresponding functors described in the lemma.

3.3 2-morphims

Now we want to make groupoid objects into a 2-category, i.e. we want to define 2-morphisms, which will be called “natural tranformations”.

Definition 3.4. (a little generalization of [M],§2.2) Given two morphisms of groupoid objects in $\mathscr{C}$:

$$(\psi, \Psi), (\phi, \Phi) : \ (R \overset{s}{\underset{t}{\rightrightarrows}} U) \to (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U')$$

a *natural transformation* $\alpha : (\psi, \Psi) \Rightarrow (\phi, \Phi)$ is the datum of a *morphism* $\alpha : U \to R'$ in $\mathscr{C}$ such that:

(i) $s' \circ \alpha = \psi$;

(ii) $t' \circ \alpha = \phi$;

using (i) and (ii) together with the definition of morphism between groupoid objects, we get that:

$$t' \circ (\alpha \circ s) = \phi \circ s = s' \circ \Phi$$
$$\text{and} \quad t' \circ \Psi = \psi \circ t = s' \circ (\alpha \circ t).$$

Hence we can consider both $(\alpha \circ s, \Phi)$ and $(\Psi, \alpha \circ t)$ as morphisms in the category $\mathscr{C}$ from R to $R'\,_{t'}\times_{s'} R'$. Then we require that the following equality between morphisms from R to R' holds:

(iii) $m' \circ (\alpha \circ s, \Phi) = m' \circ (\Psi, \alpha \circ t)$.

Lemma 3.3.1. *Let us suppose we have a natural transformation α as in the previous definition in a concrete category $\mathscr{C}$ and let us suppose that $\mathscr{R}$ and $\mathscr{R}'$ are the categories associated to $R \underset{t}{\overset{s}{\rightrightarrows}} U$ and $R' \underset{t'}{\overset{s'}{\rightrightarrows}} U'$ respectively by lemma 3.1.2; moreover, let us suppose that $\widetilde{\Psi}$ and $\widetilde{\Phi}$ are the functors associated to (ψ, Ψ) and (ϕ, Φ) respectively by lemma 3.2.1. Then we can associate to α a natural transformation of functors:*

$$\tilde{\alpha} : \widetilde{\Psi} \to \widetilde{\Phi}.$$

Proof. Since $\mathscr{C}$ is a concrete category, we have that α is a *set map* (with additional properties) from U to R'. Using (i) and (ii) of definition 3.4, we get that for every point $x \in U = \mathscr{R}_0$:

$$s'(\alpha(x)) = \psi(x) = \widetilde{\Psi}(x)$$
$$\text{and} \quad t'(\alpha(x)) = \phi(x) = \widetilde{\Phi}(x)$$

i.e. $\alpha(x) \in \mathscr{R}'(\widetilde{\Psi}(x), \widetilde{\Phi}(x)) \quad \forall x \in \mathscr{R}_0$. Moreover, let us fix any pair of objects $x, y \in \mathscr{R}_0$ and any $f \in \mathscr{R}(x, y)$ (i.e. any point $f \in R$ such that $s(f) = x$ and $t(f) = y$); using (iii) of definition 3.4 and the previuos 2 lemmas, we get that:

$$\widetilde{\Phi}(f)\circ\alpha(x)=m'(\alpha(s(f)),\widetilde{\Phi}(f))=m'(\alpha\circ s(f),\Phi(f))=m'\circ(\alpha\circ s,\Phi)(f)\overset{(iii)}{=}$$

$$\overset{(iii)}{=} m'\circ(\Psi,\alpha\circ t)(f)=m'(\Psi(f),\alpha\circ t(f))=m'(\widetilde{\Psi}(f),\alpha(y))=\alpha(y)\circ\widetilde{\Psi}(f).$$

In other words, if for any $x\in\mathscr{R}_0$ we set $\tilde{\alpha}_x:=\alpha(x)$, we get that for any $f\in\mathscr{R}(x,y)$ the following diagram is commutative in $\mathscr{R}'$:

$$\begin{tikzcd}
\widetilde{\Psi}(x) \arrow[r, "\tilde{\alpha}_x"] \arrow[d, "\widetilde{\Psi}(f)"'] \arrow[dr, phantom, "\curvearrowright"] & \widetilde{\Phi}(x) \arrow[d, "\widetilde{\Phi}(f)"] \\
\widetilde{\Psi}(y) \arrow[r, "\tilde{\alpha}_y"'] & \widetilde{\Phi}(y).
\end{tikzcd}$$

i.e: we have defined a natural transformation between functors:

$$\tilde{\alpha}:\widetilde{\Psi}\Rightarrow\widetilde{\Phi}.$$

□

Clearly all the previous 3 lemmas only work in the case where $\mathscr{C}$ is a concrete category; in the general case it does not make any sense the nototion of category associated to a groupoid object $R\overset{s}{\underset{t}{\rightrightarrows}}U$, so also the second and the third lemma loose their sense.

Definition 3.5. Let us consider 2 natural transformations as follows:

$$\begin{tikzcd}
(R\overset{s}{\underset{t}{\rightrightarrows}}U) \arrow[rr, bend left=50, "{(\psi_1,\Psi_1)}"] \arrow[rr, "{(\psi_2,\Psi_2)}"'] \arrow[rr, bend right=50, "{(\psi_3,\Psi_3)}"'] & \overset{\Downarrow\alpha}{} \underset{\Downarrow\beta}{} & (R'\overset{s'}{\underset{t'}{\rightrightarrows}}U');
\end{tikzcd}$$

using definition 3.4 for α and β we get that:

$$t' \circ \alpha = \psi_2 = s' \circ \beta,$$

hence we can consider the morphism $(\alpha, \beta) : U \to R' {}_{t'}\times_{s'} R'$, so we can define:

$$\beta \odot \alpha := m' \circ (\alpha, \beta) : U \to R'.$$

Lemma 3.3.2. *In this way we have defined a natural transformation:*

$$\beta \odot \alpha : (\psi_1, \Psi_1) \Rightarrow (\psi_3, \Psi_3).$$

Proof. Using again definition 3.4 and property (ii) of definition 3.1 we get that:

$$s' \circ (\beta \odot \alpha) = s' \circ m' \circ (\alpha, \beta) = s' \circ pr'_1 \circ (\alpha, \beta) = s' \circ \alpha = \psi_1$$

$$\text{and} \quad t' \circ (\beta \odot \alpha) = t' \circ m' \circ (\alpha, \beta) = t' \circ pr'_2 \circ (\alpha, \beta) = t' \circ \beta = \psi_2.$$

Hence conditions (i) and (ii) of definition 3.4 are satisfied. Now if we use condition (iii) for α and β we get:

$$m' \circ (\alpha \circ s, \Psi_2) = m' \circ (\Psi_1, \alpha \circ t); \tag{3.1}$$

$$m' \circ (\beta \circ s, \Psi_3) = m' \circ (\Psi_2, \beta \circ t). \tag{3.2}$$

Hence we obtain:

$$m' \circ ((\beta \odot \alpha) \circ s, \Psi_3) = m' \circ (m' \circ (\alpha, \beta) \circ s, \Psi_3) =$$
$$= m' \circ (m' \circ (\alpha \circ s, \beta \circ s), \Psi_3) \overset{*}{=} m' \circ (\alpha \circ s, m'(\beta \circ s, \Psi_3)) \overset{(3.2)}{=}$$
$$\overset{(3.2)}{=} m' \circ (\alpha \circ s, m' \circ (\Psi_2, \beta \circ t)) \overset{*}{=} m' \circ (m' \circ (\alpha \circ s, \Psi_2), \beta \circ t) \overset{(3.1)}{=}$$
$$\overset{(3.1)}{=} m' \circ (m' \circ (\Psi_1, \alpha \circ t), \beta \circ t) \overset{*}{=} m' \circ (\Psi_1, m' \circ (\alpha \circ t, \beta \circ t)) =$$
$$= m' \circ (\Psi_1, m' \circ (\alpha, \beta) \circ t) = m' \circ (\Psi_1, (\beta \odot \alpha) \circ t)$$

where all the passages denoted with $\overset{*}{=}$ are just axiom (iii) of definition 3.1. So condition (iii) of definition 3.4 is satisfied, i.e. $\beta \odot \alpha$ is a natural transformation as we claimed. □

Let us define also the *horizontal composition* of natural transformations:

Definition 3.6. Consider a diagram of this form:

$$\begin{tikzcd}
(R \underset{t}{\overset{s}{\rightrightarrows}} U) \arrow[r, bend left, "(\psi_1,\Psi_1)"] \arrow[r, bend right, "(\psi_2,\Psi_2)"'] \arrow[r, phantom, "\Downarrow \alpha"] & (R' \underset{t'}{\overset{s'}{\rightrightarrows}} U') \arrow[r, bend left, "(\phi_1,\Phi_1)"] \arrow[r, bend right, "(\phi_2,\Phi_2)"'] \arrow[r, phantom, "\Downarrow \beta"] & (R'' \underset{t''}{\overset{s''}{\rightrightarrows}} U'').
\end{tikzcd}$$

In particular, we get that:

$$s' \circ \alpha = \psi_1 \quad \text{and} \quad t' \circ \alpha = \psi_2; \tag{3.3}$$

$$s'' \circ \beta = \phi_1 \quad \text{and} \quad t'' \circ \beta = \phi_2; \tag{3.4}$$

$$m' \circ (\alpha \circ s, \Psi_2) = m' \circ (\Psi_1, \alpha \circ t); \tag{3.5}$$

$$m'' \circ (\beta \circ s', \Phi_2) = m'' \circ (\Phi_1, \beta \circ t'). \tag{3.6}$$

So:

$$t'' \circ (\Phi_1 \circ \alpha) = \phi_1 \circ t' \circ \alpha \overset{(3.3)}{=} \phi_1 \circ \psi_2 \overset{(3.4)}{=} s'' \circ (\beta \circ \psi_2);$$

hence $(\Phi_1 \circ \alpha, \beta \circ \psi_2)$ ia a morphism from U to $R''\,_{t''}\times_{s''} R''$, so we can define:

$$\beta * \alpha := m'' \circ (\Phi_1 \circ \alpha, \beta \circ \psi_2) : U \to R''.$$

Lemma 3.3.3. *$\beta * \alpha$ is a natural transformation from $(\phi_1, \Phi_1) \circ (\psi_1, \Psi_1)$ to $(\phi_2, \Phi_2) \circ (\psi_2, \Psi_2)$, called horizontal composition of α and β.*

Proof. Since all the maps involved are morphisms in $\mathscr{C}$ we get that also $\beta * \alpha$ is a morphism in this category. Now we have that:

$$s'' \circ (\beta * \alpha) = s'' \circ m'' \circ (\Phi_1 \circ \alpha, \beta \circ \psi_2) = s'' \circ \Phi_1 \circ \alpha = \phi_1 \circ s' \circ \alpha \overset{(3.3)}{=} \phi_1 \circ \psi_1$$
$$\text{and} \quad t'' \circ (\beta * \alpha) = t'' \circ m'' \circ (\Phi_1 \circ \alpha, \beta \circ \psi_2) = t'' \circ \beta \circ \psi_2 \overset{(3.4)}{=} \phi_2 \circ \psi_2;$$

hence conditions (i) and (ii) of definition 3.4 are satisfied. Let us prove (iii); in order to do this, we give the following preliminary results:

(A)

$$(\beta * \alpha) \circ s = m'' \circ (\Phi_1 \circ \alpha, \beta \circ \psi_2) \circ s = m'' \circ (\Phi_1 \circ \alpha \circ s, \beta \circ \psi_2 \circ s);$$

(B)

$$m''\circ(\beta\circ\psi_2\circ s, \Phi_2\circ\Psi_2) = m''\circ(\beta\circ s'\circ\Psi_2, \Phi_2\circ\Psi_2) = m''\circ(\beta\circ s', \Phi_2)\circ\Psi_2 \overset{(3.6)}{=}$$
$$\overset{(3.6)}{=} m''\circ(\Phi_1, \beta\circ t')\circ\Psi_2 = m''\circ(\Phi_1\circ\Psi_2, \beta\circ t'\circ\Psi_2) = m''\circ(\Phi_1\circ\Psi_2, \beta\circ\psi_2\circ t);$$

(C)

$$m'' \circ (\Phi_1 \circ \alpha \circ s, \Phi_1 \circ \Psi_2) = m'' \circ (\Phi_1 \times \Phi_1) \circ (\alpha \circ s, \Psi_2) =$$
$$= \Phi_1 \circ m' \circ (\alpha \circ s, \Psi_2) \overset{(3.5)}{=} \Phi_1 \circ m' \circ (\Psi_1, \alpha \circ t) =$$
$$= m'' \circ (\Phi_1 \times \Phi_1) \circ (\Psi_1, \alpha \circ t) = m'' \circ (\Phi_1 \circ \Psi_1, \Phi_1 \circ \alpha \circ t);$$

(D)

$$m'' \circ (\Phi_1 \circ \alpha \circ t, \beta \circ \psi_2 \circ t) = m'' \circ (\Phi_1 \circ \alpha, \beta \circ \psi_2) \circ t = (\beta * \alpha) \circ t.$$

So we have that:

$$m'' \circ ((\beta * \alpha) \circ s, \Phi_2 \circ \Psi_2) \overset{(A)}{=} m'' \circ (m'' \circ (\Phi_1 \circ \alpha \circ s, \beta \circ \psi_2 \circ s), \Phi_2 \circ \Psi_2) =$$
$$= m''\circ(\Phi_1\circ\alpha\circ s, m''\circ(\beta\circ\psi_2\circ s, \Phi_2\circ\Psi_2)) \overset{(B)}{=} m''\circ(\Phi_1\circ\alpha\circ s, m''\circ(\Phi_1\circ\Psi_2, \beta\circ\psi_2\circ t)) =$$

$$= m''\circ(m''\circ(\Phi_1\circ\alpha\circ s, \Phi_1\circ\Psi_2), \beta\circ\psi_2\circ t) \overset{(C)}{=} m''\circ(m''\circ(\Phi_1\circ\Psi_1, \Phi_1\circ\alpha\circ t), \beta\circ\psi_2\circ t) =$$

$$= m'' \circ (\Phi_1 \circ \Psi_1, m'' \circ (\Phi_1 \circ \alpha \circ t, \beta \circ \psi_2 \circ t)) \overset{(D)}{=} m'' \circ (\Phi_1 \circ \Psi_1, (\beta * \alpha) \circ t)$$

where all the passages without label are just property (iii) of groupoid objects; so also property (iii) of definition 3.4 is satisfied, hence $\beta * \alpha$ is a natural transformation from $(\phi_1 \circ \psi_1, \Phi_1 \circ \Psi_1)$ to $(\phi_2 \circ \psi_2, \Phi_2 \circ \Psi_2)$.

□

3.4 The 2-category ($\mathscr{C}$ – Groupoids)

Now we want to prove that with these definitions of groupoid objects, morphisms and natural transformations we can describe a 2-category, that we will denote with ($\mathscr{C}$ – **Groupoids**). The $\mathscr{C}$ here means that all objects and morphisms we work with are in a fixed category $\mathscr{C}$. It is clear from our notations what we mean with objects, 1-morphisms and 2-morphisms of this new 2-category.

In order to simplify the following proofs, we will denote from now on the 2-category ($\mathscr{C}$ – **Groupoids**) (that we are going to define) with the letter $\mathscr{D}$.

Definition 3.7. First of all, we define the class of objects $\mathscr{D}_0$ as the set of all groupoid objects in the category $\mathscr{C}$ we have fixed. Then for every pair of objects $R \underset{t}{\overset{s}{\rightrightarrows}} U$, $R' \underset{s'}{\overset{t'}{\rightrightarrows}} U'$ we define the small category $\mathscr{D}(R \underset{t}{\overset{s}{\rightrightarrows}} U, R' \underset{t'}{\overset{s'}{\rightrightarrows}} U')$ as follows:

- the objects are all the morphisms of the form:

$$(\psi, \Psi) : (R \underset{t}{\overset{s}{\rightrightarrows}} U) \to (R' \underset{t'}{\overset{s'}{\rightrightarrows}} U');$$

- the morphisms are all the natural transformations of the form:

$$\alpha : (\psi, \Psi) \Rightarrow (\phi, \Phi).$$

The composition in this small category will be the vertical composition $\odot$ already defined. Moreover, for every object $(\psi, \Psi) : R \overset{s}{\underset{t}{\rightrightarrows}} U \to R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'$ we define the morphism:

$$i_{(\psi,\Psi)} : (\psi, \Psi) \Rightarrow (\psi, \Psi)$$

as $i_{(\psi,\Psi)} := e' \circ \psi = \Psi \circ e$; this is a natural transformation from (ψ, Ψ) to itself; indeed:

$$s' \circ i_{(\psi,\Psi)} = s' \circ e' \circ \psi = \psi \quad \text{and} \quad t' \circ i_{(\psi,\Psi)} = t' \circ e' \circ \psi = \psi;$$

moreover,

$$m'(i_{(\psi,\Psi)} \circ s, \Psi) = m' \circ (e' \circ \psi \circ s, \Psi) = m' \circ (e' \circ s' \circ \Psi, \Psi) =$$

$$= m' \circ (e' \circ s', 1_{R'}) \circ \Psi \overset{*}{=} 1_{R'} \circ \Psi \overset{*}{=} m' \circ (1_{R'}, e' \circ t') \circ \Psi =$$

$$= m' \circ (\Psi, e' \circ t' \circ \Psi) = m' \circ (\Psi, e' \circ \psi \circ t) = m' \circ (\Psi, i_{(\psi,\Psi)} \circ t)$$

where the passages denotes with $\overset{*}{=}$ come from axiom (iv) of groupoid objects. Hence $i_{(\psi,\Psi)}$ is a morphism in $\mathscr{D}(R \overset{s}{\underset{t}{\rightrightarrows}} U, R' \overset{s'}{\underset{t'}{\rightrightarrows}} U')$.

We omit the simple proof that actually $\mathscr{D}(R \overset{s}{\underset{t}{\rightrightarrows}} U, R' \overset{s'}{\underset{t'}{\rightrightarrows}} U')$ is a small category.

Until now we have complely defined data (1) and (2) of definition 1.9, so let us also define the datum (3), i.e. the functor "composition".

Definition 3.8. For every triple of groupoid objects $(R \overset{s}{\underset{t}{\rightrightarrows}} U)$, $(R' \overset{s'}{\underset{t'}{\rightrightarrows}} U')$, $(R'' \overset{s''}{\underset{t''}{\rightrightarrows}} U'')$ we define the functor $c = c_{(R\rightrightarrows U),(R'\rightrightarrows U'),(R''\rightrightarrows U'')}$:

$$c : \mathscr{D}(R \overset{s}{\underset{t}{\rightrightarrows}} U, R' \overset{s'}{\underset{t'}{\rightrightarrows}} U') \times \mathscr{D}(R' \overset{s'}{\underset{t'}{\rightrightarrows}} U', R'' \overset{s''}{\underset{t''}{\rightrightarrows}} U'') \to \mathscr{D}(R \overset{s}{\underset{t}{\rightrightarrows}} U, R'' \overset{s''}{\underset{t''}{\rightrightarrows}} U'')$$

as follows:

- for every pair of objects:

  $(\psi, \Psi) : (R \overset{s}{\underset{t}{\rightrightarrows}} U) \to (R' \overset{t'}{\underset{s'}{\rightrightarrows}} U')$ and $(\phi, \Phi) : (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U') \to (R'' \overset{t''}{\underset{s''}{\rightrightarrows}} U'')$

  we define $c\Big((\psi, \Psi), (\phi, \Phi)\Big) := (\phi, \Phi) \circ (\psi, \Psi)$.

- for every pair of morphisms (i.e. natural transformations):

$$\begin{tikzcd}
(R \overset{s}{\underset{t}{\rightrightarrows}} U) \arrow[r, bend left, "(\psi_1{,}\Psi_1)"] \arrow[r, bend right, "(\psi_2{,}\Psi_2)"'] \arrow[r, phantom, "\Downarrow \alpha"] & (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U') \arrow[r, bend left, "(\phi_1{,}\Phi_1)"] \arrow[r, bend right, "(\phi_2{,}\Phi_2)"'] \arrow[r, phantom, "\Downarrow \beta"] & (R'' \overset{s''}{\underset{t''}{\rightrightarrows}} U'')
\end{tikzcd}$$

  we define $c(\alpha, \beta) := \beta * \alpha$.

Our aim now is to prove that c preserves compositions. As in the case of (**Pre-Orb**) this is equivalent to prove that the *interchange law* is satified; in other words, for every diagram of the form:

$$\begin{tikzcd}
(R \overset{s}{\underset{t}{\rightrightarrows}} U) \arrow[r, bend left=60, "(\psi_1{,}\Psi_1)"] \arrow[r, "(\psi_2{,}\Psi_2)"'] \arrow[r, bend right=60, "(\psi_3{,}\Psi_3)"'] \arrow[r, phantom, shift left=4, "\Downarrow \alpha"] \arrow[r, phantom, shift right=4, "\Downarrow \mu"] & (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U') \arrow[r, bend left=60, "(\phi_1{,}\Phi_1)"] \arrow[r, "(\phi_2{,}\Phi_2)"'] \arrow[r, bend right=60, "(\phi_3{,}\Phi_3)"'] \arrow[r, phantom, shift left=4, "\Downarrow \beta"] \arrow[r, phantom, shift right=4, "\Downarrow \nu"] & (R'' \overset{s''}{\underset{t''}{\rightrightarrows}} U'')
\end{tikzcd}$$

we have to prove that:

$$(\nu * \mu) \odot (\beta * \alpha) \overset{?}{=} (\nu \odot \beta) * (\mu \odot \alpha). \tag{3.7}$$

Now

$$\nu * \mu = m'' \circ (\Phi_2 \circ \mu, \nu \circ \psi_3) \quad \text{and} \quad \beta * \alpha = m'' \circ (\Phi_1 \circ \alpha, \beta \circ \psi_2)$$

so the L.H.S. of (3.7) is equal to:

$$m'' \circ (m'' \circ (\Phi_1 \circ \alpha, \beta \circ \psi_2), m'' \circ (\Phi_2 \circ \mu, \nu \circ \psi_3)) \overset{*}{=}$$

$$\overset{*}{=} m'' \circ (\Phi_1 \circ \alpha, m'' \circ (m'' \circ (\beta \circ \psi_2, \Phi_2 \circ \mu), \nu \circ \psi_3)).$$

On the other hand,

$$\nu \odot \beta = m'' \circ (\beta, \nu) \quad \text{and} \quad \mu \odot \alpha = m'' \circ (\alpha \circ \mu);$$

hence the R.H.S. of (3.7) is equal to:

$$m'' \circ (\Phi_1 \circ m'' \circ (\alpha, \mu), m'' \circ (\beta, \nu) \circ \psi_3) =$$

$$= m'' \circ (m'' \circ (\Phi_1 \circ \alpha, \Phi_1 \circ \mu), m'' \circ (\beta \circ \psi_3, \nu \circ \psi_3)) \overset{*}{=}$$

$$\overset{*}{=} m'' \circ (\Phi_1 \circ \alpha, m'' \circ (m'' \circ (\Phi_1 \circ \mu, \beta \circ \psi_3), \nu \circ \psi_3)).$$

Here $\overset{*}{=}$ comes from axiom (iii) of groupoid objects. Now if we compare the left and the right hand side of (3.7) we notice that they are equal except for the central part, so in order to prove (3.7) it suffices to prove that:

$$m'' \circ (\beta \circ \psi_2, \Phi_2 \circ \mu) \overset{?}{=} m'' \circ (\Phi_1 \circ \mu, \beta \circ \psi_3). \tag{3.8}$$

Now we recall that by hypothesis μ is a natural transformation from (ψ_2, Ψ_2) to (ψ_3, Ψ_3), and β is a natural transformation from (ϕ_1, Φ_1) to (ϕ_2, Φ_2), so in particular:

$$\psi_2 = s' \circ \mu, \quad \psi_3 = t' \circ \mu \quad \text{and} \quad m'' \circ (\Phi_1, \beta \circ t') = m'' \circ (\beta \circ s', \Phi_2).$$

Hence:

$$m'' \circ (\beta \circ \psi_2, \Phi_2 \circ \mu) = m'' \circ (\beta \circ s' \circ \mu, \Phi_2 \circ \mu) =$$

$$= m'' \circ (\beta \circ s', \Phi_2) \circ \mu = m'' \circ (\Phi_1, \beta \circ t') \circ \mu =$$

$$= m'' \circ (\Phi_1 \circ \mu, \beta \circ t' \circ \mu) = m'' \circ (\Phi_1 \circ \mu, \beta \circ \psi_3).$$

So we have proved that (3.8) is true, hence also (3.7) holds, so we have proved that c preserves compositions. In the same way one can prove that it preserves also the identities, hence c is a functor.

Definition 3.9. For every groupoid object $R \overset{s}{\underset{t}{\rightrightarrows}} U$ we define the morphism $1_{R\rightrightarrows U} := (1_U, 1_R)$ defined from $R \overset{s}{\underset{t}{\rightrightarrows}} U$ to itself. Moreover, we define the natural trasformation:

$$i_{R\rightrightarrows U} := e : U \to R$$

This is actually a natural transformation from $1_{R\rightrightarrows U}$ to itself. Together this 2 definitions are equivalent to datum (4) of definition 1.9.

Until now we have only defined the data of the 2-category $\mathscr{D}$ as in definition 1.9; we have also to prove that the 4 axioms of a 2-category are satisfied (see remark 1.5). We will give a proof only of the non trivial part, namely axiom (b): suppose we are in the following situation:

$$\begin{tikzcd}
(R_1 \overset{s_1}{\underset{t_1}{\rightrightarrows}} U_1) \arrow[r, bend left, "{(\psi_1,\Psi_1)}"] \arrow[r, bend right, "{(\psi_2,\Psi_2)}"'] \arrow[r, phantom, "\Downarrow \alpha"] & (R_2 \overset{s_2}{\underset{t_2}{\rightrightarrows}} U_2) \arrow[r, bend left, "{(\phi_1,\Phi_1)}"] \arrow[r, bend right, "{(\phi_2,\Phi_2)}"'] \arrow[r, phantom, "\Downarrow \beta"] & (R_3 \overset{s_3}{\underset{t_3}{\rightrightarrows}} U_3) \arrow[r, bend left, "{(\theta_1,\Theta_1)}"] \arrow[r, bend right, "{(\theta_2,\Theta_2)}"'] \arrow[r, phantom, "\Downarrow \gamma"] & (R_4 \overset{s_4}{\underset{t_4}{\rightrightarrows}} U_4).
\end{tikzcd}$$

We want to prove that $(\gamma * \beta) * \alpha \overset{?}{=} \gamma * (\beta * \alpha)$. Let us consider the L.H.S. of this relation: we recall that $\gamma * \beta = m_4 \circ (\Theta_1 \circ \beta, \gamma \circ \phi_2)$, so we use:

$$\begin{tikzcd}
(R_1 \overset{s_1}{\underset{t_1}{\rightrightarrows}} U_1) \arrow[r, bend left, "{(\psi_1,\Psi_1)}"] \arrow[r, bend right, "{(\psi_2,\Psi_2)}"'] \arrow[r, phantom, "\Downarrow \alpha"] & (R_2 \overset{s_2}{\underset{t_2}{\rightrightarrows}} U_2) \arrow[r, bend left, "{(\theta_1\circ\phi_1,\Theta_1\circ\Phi_1)}"] \arrow[r, bend right, "{(\theta_2\circ\phi_2,\Theta_2\circ\Phi_2)}"'] \arrow[r, phantom, "\Downarrow \gamma * \beta"] & (R_4 \overset{s_4}{\underset{t_4}{\rightrightarrows}} U_4).
\end{tikzcd}$$

in order to compute:

$$(\gamma * \beta) * \alpha = m_4 \circ (\Theta_1 \circ \Phi_1 \circ \alpha, m_4 \circ (\Theta_1 \circ \beta, \gamma \circ \phi_2) \circ \psi_2). \tag{3.9}$$

On the other hand, if we want to compute the R.H.S, we recall that $\beta * \alpha = m_3 \circ (\Phi_1 \circ \alpha, \beta \circ \psi_2)$; hence using the diagram:

$$\begin{tikzcd}
(R_1 \overset{s_1}{\underset{t_1}{\rightrightarrows}} U_1) \arrow[r, bend left, "{(\phi_1 \circ \psi_1, \Phi_1 \circ \Psi_1)}"] \arrow[r, bend right, "{(\phi_2 \circ \psi_2, \Phi_2 \circ \Psi_2)}"'] \arrow[r, phantom, "\Downarrow \beta * \alpha"] & (R_3 \overset{s_3}{\underset{t_3}{\rightrightarrows}} U_3) \arrow[r, bend left, "{(\theta_1, \Theta_1)}"] \arrow[r, bend right, "{(\theta_2, \Theta_2)}"'] \arrow[r, phantom, "\Downarrow \gamma"] & (R_4 \overset{s_4}{\underset{t_4}{\rightrightarrows}} U_4)
\end{tikzcd}$$

we get that:

$$\begin{aligned}
\gamma * (\beta * \alpha) &= m_4 \circ (\Theta_1 \circ m_3 \circ (\Phi_1 \circ \alpha, \beta \circ \psi_2), \gamma \circ \phi_2 \circ \psi_2) = \\
&= m_4 \circ (m_4 \circ (\Theta_1 \times \Theta_1) \circ (\Phi_1 \circ \alpha, \beta \circ \psi_2), \gamma \circ \phi_2 \circ \psi_2) = \\
&= m_4 \circ (m_4 \circ (\Theta_1 \circ \Phi_1 \circ \alpha, \Theta_1 \circ \beta \circ \psi_2), \gamma \circ \phi_2 \circ \psi_2) = \\
&= m_4 \circ (\Theta_1 \circ \Phi_1 \circ \alpha, m_4 \circ (\Theta_1 \circ \beta \circ \psi_2, \gamma \circ \phi_2 \circ \psi_2)) = \\
&= m_4 \circ (\Theta_1 \circ \Phi_1 \circ \alpha, m_4 \circ (\Theta_1 \circ \beta, \gamma \circ \phi_2) \circ \psi_2) \,. \qquad (3.10)
\end{aligned}$$

By comparing (3.9) and (3.10), we are done.

Proposition 3.4.1. *Hence we have completely described a 2-category, that we will onwards denote with* ($\mathscr{C}$ – ***Groupoids***).

Remark 3.3. Note that in this category *every 2-morphism is a 2-isomorphism*, (i.e. invertible with respect to $\odot$). So, according to definition 1.8, we will call also *natural equivalences* the 2-morphisms of ($\mathscr{C}$ – **Groupoids**).

In order to prove that, let us consider any natural transformation α:

$$\begin{tikzcd}(R \overset{s}{\underset{t}{\rightrightarrows}} U) \arrow[r, bend left, "{(\psi_1,\Psi_1)}"] \arrow[r, bend right, "{(\psi_2,\Psi_2)}"'] \arrow[r, phantom, "\Downarrow \alpha"] & (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U')\end{tikzcd}$$

and let us define $\beta := i' \circ \alpha : U \to R'$. We want to prove that β is a natural transformation $(\psi_2, \Psi_2) \Rightarrow (\psi_1, \Psi_1)$. First of all,

$$s' \circ \beta = s' \circ i' \circ \alpha = t' \circ \alpha = \psi_2$$

$$\text{and} \quad t' \circ \beta = t' \circ i' \circ \alpha = s' \circ \alpha = \psi_1.$$

Moreover,

$$m' \circ (\beta \circ s, \Psi_1) = m' \circ (i' \circ \alpha \circ s, i' \circ i' \circ \Psi_1) =$$

$$= m' \circ (i' \circ pr_2, i' \circ pr_1) \circ (i' \circ \Psi_1, \alpha \circ s) \overset{*}{=} i' \circ m' \circ (i' \circ \Psi_1, \alpha \circ s) =$$

$$= i' \circ m' \circ (\Psi_1 \circ i, \alpha \circ t \circ i) = i' \circ m' \circ (\Psi_1, \alpha \circ t) \circ i \overset{**}{=}$$

$$\overset{**}{=} i' \circ m' \circ (\alpha \circ s, \Psi_2) \circ i = i' \circ m' \circ (\alpha \circ s \circ i, \Psi_2 \circ i) =$$

$$= i' \circ m' \circ (\alpha \circ t, i' \circ \Psi_2) \overset{*}{=} m' \circ (i' \circ pr_2, i' \circ pr_1) \circ (\alpha \circ t, i' \circ \Psi_2) =$$

$$= m' \circ (i' \circ i' \circ \Psi_2, i' \circ \alpha \circ t) = m' \circ (\Psi_2, \beta \circ t).$$

Here $\overset{**}{=}$ is just property (iii) for α and $\overset{*}{=}$ follows from lemma 3.1.1 applied to the groupoid object $R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'$; hence β is a 2-morphism from (ψ_2, Ψ_2) to (ψ_1, Ψ_1). Now we want to prove that it is the inverse of α, i.e. that $\beta \odot \alpha = i_{(\psi_1, \Psi_1)}$ and $\alpha \odot \beta = i_{(\psi_2, \Psi_2)}$. We prove only the first relation, the second one is similar:

$$\beta \odot \alpha = m' \circ (\alpha, \beta) = m' \circ (\alpha, i' \circ \alpha) = m' \circ (1_{R'}, i') \circ \alpha =$$

$$= e' \circ s' \circ \alpha = e' \circ \psi_1 = i_{(\psi_1, \Psi_1)}.$$

3.5 The 2-category (Grp)

As we mentioned at the end of section 1, in the category (**Manifolds**) of complex manifolds (and holomorphic maps between them) in general the fiber product (as defined in chapter 1) does not exists. To be more precise, it exists if ignore the manifold structure and we work in (**Sets**), but in general the set that we obtain doesn't admit a structure of complex manifold. Even if this is the general case, we can add some extra hypothesis on the maps involved in order to ensure that the fiber product exists. One of the most interesting condition is about submersions.

We recall that an holomorphic (or smooth) map $f : M \to N$ is a *submersion* at a point $p \in M$ if its differential $Df_p : T_pM \to T_{f(p)}N$ is surjective. We will say that f is a submersion if it is a submersion at every point of M. A well known result about submersions is the following:

Proposition 3.5.1. *(normal form of a submersion) Let* $f : M \to N$ *be a holomorphic map between complex manifolds of dimensions* m *and* n *respectively. If* f *is a submersion at a point* p*, then there exists a pair of charts* (U, ϕ)*,* (V, ξ) *for open neighborhoods of* p *in* M *and* $f(p)$ *in* N *respectively, such that the the local expression in coordinates of the function* f *is given by:*

$$\xi \circ f \circ \phi^{-1}(t_1, \cdots, t_n, t_{n+1}, \cdots, t_m) = (t_1, \cdots, t_n)$$

for every point $(t_1, \cdots, t_m) \in \phi(U) \subseteq \mathbb{C}^m$

Proposition 3.5.2. *Let us fix any pair of holomorphic maps between complex manifolds* $X \xrightarrow{f} Y$ *and* $Z \xrightarrow{g} Y$*. If* f *is a submersion, then there exists their fibered product* $X \times_Y Z$ *in the category (**Manifolds**) and its complex dimension is equal to:*

$$dim(X \times_Y Z) = dim(X) + dim(Z) - dim(Y).$$

Moreover, the map $pr_2 : X \times_Y Z \to Z$ *is again a submersion.*

The previuos proposition and similar ones are cited in many articles (see for example [RB] and [M]), but I found nowhere a proof of this fact, so I had to prove it on my own.

Proof. In order to simplify the proof, let us use the following notations:

$$\dim(X) = m \quad \dim(Y) = n \quad \text{and} \quad \dim(Z) = q$$

Since f is a submersion, we have that $m \geq n$. We give to $X \times Y$ the product topology and to $X \times_Y Z$ the topology induced by the fact that it is a set contained in $X \times Y$.

Now let us fix any point $x \in X$ and let us call $y := f(x)$; if we apply the previous proposition we get that there exist charts (U_x, ϕ_x) and (V_y, ψ_y) such that the map:

$$\psi_y \circ f \circ \phi_x^{-1} : \phi_x(U_x) \to \psi_y(V_y)$$

defined between open sets of $\mathbb{C}^m$ and $\mathbb{C}^n$ is of the form:

$$(t_1, \cdots, t_n, t_{n+1}, \cdots, t_m) \to (t_1, \cdots, t_n). \tag{3.11}$$

Now $\phi_x(U_x)$ is an open neighborhood of $\phi_x(x)$ in $\mathbb{C}^m$, so there exists a polydisk $\Delta = \{x' \in \mathbb{C}^m \text{ s.t. } |x'_i - x_i| < \delta \quad \forall i = 1, \cdots, m\}$ (for some positive radius δ) completely contained in $\phi_x(U_x)$. Let us consider the affine automorphism η on $\mathbb{C}^n$ given by:

$$\eta(x') := \frac{1}{\delta}(x' - x);$$

so we get that η maps Δ into the standard polydisk $\Delta^n = \{x' \in \mathbb{C}^m \text{ s.t.} |x'_i| < 1 \quad \forall i = 1, \cdots, m\}$. Now $\eta \circ \phi_x$ is again an homeomorphism, so it makes sense to consider the chart $(\widetilde{U}_x, \tilde{\phi}_x)$ around x in X, where $\widetilde{U}_x := \phi_x^{-1}(\Delta) = \phi_x^{-1}(\eta^{-1}(\Delta^n))$ and $\tilde{\phi}_x := \eta \circ \phi_x$. Now let we call $\bar{\eta}$ the affine automorphisms on $\mathbb{C}^n$ given by:

$$(\overline{\eta})(y') := \frac{1}{\delta}(y' - y);$$

using (3.11) we get that it makes sense to consider the open set $\tilde{V}_y := \psi_y^{-1}(\overline{\eta}^{-1}(\Delta^n))$; if we define $\tilde{\psi}_y := \overline{\eta} \circ \psi_y$, we get that $(\widetilde{V}_y, \tilde{\psi}_y)$ is again a chart around y in Y. Moreover, if we express the map f with respect to these charts, we get the same expression of (3.11) but in this case the map will be defined from Δ^m to Δ^n. So *there is no loss of generality in assuming that the charts* (U_x, ϕ_x) *and* (V_y, ψ_y) *are such that* (3.11) *holds and* $\phi_x(\widetilde{U}_x) = \Delta^m$, $\psi_y(V_y) = \Delta^n$. From now on we will always assume that for every point x in X we have chosen a pair of charts of this form.

Now let us fix any point $(x, z) \in X \times_Y Z$. Since $g(z) = f(x) =: y$ and g is holomorphic (hence continuous), we get that $g^{-1}(V_y)$ is an open neighborhood of z in Z. Now let us choose any chart (W_z, ξ_z) around z: eventually by restricting to $\widetilde{W}_z = W_z \cap g^{-1}(V_y)$, we can assume that $g(W_z) \subseteq V_y$. Our aim now is to construct a chart around (x, z) in the fiber product. First of all, we consider it as a point of $X \times Y$, so a chart around it is (for example) $(U_x \times W_z, \phi_x \times \xi_z)$. In particular $U_x \times W_z$ is an open neighborhood of (x, z) in $X \times Z$. By definition of induced topology, the set:

$$A_{x,z} := (U_x \times W_z) \cap (X \times_Y Z)$$

is open in $X \times_Y Z$; explicitly, this set is given by:

$$A_{x,z} = \{(x', z') \text{ s.t. } x' \in U_x, z' \in W_z \quad \text{and} \quad f(x') = g(z')\}.$$

Then we can define a set map:

$$\begin{array}{rccl} \theta_{x,z}: & A_{x,z} & \rightarrow & \Delta^{m-n} \times \xi_z(W_z) \subset \mathbb{C}^{m-n} \times \mathbb{C}^q \\ & (x', z') & \rightarrow & \Big((\phi_x^{n+1}(x'), \cdots, \phi_x^m(x')), z'\Big) \end{array}$$

where ϕ_x^i is the i-th component of ϕ_x. This map is well defined because ϕ_x has values in Δ^m, so $|\phi_x^i(x')| < 1$ for every $i = 1, \cdots, m$ hence

$(\phi_x^{n+1}(x') \cdots \phi_x^m(x'))$ actually belongs to Δ^{m-n}. This map is continuous because it is the composition of the following continuous maps:

$$\begin{tikzcd}
U_x \times W_z \arrow[r, "(\phi_x \times \xi_z)"] & \phi(U_x) \times \xi_z(W_z) \subseteq \mathbb{C}^{m+q} & = \mathbb{C}^{n+(m-n+q)} \arrow[dl, "pr_2"] \\
A_{x,z} \arrow[u, "i"] \arrow[r, "\theta_{x,y}"'] \arrow[ur, phantom, "\curvearrowright"] & \Delta^{m-n} \times \xi_z(W_z) &
\end{tikzcd}$$

where i is just the inclusion map. Moreover, we can define the set map:

$$\begin{array}{rccc}
\gamma_{x,z}: & \Delta^{m-n} \times \xi_z(W_z) & \to & A_{x,z} \\
 & \Big((t_{n+1}, \cdots, t_m), l\Big) & \to & \Big(\phi_x^{-1}\big(\psi_y \circ g \circ \xi_z^{-1}(l), t_{n+1}, \cdots, t_m\big), \xi_z^{-1}(l)\Big).
\end{array}$$

A priori this map has values in $U_x \times W_z$, but actually it has values in $A_{x,z}$. Indeed, $\xi_z^{-1}(l) \in W_z$, hence $\phi_y \circ g \circ \xi_z^{-1}(l) \in \Delta^n$ and $(t_{m+1}, \cdots, t_n) \in \Delta^{m-n}$, so the first of the two points of the image is well defined. Moreover. a direct check proves that f applied on the first point is equal to g applied to the second one. In addition, this map is continuous if considered as a map with values in $U_x \times W_z$, so using the universal property of the induced topology, we get that it is also continuous if considered with values in $A_{x,z}$.

Moreover, one can easily see that this map is the inverse of $\theta_{x,z}$, hence $\theta_{x,z}$ is an homeomorphism. So it makes sense to define the family of charts:

$$\mathcal{F} := \{(A_{x,z}, \theta_{x,z})\}_{(x,z) \in X \times_Y Z}$$

where for every point $(x, z) \in X \times_Y Z$ we have chosen *one* chart (W_z, ξ_z) such that $W_z \subseteq g^{-1}(V_{f(x)})$; we want to verify that this family is an atlas on $X \times_Y Z$. First of all, let us consider the family:

$$\mathcal{F}_X := \{(U_x, \phi_x)\}_{x \in X}$$

where the charts of this family are constructed as before; this is clearly an atlas on X, hence for every pair of points x, x' in X the transition map $\phi_{x'} \circ \phi_x^{-1}$ is holomorphic. Moreover, the family

$$\mathcal{F}_Z := \{(W_z, \xi_z)\}_{z \in Z}$$

is (part of) an atlas on Z, so for every pair of points z, z' in Z the transition map $\phi_{z'} \circ \phi_z^{-1}$ is holomorphic. Now let us fix any pair of points $(x, z), (x', z')$ in $X \ \times_Y \ Z$, let us suppose that $A_{x,z} \cap A_{x',z'} \neq \varnothing$ and let us consider the transition map:

$$\sigma := \theta_{x',z'} \circ \theta_{x,z}^{-1} : \Delta^{m-n} \times W_z \to \Delta^{m-n} \times W_{z'}$$

and let us fix any point $(t_{n+1}, \cdots, t_m, l_1 \cdots, l_q)$ in its domain. Then the first components of its image via σ are of the form:

$$\phi_{x'}^i \Big(\phi_x^{-1} \big(\psi_y \circ g \circ \xi_z^{-1}(l_1, \cdots, l_q), t_{n+1}, \cdots, t_m \big) \Big)$$

for $i = n+1, \cdots, m$; the other components are of the form:

$$\xi_{z'}^j (\xi_z^{-1}(l_1, \cdots, l_q)$$

for $j = 1, \cdots, q$. Hence, all the components of σ are holomorphic maps using the fact that the transition maps and the function g (expressed in coordinates) are so. Since this holds for every pair of points (x, z) and (x', z') in $X \ \times_Y \ Z$, then also $\theta_{x,z} \circ \theta_{x',z'}^{-1}$ is holomorphic; hence σ is biholomorphic.

So we have proved the compatibility condition on the intersection on any pair of two maps. Moreover, by construction the domains of the charts of $\mathcal{F}$ cover all $X \ \times_Y \ Z$; so we have proved that this set is actually a complex manifold of dimension equal to $m + q - n$.

Now let us consider the square:

$$\begin{tikzcd}
X \times_Y Z \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd} \tag{3.12}$$

where for every point $(x, z) \in X \times_Y Z$ we define $pr_1(x, z) := x$ and $pr_2(x, z) := z$. Clearly this diagram is commutative; our aim is to prove that both pr_1 and pr_2 are holomorphic. Let us consider the first one and let us fix any point (x, z) in the fiber product. Then we can choose the charts $(A_{x,z}, \theta_{x,z})$ around (x, z) and (U_x, ϕ_x) around x; if we express pr_1 with respect to these cordinates, we get a map of the form:

$$\Big((t_{n+1}, \cdots, t_m), l\Big) \mapsto \Big(\psi_y \circ g \circ \xi_z^{-1}(l), t_{n+1}, \cdots, t_m\Big)$$

which is clearly holomorphic. Moreover, we can choose the chart (W_z, ξ_z) around the point $z = pr_2(x, z)$ and again $(A_{x,z}, \theta_{x,z})$ around (x, z); then in these coordinates pr_2 has the form:

$$\Big((t_{n+1}, \cdots, t_m), l\Big) \mapsto l$$

which is clearly holomorphic. Moreover, its differential has maximum rank, and this property does not depend on the choice of charts in domain or codomain, hence pr_2 *is a submersion.*

Until now we have proved that (3.12) is a commutative diagram in the category (**Manifolds**), hence in order to prove the proposition it suffices to prove that the **UP** of fiber products holds. So let us consider any other complex manifold G together with a pair of holomorphic maps $a : G \to X$ and $b : G \to Z$ such that $f \circ a = g \circ b$. Since $X \times_Y Z$ is a fiber product in (**Sets**), *there exists a unique set map* $\gamma : G \to X \times_Y Z$ making the following diagram commute:

$$\begin{tikzcd}
G \arrow[rrd, bend left, "b"] \arrow[rd, dashed, "\gamma"] \arrow[rdd, bend right, "a"'] \arrow[rrd, phantom, "\curvearrowright" near start] \arrow[rdd, phantom, "\curvearrowright" near start] & & \\
& X \times_Y Z \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[rd, phantom, "\curvearrowright"] & Z \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Using the results of chapter 1, we know that the set map γ acts on every point g of G as:

$$\gamma(g) = (a(g), b(g)).$$

Our aim is to prove that γ is holomorphic, so let us fix any point g in G and for simplicity let us define $x := a(g)$ and $z := b(g)$. We know that both a and b are holomorphic, hence in particular they are continuous, so $A := a^{-1}(U_x)$ and $B := b^{-1}(W_z)$ are both open neighborhoods of g in G. Let us fix a chart (D_g, η_g) around g in G. Eventually by restricting D_g, we can suppose that $D_g \subseteq A \cap B$. Now a and b are holomorphic, hence their expressions in coordinates:

$$\phi_x \circ a \circ \eta_g^{-1} \quad \text{and} \quad \xi_z \circ b \circ \eta_g^{-1}$$

are both holomorphic maps. Now let us choose the charts (D_g, η_g) around $g \in G$ and $(A_{x,z}, \theta_{x,z})$ around $\gamma(g) = (x, z)$ in $X \times_Y Z$. Then for every point $g' \in D_g$ the local expression of γ with respect to these coordinates has the following form: it associates to every $g' \in \eta_g(D_g)$ the point:

$$\Big(\phi_x^{n+1}(a \circ \eta_g^{-1}(g')), \cdots, \phi_x^m(a \circ \eta_g^{-1}(g')), \xi_z^{-1}(b \circ \eta_g^{-1}(g')) \cdots, \xi_z^{-1}(b \circ \eta_g^{-1}(h))\Big)$$

hence γ is an holomorphic map, so the **UP** of fiber products in (**Manifolds**) is satisfied. □

Definition 3.10. A groupoid object in (**Manifolds**) is called *Lie groupoid* if both the source and the target map are submersions.

Remark 3.4. Using the previuos proposition, we get that the fiber product used in the first point of the definition of groupoid objects is clearly satisfied. Moreover, the resulting maps pr_1 and pr_2 are again both submersions. Hence also the fiber product $R_t \times_s R_t \times_s R$ exists in (**Manifolds**). Indeed:

$$R_t \times_s R_t \times_s R = \{(r, r', r'') \in R \times R \times R \text{ s.t. } s(r') = t(r) \, \text{and} \, s(r'') = t(r')\} =$$

$$= (R_t \times_s R)_{\tilde{t}} \times_s R$$

where $\tilde{t} := t \circ pr_2$. Here the fiber product behind parenthesis exists, $\tilde{t}$ is a submersion (because t is so by hypothesis and pr_2 is so using the previous proposition) and also s is a submersion, hence the whole fiber product exists in (**Manifolds**) and again the projection maps are submersions, so by induction we can prove that there exists fiber products of the form:

$$R_t \times_s \cdots \quad {}_t \times_s R$$

for arbitrary (but finite) number of factors.

Definition 3.11. An holomorphic map $f : X \to Y$ between complex manifolds is *étale* if it is *locally* a biholomorphism, i.e. if for every point $x \in X$ there exists an open neighborhood A of x in X such that f restricted to A and to $f(A)$ is a biholomorphism.

Remark 3.5. If $f : X \to Y$ is étale, then the complex dimension of X and Y is the same. Indeed the notion of dimension can be checked locally, where X and Y are biholomorphic.

Corollary 3.5.3. *Let us suppose we have a pair of holomorphic maps* $f : X \to Y$ *and* $g : Z \to Y$ *and let us suppose that* f *is étale. Then the fiber product* $X \times_Y Z$ *exists in (**Manifolds**) and its complex dimension is equal to the dimension of* Z*. Moreover in this case also the maps* pr_2 *is étale.*

Proof. Every étale map is a submersion between manifolds of the same dimension, hence the first part of the proof follows directly from proposition 3.5.2. Moreover, in the proof of that proposition we have seen that a local expression in coordinates for pr_2 is of the form:

$$(t_{n+1}, \cdots, t_m, l_1, \cdots, l_q) \mapsto (l_1, \cdots, l_q)$$

but in our case $m = n$ and the set Δ^{m-n} is just a single point, hence in these coordinates pr_2 is just the identity. Hence pr_2 is étale. □

This property is known in the following way: *the étaleness property is stable under under fiber products.*

Now we can give the following definition:

Definition 3.12. ([M],§1.2) An *étale groupoid* is a groupoid object $R \overset{s}{\underset{t}{\rightrightarrows}} U$ in the category (**Manifolds**) such that both the maps s and t are étale. Using remark 3.4, the fiber products $R\,{}_t{\times}_s\, R$ and $R\,{}_t{\times}_s\, R\,{}_t \times_s R$ exist, so all definition 3.1 still makes sense.(Clearly every étale groupoid is also a Lie groupoid).

Remark 3.6. Let us fix any groupoid object $R \overset{s}{\underset{t}{\rightrightarrows}} U$ in (**Manifold**), i.e. let us suppose that the fiber product $R\,{}_s{\times}_t\, R$ already exists. *Then if s is étale, also t is so, and conversly.*

Indeed using axiom (v) of definition 3.1 we get that $i \circ i = 1_R$ and by definition of groupoid object the map $i : R \to R$ is holomorphic, so i is biholomorphic. Moreover, the same axiom requires also that $t = s \circ i$, so t is the composition of an étale map with a biholomorphic one, so it is étale.

Definition 3.13. A groupoid object in (**Manifolds**) is *proper* if the map $(s,t) : R \to U \times U$ is proper, i.e. if the pre-image of any compact set in $U \times U$ is compact in R. The map (s,t) is usually known as *relative diagonal* of the groupoid object.

Definition 3.14. The objects we are interested in are the *étale proper groupoid objects in* (**Manifolds**). These objects will be the objects of a 2-category, that we will call (**Grp**).

Definition 3.15. A morphism between étale proper groupoids in (**Manifolds**) is just any morphisms between groupoid objects in (**Manifolds**) in the sense of the previous definitions and we define the 2-morphisms in the same way.

This 2-category differs from ((**Manifolds**)-**Groupoids**) just for the fact that we require some additional conditions on the source and target maps of the objects. All the previous proofs works also with this additional condition (just on the level of objects), so (**Grp**) *is actually a 2-category.* To be more precise, in the language of category theory, (**Grp**) is a *full* sub-2-category of ((**Manifolds**)-**Groupoids**), but this is not essential for this work.

This 2-category appears to be as the natural target for the 2-functor F that we will describe in the following chapter.

The following 4 examples are taken almost under verbatim from [M].

Example 3.1. Any complex manifold $\mathcal{M}$ on a topological space M can be viewed as a Lie groupoid as follows: we set $R := M$, $U := M$ and we define the five structure maps as follows:

- $s = t = id : R \rightarrow U$;
- $i := id : R \rightarrow R$;
- $e := id : U \rightarrow R$;
- since both s and t are the identity, we have that $M\,{}_t\!\times_s M = \Delta_M$ (i.e. the diagonal of $M \times M$), so we can define the multiplication map $m : \Delta_M \rightarrow M$ as $m(x, x) := x$.

It is obvious that all these maps are holomorphic, that s and t are both étale and that the axioms of a groupoid are all satified. Moreover, the map

(s,t) is just the diagonal map $M \to M \times M$, so it is proper. Hence we have completely described an étale proper groupoid in (**Manifolds**). This groupoid is called *unit groupoid* associated to $\mathcal{M}$.

Example 3.2. To any complex manifold $\mathcal{M}$ on M we can also associate another groupoid, called *pair groupoid* as follows: we define $R := M \times M$, $U := M$ and:

- $s : M \times M \to M$, $s(x,y) := x$;
- $t : M \times M \to M$, $t(x,y) := y$;
- $i : M \times M \to M \times M$, $i(x,y) := (y,x)$;
- $e : M \to M \times M$, $e(x) := (x,x)$:
- by definition of s and t we have that:

  $$R\,{}_t\times_s\, R = (M \times M)\,{}_t\times_s\,(M \times M) = \Big\{\Big((x,y),(y,z)\Big), x, y, z, \in M\Big\}$$

  so we define the multiplication map $m : R\,{}_t\times_s\, R \to R$ as:

  $$m\left((x,y),(y,z)\right) := (x,z)$$

As before, this defines a groupoid object in (**Manifolds**), which is a Lie groupoid because clearly both s and t are submersions, but is not étale because these two maps are defined between manifolds of different dimensions. We can think to the space R as a set which contains *exactly* one arrow for every pair of points (x,y) of the manifold M. Using this interpretation, the multiplication m just links every pair of arrows $x \to y$ and $y \to z$ and the inversion map just reverses the directions of the arrows.

Example 3.3. We recall that a *Lie group* is a (complex) manifold $\mathcal{M}$ (on a topological space M) which is also a (multiplicative) group, such that multiplication and inversion are both holomorphic maps. To any object of this type we can associate a Lie groupoid as follows: we set $U := \{pt\}$ (a topological space with a single point), $R := M$ and the structure maps:

- s and t are both defined from M to $\{pt\}$ as the trivial maps;
- $i : M \to M$ is the inversion map of the group;
- using the definition of source and target, we have that $R\,{}_t\!\times_s M = M \times M$, so we can define $m : M \times M \to M$ as the multiplication map of the group;
- we define $e : \{pt\} \to M$ as $e(pt) := 1$ where 1 is the neutral element of the multiplicative group M.

The axioms of a multiplicative associative group imply easily the axioms of a Lie groupoid and give an explanation for the word "multiplication" used in the definition of groupoids at the beginning of this chapter. Moreover, this explains also the term "Lie groupoid" as a generalization of Lie group.

Example 3.4. Let us suppose that a Lie group K *acts (holomorphically) on the left* on a manifold $\mathcal{M}$ (defined on a space M). Then we define $U := M$, $R := K \times M$ and the maps:

- $s : K \times M \to M$, $s(k, x) := x$;
- $t : K \times M \to M$, $t(k, x) := k \cdot x$;
- $i : K \times M \to K \times M$, $i(k, x) := (k^{-1}, x)$ where k^{-1} is the inverse of k with respect to the multiplication map on K.
- usign the previous definitions, we get that:

  $$R\,{}_t\!\times_s R = (K \times M)\,{}_t\!\times_s (K \times M) = \Big\{\Big((k, x), (h, y)\Big) \text{ s.t. } k \cdot x = y\Big\}$$

  so for every point in it we define:

  $$m\left((k, x), (h, y)\right) := (h \cdot k, x);$$

- $e : M \to K \times M$, $e(x) := (1, x)$.

This Lie groupoid is called *translation groupoid* or also *action groupoid* associated to the action of K on M and is denoted by $K \ltimes M$. A similar construction applied to a right action of a Lie group K on M gives us the translation groupoid $K \rtimes M$.

> “La poesia è l’arte di dare nomi diversi alla stessa cosa; la matematica è l’arte di dare lo stesso nome a cose diverse.”
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>
> Henri Poincaré

Chapter 4

From orbifolds to groupoids

The aim of this chapter is to describe a 2-functor F from (**Pre-Orb**) to (**Grp**). In order to do that, first of all we will follows the construction due to Dorette Pronk, in order to associate to every orbifold atlas an object of the 2-category (**Grp**), i.e. a groupoid object in the category (**Manifolds**) with source and target étale and proper relative diagonal $(s,t) : R \to U \times U$ (as described in the previous chapter). This will be done first by defining two *sets* U and R and five *set maps* s,t,m,i,e that make the pair (R,U) into a groupoid object in (**Sets**). Then we will describe the topology on R and U and we prove that actually they are both complex manifolds such that the five structure maps are holomorphic; this proves that we have obtained a groupoid object in (**Manifolds**). Moreover, we will verify that s and t are both étale and that the relative diagonal is proper; hence $R \overset{s}{\underset{t}{\rightrightarrows}} U$ will be an object in (**Grp**).

The original part of this chapter consists in the proof that we can also associate to every compatible system a morphism between the corresponding groupoid objects and to every natural transformation in (**Pre-Orb**) a natural transformation in (**Grp**). Moreover, we will prove that this construction satisfies the axioms for a covariant 2-functor given in definition 1.12.

4.1 Objects

We will follow [Pr] with minor changes for the description on the level of objects from orbifolds atlases to groupoids.

Let us consider an orbifold atlas $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i\in I}$ of dimension n on a paracompact and second countable Hausdorff topological space X; we want to associate to it a groupoid $R \overset{s}{\underset{t}{\rightrightarrows}} U$ which "encodes" the information about the underlying topological space X and the atlas $\mathcal{U}$. First of all, we define:

$$U := \coprod_{i\in I} \widetilde{U}_i$$

with the topology of the disjoint union. Since the $\widetilde{U}_i$'s are all open subsets of $\mathbb{C}^n$ and U is their disjoint union, then U is a complex manifolds of dimension n.

Remark 4.1. The points of this manifold will be always denoted as $(\tilde{x}_i, \widetilde{U}_i)$ if $\tilde{x}_i \in \widetilde{U}_i \subseteq U$. In the following constructions we will tacictly assume that if we take a generic point $\tilde{x}_i$, then this point belongs to $\widetilde{U}_i$.

Now the idea is that whenever we have U defined in this way, we would like to recover both the underlying topological space X and the atlas; how to do this must be encoded in R, that we are going to define. If we want to recover X set-theoretically, we have to identify on U any two pair of points $(\tilde{x}_i, \widetilde{U}_i)$ and $(\tilde{x}_j, \widetilde{U}_j)$ such that $\pi_i(\tilde{x}_i) = \pi_j(\tilde{x}_i)$ on X. Now suppose that we have fixed such a pair of points, call x their common image in X and apply the definition of orbifold atlas (definition 2.4). So we get that there exists an open set $U_k \subseteq U_i \cap U_j$ on X which contains x, a uniformizing system $(\widetilde{U}_k, G_k, \pi_k) \in \mathcal{U}$ for it, and embeddings:

$$(\widetilde{U}_i, G_i, \pi_i) \overset{\lambda_{ki}}{\longleftarrow} (\widetilde{U}_k, G_k, \pi_k) \overset{\lambda_{kj}}{\longrightarrow} (\widetilde{U}_j, G_j, \pi_j); \tag{4.1}$$

using remark 2.7 we can assume that we have chosen $\tilde{x}_k \in \widetilde{U}_k$ such that:

$$\lambda_{ki}(\tilde{x}_k) = \tilde{x}_i \quad \text{and} \quad \lambda_{kj}(\tilde{x}_k) = \tilde{x}_j. \tag{4.2}$$

Conversely, if there exists a pair of embeddings (4.1) such that (4.2) is satisfied, we get immediately that $(\tilde{x}_i, \widetilde{U}_i)$ and $(\tilde{x}_j, \widetilde{U}_j)$ will be identified in X. So the first idea could be to define a set R whose elements are of the form $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj})$ together with two morphisms "source" and "target" from R to U and an "inverse" as follows:

$$s(\lambda_{ki}, \tilde{x}_k, \lambda_{kj}) := (\lambda_{ki}(\tilde{x}_k), \widetilde{U}_i)$$

$$t(\lambda_{ki}, \tilde{x}_k, \lambda_{kj}) := (\lambda_{kj}(\tilde{x}_k), \widetilde{U}_j)$$

$$i(\lambda_{ki}, \tilde{x}_k, \lambda_{kj}) := (\lambda_{kj}, \tilde{x}_k, \lambda_{ki}).$$

Then we would easily recover the set X as $U/\sim$ where $\sim$ is the relation defined by:

$$(\tilde{x}, \widetilde{U}_i) \sim (\tilde{y}, \widetilde{U}_j) \overset{def}{\Longleftrightarrow} \begin{cases} \exists (\lambda_{ki}, \tilde{x}_k, \lambda_{kj}) \in R \text{ s.t.} \\ s(\lambda_{ki}, \tilde{x}_k, \lambda_{kj}) = (\tilde{x}_i, \widetilde{U}_i) \,\text{and}\, t(\lambda_{ki}, \tilde{x}_k, \lambda_{kj}) = (\tilde{x}_j, \widetilde{U}_j). \end{cases}$$

However, with this definition some problems arise about the definition of multiplication m on $R\,{}_s\!\times_t R$. The best way to solve the problem is to proceed as Dorette Pronk does in [Pr]: first of all, as before we will use the notation λ_{ki} to denote any embedding from $(\widetilde{U}_k, G_k, \pi_k)$ to $(\widetilde{U}_i, G_i, \pi_i)$; whenever we have embeddings:

$$(\widetilde{U}_i, G_i, \pi_i) \overset{\lambda_{ki}}{\leftarrow} (\widetilde{U}_k, G_k, \pi_k) \overset{\lambda_{kj}}{\rightarrow} (\widetilde{U}_j, G_j, \pi_j) \tag{4.3}$$

we use the notations:

$$\widetilde{U}_k^{ij} \quad \text{or} \quad (\lambda_{ki}, \lambda_{kj})$$

to denote a copy of the manifold $\widetilde{U}_k$ indexed by the embeddings λ_{ki} and λ_{kj}. *To be more precise, we have to write* $\widetilde{U}_k^{\lambda_{ki}, \lambda_{kj}}$ *whenever we have such an object because in general there is not a unique embedding from a uniformizing system to another one. However, we will use this complicated notation only* *when there is some ambiguity on the embeddings used.* Then we define the set:

$$\hat{R} := \coprod \widetilde{U}_k^{ij}$$

where the disjoint union is taken *over all triples of uniformizing systems of the form* (4.3). In other words, the disjoint union is taken over all the uniformizing systems $(\widetilde{U}_k, G_k, \pi_k)$ and over all the possible embeddings of them into other uniformizing sytems (possibly coinciding) as in (4.3). Any point $\tilde{x} \in \widetilde{U}_k^{ij} \subseteq \hat{R}$ will be denoted by:

$$(\lambda_{ki}, \tilde{x}_k, \lambda_{kj}).$$

$\hat{R}$ is a complex manifold of dimension n because it is a disjoint union of open sets in $\mathbb{C}^n$; on it we give the following:

Definition 4.1. Two points $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj})$ and $(\lambda_{li}, \tilde{x}_l, \lambda_{lj})$ are said to be *equivalent* in $\hat{R}$ and we write:

$$(\lambda_{ki}, \tilde{x}_k, \lambda_{kj}) \sim (\lambda_{li}, \tilde{x}_l, \lambda_{lj}) \tag{4.4}$$

iff there exists an open connected set U_m in X, a uniformizing system $(\widetilde{U}_m, G_m, \pi_m) \in \mathcal{U}$, a point $\tilde{x}_m \in \widetilde{U}_m$ and two embeddings:

$$(\widetilde{U}_k, G_k, \pi_k) \overset{\lambda_{mk}}{\longleftarrow} (\widetilde{U}_m, G_m, \pi_m) \overset{\lambda_{ml}}{\longrightarrow} (\widetilde{U}_l, G_l, \pi_l)$$

such that the following diagrams are commutative:

$$\begin{tikzcd}
& \widetilde{U}_k \arrow[dl, "\lambda_{ki}"'] \arrow[dr, "\lambda_{kj}"] & & & \tilde{x}_k \arrow[dl, "\lambda_{ki}"'] \arrow[dr, "\lambda_{kj}"] & \\
\widetilde{U}_i \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_m \arrow[u, "\lambda_{mk}"'] \arrow[d, "\lambda_{ml}"] \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_j & \tilde{x}_i \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_m \arrow[u, "\lambda_{mk}"'] \arrow[d, "\lambda_{ml}"] \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_j \\
& \widetilde{U}_l \arrow[ul, "\lambda_{li}"] \arrow[ur, "\lambda_{lj}"'] & & & \tilde{x}_l \arrow[ul, "\lambda_{li}"] \arrow[ur, "\lambda_{lj}"'] &
\end{tikzcd}$$

(4.5)

Note that in particular this implies that U_m is an open neighborhood in X of the point

$$\pi_i(\tilde{x}_i) = \pi_j(\tilde{x}_j) = \pi_k(\tilde{x}_k) = \pi_l(\tilde{x}_l) = \pi_m(\tilde{x}_m).$$

Remark 4.2. In order to simplify the notations, *here and from now on we omit the groups G_i's and the maps π_i's*; in other words from now on every map λ_{ij} will be an embedding between uniformizing systems even if we write only its source and target as open sets of $\mathbb{C}^n$ and not as uniformizing systems.

Now our aim is to prove that (4.4) *is an equivalence relation on R'.* In order to do this, let us state and prove the following 2 lemmas.

Lemma 4.1.1. *Let us fix 4 uniformizing systems $(\widetilde{U}_l, G_l, \pi_l)$, $(\widetilde{U}_i, G_i, \pi_i)$, $(\widetilde{U}_j, G_j, \pi_j)$, $(\widetilde{U}_k, G_k, \pi_k)$ together with 4 embeddings λ_{ik}, λ_{jk}, $\tilde{\lambda}_{li}$ and $\tilde{\lambda}_{lj}$ such that $\lambda_{ik}(\widetilde{U}_i) \cap \lambda_{jk}(\widetilde{U}_j) \subseteq \widetilde{U}_k$ is non-empty. Then there exist embeddings between uniforming systems λ_{li} and λ_{lj} such that:*

$$\begin{tikzcd}
 & \widetilde{U}_l \arrow[dl, "\lambda_{li}"'] \arrow[dr, "\lambda_{lj}"] & \\
\widetilde{U}_i \arrow[dr, "\lambda_{ik}"'] & \curvearrowright & \widetilde{U}_j \arrow[dl, "\lambda_{jk}"] \\
 & \widetilde{U}_k &
\end{tikzcd}$$

is commutative. Moreover, if we fix any point $\tilde{x}_k \in \lambda_{ik}(\widetilde{U}_i) \cap \lambda_{jk}(\widetilde{U}_j)$ *such that* $\pi_k(\tilde{x}_k) \in U_l$*, these embeddings can be chosen such that* $\tilde{x}_k \in \lambda_{jk} \circ \lambda_{lj}(\widetilde{U}_l)$*.*

Proof. ([Pr], lemma 4.4.1 with some changes) Since $\pi_k(\tilde{x}_k) \in U_l$, as in remark 2.7 we can assume that $\lambda_{ik}^{-1}(\tilde{x}_k) \in \tilde{\lambda}_{li}(\widetilde{U}_i)$ and $\lambda_{jk}^{-1}(\tilde{x}_k) \in \tilde{\lambda}_{lj}(\widetilde{U}_l)$ (at least we substitute the embeddings $\tilde{\lambda}_{li}$ and $\tilde{\lambda}_{lj}$, and this does not change the result).

Now let us consider the embeddings:

$$\alpha := \lambda_{ik} \circ \tilde{\lambda}_{li} \quad \text{and} \quad \beta := \lambda_{jk} \circ \tilde{\lambda}_{lj}$$

both defined from $(\widetilde{U}_l, G_l, \pi_l)$ to $(\widetilde{U}_k, G_k, \pi_k)$. Using lemma 2.1.7 we get that there exists a unique $g \in G_k$ such that $g \circ \alpha = \beta$.

Now $\alpha(\widetilde{U}_l) \cap g \circ \alpha(\widetilde{U}_l) \neq \varnothing$ since it contains $\tilde{x}_k$, hence we can apply lemma 2.1.10, so there exists a unique $h \in G_l$ such that $\Lambda_{ki} \circ \tilde{\Lambda}_{li}(h) = g$. Now let us define:

$$\lambda_{li} := \tilde{\lambda}_{li} \circ h \quad \text{and} \quad \lambda_{lj} := \tilde{\lambda}_{lj}.$$

Then we get that:

$$\lambda_{ik} \circ \lambda_{li} = \lambda_{ik} \circ \tilde{\lambda}_{li} \circ h = \lambda_{ik} \circ \tilde{\Lambda}_{li}(h) \circ \tilde{\lambda}_{li} = (\Lambda_{ik} \circ \tilde{\Lambda}_{li}(h)) \circ \lambda_{ik} \circ \tilde{\lambda}_{li} =$$

$$= g \circ \lambda_{ik} \circ \tilde{\lambda}_{li} = g \circ \alpha = \beta = \lambda_{jk} \circ \lambda_{lj}$$

and $\tilde{x}_k$ belongs to the image of this embedding, as required. □

Lemma 4.1.2. *Let us fix an atlas $\mathcal{U}$, a pair of embeddings $\lambda_{ik} : (\widetilde{U}_i, G_i, \pi_i) \rightarrow (\widetilde{U}_k, G_k, \pi_k)$ and $\lambda_{jk} : (\widetilde{U}_j, G_j, \pi_j) \rightarrow (\widetilde{U}_k, G_k, \pi_k)$ and a pair of points $\tilde{x}_i \in \widetilde{U}_i$, $\tilde{x}_j \in \widetilde{U}_j$ such that $\lambda_{ik}(\tilde{x}_i) = \lambda_{jk}(\tilde{x}_j)$. Then there exists a uniformizing system $(\widetilde{U}_l, G_l, \pi_l) \in \mathcal{U}$, a pair of embeddings $\lambda_{li}, \lambda_{lj}$ and a point $\tilde{x}_l \in \widetilde{U}_l$ such that the following diagrams are commutative:*

$$\begin{tikzcd}
& \widetilde{U}_l \arrow[dl, "\lambda_{li}"'] \arrow[dr, "\lambda_{lj}"] & & & & \tilde{x}_l \arrow[dl, "\lambda_{li}"'] \arrow[dr, "\lambda_{lj}"] & \\
\widetilde{U}_i \arrow[dr, "\lambda_{ik}"'] & \curvearrowright & \widetilde{U}_j \arrow[dl, "\lambda_{jk}"] & & \tilde{x}_i \arrow[dr, "\lambda_{ik}"'] & \curvearrowright & \tilde{x}_j \arrow[dl, "\lambda_{jk}"] \\
& \widetilde{U}_k & & & & \tilde{x}_k &
\end{tikzcd} \qquad (4.6)$$

Proof. By hypothesis $\lambda_{ik}(\tilde{x}_i) = \lambda_{jk}(\tilde{x}_j)$, hence:

$$\pi_i(\tilde{x}_i) = \pi_k \circ \lambda_{ik}(\tilde{x}_i) = \pi_k \circ \lambda_{jk}(\tilde{x}_j) = \pi_j(\tilde{x}_j)$$

so by definition of atlas we get that there exists an open neighborhood U_l of this point, a uniformizing system $(\widetilde{U}_l, G_l, \pi_l) \in \mathcal{U}$ for it and embeddings $\tilde{\lambda}_{li}$ and $\tilde{\lambda}_{lj}$ of it into $(\widetilde{U}_i, G_i, \pi_i)$ and $(\widetilde{U}_j, G_j, \pi_j)$ respectively. Since by hypothesis $\lambda_{ik}(\widetilde{U}_i) \cap \lambda_{jk}(\widetilde{U}_j) \neq \varnothing$, we can apply the previous lemma for the point $\lambda_{ik}(\tilde{x}_i) = \lambda_{jk}(\tilde{x}_j)$, so we get that there exists a point $\tilde{x}_l \in \widetilde{U}_l$ and embeddings λ_{li} and λ_{lj} such that (4.6) holds. □

Lemma 4.1.3. *The relation (4.4) is an equivalence relation on $\hat{R}$.*

Proof. ([Pr], lemma 4.4.2) The relation is clearly reflexive and symmetric. To prove transitivity, suppose we have:

$$(\lambda_{ki}, \tilde{x}_k, \lambda_{kj}) \sim (\lambda_{li}, \tilde{x}_l, \lambda_{lj}) \sim (\lambda_{mi}, \tilde{x}_m, \lambda_{mj})$$

In other words, using definition 4.1, there exist uniformizing systems $(\widetilde{U}_n, G_n, \pi_n), (\widetilde{U}_p, G_p, \pi_p) \in \mathcal{U}$, two points $\tilde{x}_n \in \widetilde{U}_n$, $\tilde{x}_p \in \widetilde{U}_p$ and embeddings $\lambda_{nk}, \lambda_{nl}$, $\lambda_{pl}, \lambda_{pm}$ making the following diagrams commute:

$$\begin{tikzcd}
 & \widetilde{U}_k \arrow[ddl, "\lambda_{ki}"'] \arrow[ddr, "\lambda_{kj}"] & \\
\curvearrowright & \widetilde{U}_n \arrow[u, "\lambda_{nk}"'] \arrow[d, "\lambda_{nl}"] & \curvearrowright \\
\widetilde{U}_i & \widetilde{U}_l \arrow[l, "\lambda_{li}"'] \arrow[r, "\lambda_{lj}"] & \widetilde{U}_j \\
\curvearrowright & \widetilde{U}_p \arrow[u, "\lambda_{pl}"'] \arrow[d, "\lambda_{pm}"'] & \curvearrowright \\
 & \widetilde{U}_m \arrow[uul, "\lambda_{mi}"] \arrow[uur, "\lambda_{mj}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \tilde{x}_k \arrow[ddl, "\lambda_{ki}"'] \arrow[ddr, "\lambda_{kj}"] & \\
\curvearrowright & \tilde{x}_n \arrow[u, "\lambda_{nk}"'] \arrow[d, "\lambda_{nl}"] & \curvearrowright \\
\tilde{x}_i & \tilde{x}_l \arrow[l, "\lambda_{li}"'] \arrow[r, "\lambda_{lj}"] & \tilde{x}_j. \\
\curvearrowright & \tilde{x}_p \arrow[u, "\lambda_{pl}"'] \arrow[d, "\lambda_{pm}"'] & \curvearrowright \\
 & \tilde{x}_m \arrow[uul, "\lambda_{mi}"] \arrow[uur, "\lambda_{mj}"'] &
\end{tikzcd}
\tag{4.7}$$

In particular, $\lambda_{nl}(\tilde{x}_n) = \lambda_{pl}(\tilde{x}_p)$, so we can apply lemma 4.1.2 and we get that there exists a uniformizing system $(\widetilde{U}_q, G_q, \pi_q)$, a point $\tilde{x}_q \in \widetilde{U}_q$ and a pair of embeddings $\lambda_{qn}, \lambda_{qp}$ making the following diagrams commute:

$$\begin{tikzcd}
 & \widetilde{U}_q \arrow[dl, "\lambda_{qn}"'] \arrow[dr, "\lambda_{qp}"] & \\
\widetilde{U}_n \arrow[dr, "\lambda_{nl}"'] & \curvearrowright & \widetilde{U}_p \arrow[dl, "\lambda_{pl}"] \\
 & \widetilde{U}_l &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \tilde{x}_q \arrow[dl, "\lambda_{qn}"'] \arrow[dr, "\lambda_{qp}"] & \\
\tilde{x}_n \arrow[dr, "\lambda_{nl}"'] & \curvearrowright & \tilde{x}_p. \arrow[dl, "\lambda_{pl}"] \\
 & \tilde{x}_l &
\end{tikzcd}
\tag{4.8}$$

Now using (4.7) and (4.8) we get that:

$$\lambda_{ki} \circ \lambda_{nk} \circ \lambda_{qn} = \lambda_{li} \circ \lambda_{nl} \circ \lambda_{qn} = \lambda_{li} \circ \lambda_{pl} \circ \lambda_{qp} = \lambda_{mi} \circ \lambda_{pm} \circ \lambda_{qp}$$

and

$$\lambda_{kj} \circ \lambda_{nk} \circ \lambda_{qn} = \lambda_{lj} \circ \lambda_{nl} \circ \lambda_{qn} = \lambda_{lj} \circ \lambda_{pl} \circ \lambda_{qp} = \lambda_{mj} \circ \lambda_{pm} \circ \lambda_{qp};$$

hence we get a commutative diagram:

$$\begin{tikzcd}
 & \widetilde{U}_k \arrow[dl, "\lambda_{ki}"'] \arrow[dr, "\lambda_{kj}"] & \\
\widetilde{U}_i \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_q \arrow[u, "\lambda_{nk}\circ\lambda_{qn}"'] \arrow[d, "\lambda_{pm}\circ\lambda_{qp}"'] \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_j. \\
 & \widetilde{U}_m \arrow[ul, "\lambda_{mi}"] \arrow[ur, "\lambda_{mj}"'] &
\end{tikzcd} \tag{4.9}$$

Now using the second part of (4.8) we have that $\lambda_{qn}(\tilde{x}_q) = \tilde{x}_n$; hence:

$$\lambda_{nk} \circ \lambda_{qn}(\tilde{x}_q) = \lambda_{nk}(\tilde{x}_n) = \tilde{x}_k. \tag{4.10}$$

In the same way, since $\lambda_{qp}(\tilde{x}_q) = \tilde{x}_p$, we have:

$$\lambda_{pm} \circ \lambda_{qp}(\tilde{x}_q) = \lambda_{pm}(\tilde{x}_p) = \tilde{x}_m. \tag{4.11}$$

Hence conditions (4.10) and (4.11) together with (4.7) show that the following diagram is commutative:

$$\begin{tikzcd}
 & \tilde{x}_k \arrow[dl, "\lambda_{ki}"'] \arrow[dr, "\lambda_{kj}"] & \\
\tilde{x}_i \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_q \arrow[u, "\lambda_{nk}\circ\lambda_{qn}"'] \arrow[d, "\lambda_{pm}\circ\lambda_{qp}"'] \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_j. \\
 & \tilde{x}_m \arrow[ul, "\lambda_{mi}"] \arrow[ur, "\lambda_{mj}"'] &
\end{tikzcd} \tag{4.12}$$

Diagrams (4.9) and (4.12) together prove that $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj}) \sim (\lambda_{mi}, \tilde{x}_m, \lambda_{mj})$, so $\sim$ is transitive. Hence we have proved that $\sim$ is a relation of equivalence. □

Definition 4.2. For any fixed atlas $\mathcal{U}$ over X, we define the set $R := \hat{R}/\sim$. For simplicity of notation, we will denote the class of any point $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj})$ as $[\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]$ (instead of $[(\lambda_{ki}, \tilde{x}_k, \lambda_{kj})]_\sim$).

Remark 4.3. In [MP2] the relation $\sim$ is defined to be the relation *generated* by considering equivalent $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj})$ and $(\lambda_{li}, \tilde{x}_l, \lambda_{lj})$ whenever there exists an embedding λ_{lk} such that the following diagrams are commutative:

$$\begin{tikzcd}
& \widetilde{U}_l \arrow[dl, "\lambda_{li}"'] \arrow[d, "\lambda_{lk}"] \arrow[dr, "\lambda_{lj}"] \arrow[dl, phantom, "\curvearrowright" near end, shift left=4] \arrow[dr, phantom, "\curvearrowright" near end, shift right=4] & & & & \tilde{x}_l \arrow[dl, "\lambda_{li}"'] \arrow[d, "\lambda_{lk}"] \arrow[dr, "\lambda_{lj}"] \arrow[dl, phantom, "\curvearrowright" near end, shift left=4] \arrow[dr, phantom, "\curvearrowright" near end, shift right=4] & \\
\widetilde{U}_i & \widetilde{U}_k \arrow[l, "\lambda_{ki}"] \arrow[r, "\lambda_{kj}"'] & \widetilde{U}_j & & \tilde{x}_i & \tilde{x}_k \arrow[l, "\lambda_{ki}"] \arrow[r, "\lambda_{kj}"'] & \tilde{x}_j .
\end{tikzcd} \tag{4.13}$$

In [LU] this relation is assumed to be an equivalence relation, but in general this is not true, since it is not simmetric: indeed the embedding λ_{lk} used in general will not be invertible. Hence in order to define an equivalence relation we have to pass to the relation generated by the previous definition (as in [MP2]). In other words, two points $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj})$ and $(\lambda_{li}, \tilde{x}_l, \lambda_{lj})$ will be equivalent with respect to this definition iff there exists a *finite* graph of uniformizing systems and embeddings of this form:

$$\begin{tikzcd}
 & & \widetilde{U}_k \arrow[dddll] \arrow[dddrr] \arrow[d, dash] & & \\
 & \curvearrowright & \widetilde{U}_1 \arrow[ddll] \arrow[ddrr] \arrow[d, dash] & \curvearrowright & \\
 & \curvearrowright & \widetilde{U}_2 \arrow[dll] \arrow[drr] & \curvearrowright & \\
\widetilde{U}_i & & \vdots & & \widetilde{U}_j \\
 & \curvearrowright & \widetilde{U}_{n-1} \arrow[ull] \arrow[urr] \arrow[d, dash] & \curvearrowright & \\
 & \curvearrowright & \widetilde{U}_n \arrow[uull] \arrow[uurr] \arrow[d, dash] & \curvearrowright & \\
 & & \widetilde{U}_l \arrow[uuull] \arrow[uuurr] & &
\end{tikzcd} \tag{4.14}$$

together with a similar diagram for points. Note that we don't have specified the directions of vertical embeddings; in other words we can have any finite combination of directions in the central column of the graph. It is clear that whenever we have two composable arrows we can substitute them with their composition, which is again an embedding and make the diagram commute; so we can always reduce to the case where any two consecutive arrows in the central column have opposite directions. For example, (4.7) is a diagram of this form.

Now the equivalence relation so defined coincide with the previous: in other words, if we call $\sim_A$ the equivalence relation defined in (4.4) and $\sim_B$ the equivalence relation generated by (4.13), we get that:

Proposition 4.1.4. *The relations $\sim_A$ and $\sim_B$ coincide on $\hat{R}$.*

Proof. Let $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj}) \sim_A (\lambda_{li}, \tilde{x}_l, \lambda_{lj})$, i.e. let us suppose that there exists $(\widetilde{U}_m, G_m, \pi_m) \in \mathcal{U}$, $\tilde{x}_m \in \widetilde{U}_m$ and embeddings $\lambda_{mk}, \lambda_{ml}$ which make the following diagrams commute:

$$\begin{tikzcd}
 & & \widetilde{U}_k \arrow[dll, "\lambda_{ki}"'] \arrow[drr, "\lambda_{kj}"] & & \\
\widetilde{U}_i & \curvearrowright & \widetilde{U}_m \arrow[u, "\lambda_{mk}"'] \arrow[d, "\lambda_{ml}"] & \curvearrowright & \widetilde{U}_j \\
 & & \widetilde{U}_l \arrow[ull, "\lambda_{li}"] \arrow[urr, "\lambda_{lj}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \tilde{x}_k \arrow[dll, "\lambda_{ki}"'] \arrow[drr, "\lambda_{kj}"] & & \\
\tilde{x}_i & \curvearrowright & \tilde{x}_m \arrow[u, "\lambda_{mk}"'] \arrow[d, "\lambda_{ml}"] & \curvearrowright & \tilde{x}_j. \\
 & & \tilde{x}_l \arrow[ull, "\lambda_{li}"] \arrow[urr, "\lambda_{lj}"'] & &
\end{tikzcd}$$

Hence if we call:

$$\lambda_{mi} := \lambda_{ki} \circ \lambda_{mk} = \lambda_{li} \circ \lambda_{ml} \quad \text{and} \quad \lambda_{mj} := \lambda_{kj} \circ \lambda_{mk} = \lambda_{lj} \circ \lambda_{ml},$$

we get that the following two diagrams are commutative:

$$\begin{tikzcd}
 & & \widetilde{U}_k \arrow[ddll, "\lambda_{ki}"'] \arrow[ddrr, "\lambda_{kj}"] & & \\
 & \curvearrowright & & \curvearrowright & \\
\widetilde{U}_i & & \widetilde{U}_m \arrow[ll, "\lambda_{mi}"'] \arrow[rr, "\lambda_{mj}"] \arrow[uu, "\lambda_{mk}"'] \arrow[dd, "\lambda_{ml}"] & & \widetilde{U}_j \\
 & \curvearrowright & & \curvearrowright & \\
 & & \widetilde{U}_l \arrow[uull, "\lambda_{li}"] \arrow[uurr, "\lambda_{lj}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \tilde{x}_k \arrow[ddll, "\lambda_{ki}"'] \arrow[ddrr, "\lambda_{kj}"] & & \\
 & \curvearrowright & & \curvearrowright & \\
\tilde{x}_i & & \tilde{x}_m \arrow[ll, "\lambda_{mi}"'] \arrow[rr, "\lambda_{mj}"] \arrow[uu, "\lambda_{mk}"'] \arrow[dd, "\lambda_{ml}"] & & \tilde{x}_j; \\
 & \curvearrowright & & \curvearrowright & \\
 & & \tilde{x}_l \arrow[uull, "\lambda_{li}"] \arrow[uurr, "\lambda_{lj}"'] & &
\end{tikzcd}$$

hence we have proved that $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj}) \sim_B (\lambda_{li}, \tilde{x}_l, \lambda_{lj})$.

Conversely, let us suppose we have fixed two points which are equivalent via $\sim_B$ i.e. we have a commutative diagram of the form (4.14); this means that we have a *finite* number of diagrams of the form (4.13). Since we have already proved that $\sim_A$ is an equivalence relation, if we want to prove that these two points are also equivalent with respect to $\sim_A$, it suffices to prove that this is true whenever we have a diagram of the form (4.13). Now such a diagram can be interpreted as:

$$\begin{tikzcd}
 & & \widetilde{U}_l \arrow[dll, swap, "\lambda_{li}"] \arrow[drr, "\lambda_{lj}"] & & \\
\widetilde{U}_i & \curvearrowright & \widetilde{U}_l \arrow[u, "\lambda_{ll}=1_{\widetilde{U}_l}"] \arrow[d, "\lambda_{lk}"] & \curvearrowright & \widetilde{U}_j \\
 & & \widetilde{U}_k \arrow[ull, "\lambda_{ki}"] \arrow[urr, swap, "\lambda_{kj}"] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \tilde{x}_l \arrow[dll, swap, "\lambda_{li}"] \arrow[drr, "\lambda_{lj}"] & & \\
\tilde{x}_i & \curvearrowright & \tilde{x}_l \arrow[u, "\lambda_{ll}=1_{\widetilde{U}_l}"] \arrow[d, "\lambda_{lk}"] & \curvearrowright & \tilde{x}_j \\
 & & \tilde{x}_k \arrow[ull, "\lambda_{ki}"] \arrow[urr, swap, "\lambda_{kj}"] & &
\end{tikzcd}$$

hence we have proved that $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj}) \sim_A (\lambda_{li}, \tilde{x}_l, \lambda_{lj})$. □

Remark 4.4. Having proved this result, from now on we will use without distinction the first and the second equivalence and we will refer to the equivalence classes in the same way, i.e. we will use always the notation $[\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]$.

In the following pages we will often have to define set maps from $R = \hat{R}/\sim$ using representatives of equivalences classes; since any two representative can be joined by a *finite* graph as in (4.14), in order to prove that these set maps are well defined, it will be sufficient to prove that they are well defined using a graph of the form (4.13) or of the form (4.5).

Our aim is first of all to make the pair (R, U) into a groupoid object in (**Sets**), so for the moment we don't care about the topology on R and we consider it just as a set. Four of the five structural morphisms of a groupoid object are easy to define:

$$s : R \to U \quad s([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]) := (\lambda_{ki}(\tilde{x}_k), \widetilde{U}_i);$$

$$t : R \to U \quad t([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]) := (\lambda_{kj}(\tilde{x}_k), \widetilde{U}_j);$$

$$i : R \to R \quad i([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]) := [\lambda_{kj}, \tilde{x}_k, \lambda_{ki}];$$

$$e : U \to R \quad e(\tilde{x}_i, \widetilde{U}_i) := [1_{\widetilde{U}_i}, \tilde{x}_i, 1_{\widetilde{U}_i}].$$

Note that s, t and i are well defined, i.e. they don't depend on the choice of a representative for $[\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]$. Indeed, let us consider a diagram of the form (4.13): then we have:

$$s([\lambda_{li}, \tilde{x}_l, \lambda_{lj}]) = (\lambda_{li}(\tilde{x}_l), \widetilde{U}_i) = (\lambda_{ki}(\lambda_{lk}(\tilde{x}_l)), \widetilde{U}_i) =$$
$$= (\lambda_{ki}(\tilde{x}_k), \widetilde{U}_i) = s([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]).$$

Analogous equations prove that also t and i are well defined.

Now we want to define the "multiplication" on R, so let us consider any pair of "composable arrows" $[\lambda_{ih}, \tilde{x}_i, \lambda_{ij}]$ and $[\lambda_{kj}, \tilde{x}_k, \lambda_{kl}]$ in the fiber product $R\,{}_t\!\times_s R$ in (**Sets**) (note that this fiber product always exists in (**Sets**) because of example 1.10). In other words, let us assume that:

$$t([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}]) = s([\lambda_{kj}, \tilde{x}_k, \lambda_{kl}]) \text{ i.e. } \lambda_{ij}(\tilde{x}_i) = \lambda_{kj}(\tilde{x}_k);$$

equivalently, we are in the following situation:

$$\begin{tikzcd}
 & \tilde{x}_i \arrow[d, phantom, "\Cap"] & \lambda_{ij}(\tilde{x}_i) = \lambda_{kj}(\tilde{x}_k) \arrow[d, phantom, "\Cap"] & \tilde{x}_k \arrow[d, phantom, "\Cap"] & \\
\widetilde{U}_h & \widetilde{U}_i \arrow[l, "\lambda_{ih}"'] \arrow[r, "\lambda_{ij}"] & \widetilde{U}_j & \widetilde{U}_k \arrow[l, "\lambda_{kj}"'] \arrow[r, "\lambda_{kl}"] & \widetilde{U}_l.
\end{tikzcd} \tag{4.15}$$

Since $\lambda_{ij}(\tilde{x}_i) = \lambda_{kj}(\tilde{x}_k)$, we can apply lemma 4.1.2, so we get that there exists a uniformizing system $(\widetilde{U}_f, G_f, \pi_f)$, a point $\tilde{x}_f \in \widetilde{U}_f$ and embeddings $\lambda_{fi}, \lambda_{fk}$ such that the following diagrams commute:

$$\begin{tikzcd}
& \widetilde{U}_f \arrow[dl, "\lambda_{fi}"'] \arrow[dr, "\lambda_{fk}"] & \\
\widetilde{U}_i \arrow[dr, "\lambda_{ij}"'] & \curvearrowright & \widetilde{U}_k \arrow[dl, "\lambda_{kj}"] \\
& \widetilde{U}_j &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \tilde{x}_f \arrow[dl, "\lambda_{fi}"'] \arrow[dr, "\lambda_{fk}"] & \\
\tilde{x}_i \arrow[dr, "\lambda_{ij}"'] & \curvearrowright & \tilde{x}_k. \arrow[dl, "\lambda_{kj}"] \\
& \tilde{x}_j &
\end{tikzcd}$$

So whenever we have a diagram of the form (4.15), we can "complete" it to a diagram of the form:

$$\begin{tikzcd}
& \tilde{x}_i & \lambda_{ij}(\tilde{x}_i) = \lambda_{kj}(\tilde{x}_k) & \tilde{x}_k & \\
& \rotatebox{-90}{$\in$} & \rotatebox{-90}{$\in$} & \rotatebox{-90}{$\in$} & \\
\widetilde{U}_h & \widetilde{U}_i \arrow[l, "\lambda_{ih}"'] \arrow[r, "\lambda_{ij}"] & \widetilde{U}_j & \widetilde{U}_k \arrow[l, "\lambda_{kj}"'] \arrow[r, "\lambda_{kl}"] & \widetilde{U}_l. \\
& & \curvearrowright & & \\
& & \widetilde{U}_f \arrow[uul, "\lambda_{fi}"] \arrow[uur, "\lambda_{fk}"'] & & \\
& & \rotatebox{90}{$\in$} & & \\
& & \tilde{x}_f & &
\end{tikzcd} \tag{4.16}$$

Now the idea is just to substitute the central horizontal part of it with the lower one, in order to have composable embeddings, and then define:

$$m([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], [\lambda_{kj}, \tilde{x}_k, \lambda_{kl}]) := [\lambda_{ih} \circ \lambda_{fi}, \tilde{x}_f, \lambda_{kl} \circ \lambda_{fk}].$$

Lemma 4.1.5. *The map m is well defined.*

Proof. In order prove the statement, we have to solve 2 problems:

(i) first of all, let us fix representatives $(\lambda_{ih}, \tilde{x}_i, \lambda_{ij})$ and $(\lambda_{kj}, \tilde{x}_k, \lambda_{kl})$ for the 2 points we have to "multiply". Our previous description of the multiplication map requires to *choose* a uniformizing system $(\widetilde{U}_f, G_f, \pi_f)$, a point $\tilde{x}_f \in \widetilde{U}_f$ and embeddings $\lambda_{fi}, \lambda_{fk}$ making (4.16) commute. However, this construction uses lemma 4.1.2, which gives only the existence of such data, but not the uniqueness; indeed it uses property (ii) of orbifold atlases, which guarantees only the existence of the uniformizing system and of the 2 embeddings. So we have to verify that our construction does not depend on this choice.

(ii) we have to prove that the class $[\lambda_{ih} \circ \lambda_{fi}, \tilde{x}_f, \lambda_{kl} \circ \lambda_{fk}]$ does not depend on the representatives chosen for $[\lambda_{ih}, \tilde{x}_i, \lambda_{ij}]$ and for $[\lambda_{kj}, \tilde{x}_k, \lambda_{kl}]$.

Let us solve these problems separately.

(i) Let us suppose we can "complete" a diagram (4.15) in two different ways:

$$\begin{tikzcd}
& & \tilde{x}_r \arrow[d, phantom, sloped, "\in"] & & \\
& & \widetilde{U}_r \arrow[dl, "\lambda_{ri}"'] \arrow[dr, "\lambda_{rk}"] \arrow[d, phantom, "\curvearrowright"] & & \\
\widetilde{U}_h & \widetilde{U}_i \arrow[l, "\lambda_{ih}"'] \arrow[r, "\lambda_{ij}"] & \widetilde{U}_j \arrow[d, phantom, "\curvearrowright"] & \widetilde{U}_k \arrow[l, "\lambda_{kj}"'] \arrow[r, "\lambda_{kl}"] & \widetilde{U}_l \\
& & \widetilde{U}_f \arrow[ul, "\lambda_{fi}"] \arrow[ur, "\lambda_{fk}"'] & & \\
& & \tilde{x}_f \arrow[u, phantom, sloped, "\in"] & &
\end{tikzcd} \tag{4.17}$$

Now we have that $\lambda_{ri}(\tilde{x}_r) = \lambda_{fi}(\tilde{x}_f)$, so we can apply again lemma 4.1.2 and we get that there exist a uniformizing system $(\widetilde{U}_s, G_s, \pi_s)$, a point $\tilde{x}_s \in \widetilde{U}_s$ and a pair of embeddings $\lambda_{sf}, \lambda_{sr}$ which make the following diagrams commutate:

$$\begin{tikzcd}
& \widetilde{U}_s \arrow[dl, "\lambda_{sr}"'] \arrow[dr, "\lambda_{sf}"] & & & & \tilde{x}_s \arrow[dl, "\lambda_{sr}"'] \arrow[dr, "\lambda_{sf}"] & \\
\widetilde{U}_r \arrow[dr, "\lambda_{ri}"'] & \curvearrowright & \widetilde{U}_f \arrow[dl, "\lambda_{fi}"] & & \tilde{x}_r \arrow[dr, "\lambda_{ri}"'] & \curvearrowright & \tilde{x}_f \arrow[dl, "\lambda_{fi}"] \\
& \widetilde{U}_i & & & & \tilde{x}_i &
\end{tikzcd} \tag{4.18}$$

Now using (4.17) and (4.18) together we get that:

$$\lambda_{kj} \circ \lambda_{fk} \circ \lambda_{sf} = \lambda_{ij} \circ \lambda_{fi} \circ \lambda_{sf} =$$
$$= \lambda_{ij} \circ \lambda_{ri} \circ \lambda_{sr} = \lambda_{kj} \circ \lambda_{rk} \circ \lambda_{sr}$$

and we recall that λ_{kj} is an embedding, hence in particular it is injective, so we have that:

$$\lambda_{fk} \circ \lambda_{sf} = \lambda_{rk} \circ \lambda_{sr} \tag{4.19}$$

Now if we combine together diagram (4.18) and equation (4.19), we get commutative diagrams:

$$\begin{tikzcd}
& \widetilde{U}_r \arrow[dl, "\lambda_{ih}\circ\lambda_{ri}"'] \arrow[dr, "\lambda_{kl}\circ\lambda_{rk}"] & & & \tilde{x}_r \arrow[dl, "\lambda_{ih}\circ\lambda_{ri}"'] \arrow[dr, "\lambda_{kl}\circ\lambda_{rk}"] & \\
\widetilde{U}_h \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_s \arrow[u, "\lambda_{sr}"'] \arrow[d, "\lambda_{sf}"] \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_l & \tilde{x}_h := \lambda_{hi}(\tilde{x}_i) \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_s \arrow[u, "\lambda_{sr}"'] \arrow[d, "\lambda_{sf}"] \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_l := \lambda_{kl}(\tilde{x}_k). \\
& \widetilde{U}_f \arrow[ul, "\lambda_{ih}\circ\lambda_{fi}"] \arrow[ur, "\lambda_{kl}\circ\lambda_{fk}"'] & & & \tilde{x}_f \arrow[ul, "\lambda_{ih}\circ\lambda_{fi}"] \arrow[ur, "\lambda_{kl}\circ\lambda_{fk}"'] &
\end{tikzcd} \tag{4.20}$$

This means that:

$$(\lambda_{ih} \circ \lambda_{fi}, \tilde{x}_f, \lambda_{kl} \circ \lambda_{fk}) \sim (\lambda_{ih} \circ \lambda_{ri}, \tilde{x}_r, \lambda_{kl} \circ \lambda_{rk}),$$

hence (i) is solved.

(ii) Let us suppose we have chosen another representative $(\lambda_{sh}, \tilde{x}_s, \lambda_{sj})$ for $[\lambda_{ih}, \tilde{x}_i, \lambda_{ij}]$. Using remark 4.4 it suffices to consider the case when the two representative are equivalent via a diagram of the form (4.13); in other words, we can assume there exists an embedding λ_{si} such that:

$$\lambda_{sh} = \lambda_{ih} \circ \lambda_{si}, \quad \lambda_{sj} = \lambda_{ij} \circ \lambda_{si} \quad \text{and} \quad \lambda_{si}(\tilde{x}_s) = \tilde{x}_i.$$

Now if we want to compute $m([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], [\lambda_{kj}, \tilde{x}_k, \lambda_{kl}])$ using this new representative for the first point, we have to use lemma 4.1.2 in order to choose a uniformizing system $(\widetilde{U}_r, G_r, \pi_r)$ together with a point $\tilde{x}_r \in \widetilde{U}_r$ and a pair of embeddings $\lambda_{rs}, \lambda_{rk}$ such that:

$$\lambda_{sj} \circ \lambda_{rs} = \lambda_{kj} \circ \lambda_{rk} \quad \lambda_{rs}(\tilde{x}_r) = \tilde{x}_s \quad \text{and} \quad \lambda_{rk}(\tilde{x}_r) = \tilde{x}_k.$$

Note that there are no problems in choosing all these data, since we have already proved (i). In other words, we are using commutative diagrams of the form:

$$\begin{tikzcd}
\arrow[dr, phantom, "\curvearrowright"] & \widetilde{U}_s \arrow[dl, "\lambda_{sh}"'] \arrow[d, "\lambda_{si}"] \arrow[dr, "\lambda_{sj}"] \arrow[dr, phantom, "\curvearrowright"', near start] & \widetilde{U}_r \arrow[l, "\lambda_{rs}"'] \arrow[dr, "\lambda_{rk}"] \arrow[d, phantom, "\curvearrowright"] & & \\
\widetilde{U}_h & \widetilde{U}_i \arrow[l, "\lambda_{ih}"'] \arrow[r, "\lambda_{ij}"] & \widetilde{U}_j \arrow[d, phantom, "\curvearrowright"] & \widetilde{U}_k \arrow[l, "\lambda_{kj}"'] \arrow[r, "\lambda_{kl}"] & \widetilde{U}_l \\
& & \widetilde{U}_f \arrow[ul, "\lambda_{fi}"] \arrow[ur, "\lambda_{fk}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \tilde{x}_s \arrow[ddll, "\lambda_{sh}"'] \arrow[dd, "\lambda_{si}"] \arrow[ddrr, "\lambda_{sj}"] & & \tilde{x}_r \arrow[ll, "\lambda_{rs}"'] \arrow[ddrr, "\lambda_{rk}"] & & & & \\
 & \curvearrowright & & \curvearrowright & \curvearrowright & & & & \\
\tilde{x}_h & & \tilde{x}_i \arrow[ll, "\lambda_{ih}"'] \arrow[rr, "\lambda_{ij}"] & & \tilde{x}_j & & \tilde{x}_k \arrow[ll, "\lambda_{kj}"'] \arrow[rr, "\lambda_{kl}"] & & \tilde{x}_l \\
 & & & & \curvearrowright & & & & \\
 & & & & \tilde{x}_f \arrow[uull, "\lambda_{fi}"] \arrow[uurr, "\lambda_{fk}"'] & & & &
\end{tikzcd}$$

where for simplicity we have used the following notations:

$$\tilde{x}_h := \lambda_{sh}(\tilde{x}_s) = \lambda_{ih} \circ \lambda_{si}(\tilde{x}_s) \quad \tilde{x}_l := \lambda_{kl}(\tilde{x}_k) \quad \text{and} \quad \tilde{x}_j := \lambda_{ij}(\tilde{x}_i).$$

So we get a diagram of the form:

$$\begin{tikzcd}
 & & & & \tilde{x}_r & & & & \\
 & & & & \rotatebox[origin=c]{90}{$\in$} & & & & \\
 & & & & \widetilde{U}_r \arrow[ddll, "\lambda_{ri}:=\lambda_{si}\circ\lambda_{rs}"'] \arrow[ddrr, "\lambda_{rk}"] & & & & \\
 & & & & \curvearrowright & & & & \\
\widetilde{U}_h & & \widetilde{U}_i \arrow[ll, "\lambda_{ih}"'] \arrow[rr, "\lambda_{ij}"] & & \widetilde{U}_j & & \widetilde{U}_k \arrow[ll, "\lambda_{kj}"'] \arrow[rr, "\lambda_{kl}"] & & \widetilde{U}_l \\
 & & & & \curvearrowright & & & & \\
 & & & & \widetilde{U}_f \arrow[uull, "\lambda_{fi}"] \arrow[uurr, "\lambda_{fk}"'] & & & & \\
 & & & & \rotatebox[origin=c]{270}{$\in$} & & & & \\
 & & & & \tilde{x}_f & & & &
\end{tikzcd}$$

with $\lambda_{ri}(\tilde{x}_r) = \lambda_{fi}(\tilde{x}_f)$, so we can repeat the came construction of (i) in order to get a diagram of the form (4.20), so we obtain:

$$(\lambda_{ih} \circ \lambda_{ri}, \tilde{x}_r, \lambda_{kl} \circ \lambda_{rk}) \sim (\lambda_{ih} \circ \lambda_{fi}, \tilde{x}_f, \lambda_{kl} \circ \lambda_{fk}).$$

Now by definition of λ_{ri} we have $\lambda_{ih} \circ \lambda_{ri} = \lambda_{ih} \circ \lambda_{si} \circ \lambda_{rs} = \lambda_{sh} \circ \lambda_{rs}$, hence:

$$(\lambda_{sh} \circ \lambda_{rs}, \tilde{x}_r, \lambda_{kl} \circ \lambda_{rk}) \sim (\lambda_{ih} \circ \lambda_{fi}, \tilde{x}_f, \lambda_{kl} \circ \lambda_{fk})$$

so the multiplication does not depend on the representative chosen for $[\lambda_{ih}, \tilde{x}_i, \lambda_{ij}]$. In the same way one can also prove that the multiplication doesn't depend on the representative chosen for the point $[\lambda_{kj}, \tilde{x}_k, \lambda_{kl}]$.

□

Until now we have proved that the 5 set maps s, t, m, i, e are all well defined. Moreover, we have that:

Proposition 4.1.6. *$(R \overset{s}{\underset{t}{\rightrightarrows}} U)$ is a groupoid object in (**Sets**).*

Proof. We have to prove that all the axioms for a groupoid object given in definition 3.1 are satisfied:

- For any point $(\tilde{x}_i, \widetilde{U}_i) \in U$, we have $s \circ e(\tilde{x}_i, \widetilde{U}_i) = s[1_{\widetilde{U}_i}, \tilde{x}_i, 1_{\widetilde{U}_i}] = (\tilde{x}_i, \widetilde{U}_i)$, hence $e \circ s = 1_U$; in the same way we prove also that $e \circ t = 1_U$.
- Let us take any pair of composable arrows: $\Big([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], [\lambda_{kj}, \tilde{x}_k, \lambda_{kl}]\Big)$ in $R\,_t\!\times_s R$. Then using the notations of the previous construction, we get:

$$s \circ m([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], [\lambda_{kj}, \tilde{x}_k, \lambda_{kl}]) = s([\lambda_{ih} \circ \lambda_{fi}, \tilde{x}_f, \lambda_{kl} \circ \lambda_{fk}]) =$$
$$= (\lambda_{ih} \circ \lambda_{fi}(\tilde{x}_f), \widetilde{U}_h) = (\lambda_{ih}(\tilde{x}_i), \widetilde{U}_h) = s([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}]) =$$
$$= s \circ pr_1([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], [\lambda_{kj}, \tilde{x}_k, \lambda_{kl}]).$$

In an analogous way we can also prove that $t \circ m = t \circ pr_2$.

- Now let us choose any triple of composable arrows:

$$([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], [\lambda_{kj}, \tilde{x}_k, \lambda_{kl}], [\lambda_{ml}, \tilde{x}_m, \lambda_{mn}]) \in R_t \times_s R_t \times_s R$$

$$\text{i.e. } \lambda_{ij}(\tilde{x}_i) = \lambda_{kj}(\tilde{x}_k) \quad \text{and} \quad \lambda_{kl}(\tilde{x}_k) = \lambda_{ml}(\tilde{x}_m).$$

In order to multiply the first two, we consider a uniformizing system $(\widetilde{U}_f, G_f, \pi_f)$, a point $\tilde{x}_f \in \widetilde{U}_f$ and embeddings $\lambda_{fi}, \lambda_{fk}$ while in order to compose the second and the third arrow, we consider a uniformizing system $(\widetilde{U}_s, G_s, \pi_s)$, a point $\tilde{x}_s \in \widetilde{U}_s$ and embeddings $\lambda_{sk}, \lambda_{sm}$ such that:

$$\begin{tikzcd}
 & \tilde{x}_i \arrow[d, phantom, sloped, "\in"] & & \tilde{x}_k \arrow[d, phantom, sloped, "\in"] & & \tilde{x}_m \arrow[d, phantom, sloped, "\in"] & \\
\widetilde{U}_h & \widetilde{U}_i \arrow[l, "\lambda_{ih}"'] \arrow[r, "\lambda_{ij}"] & \widetilde{U}_j \arrow[d, phantom, "\curvearrowright"] & \widetilde{U}_k \arrow[l, "\lambda_{kj}"'] \arrow[r, "\lambda_{kl}"] & \widetilde{U}_l \arrow[d, phantom, "\curvearrowright"] & \widetilde{U}_m \arrow[l, "\lambda_{ml}"'] \arrow[r, "\lambda_{mn}"] & \widetilde{U}_n \\
 & & \widetilde{U}_f \arrow[ul, "\lambda_{fi}"] \arrow[ur, "\lambda_{fk}"'] \arrow[d, phantom, sloped, "\ni"] & & \widetilde{U}_s \arrow[ul, "\lambda_{sk}"] \arrow[ur, "\lambda_{sm}"'] \arrow[d, phantom, sloped, "\ni"] & & \\
 & & \tilde{x}_f & & \tilde{x}_s & &
\end{tikzcd}$$

is commutative and $\lambda_{fi}(\tilde{x}_f) = \tilde{x}_i$, $\lambda_{fk}(\tilde{x}_f) = \tilde{x}_k = \lambda_{sk}(\tilde{x}_s)$ and $\lambda_{sm}(\tilde{x}_s) = \tilde{x}_m$. In particular, using the second relation we can apply again lemma 4.1.2 in order to get a uniformizing system $(\widetilde{U}_r, G_r, \pi_r)$, a point $\tilde{x}_r \in \widetilde{U}_r$ and embeddings $\lambda_{rf}, \lambda_{rs}$ such that:

$$\begin{tikzcd}
& \tilde{x}_i \arrow[d, phantom, sloped, "\in"] & & \tilde{x}_k \arrow[d, phantom, sloped, "\in"] & & \tilde{x}_m \arrow[d, phantom, sloped, "\in"] & \\
\widetilde{U}_h & \widetilde{U}_i \arrow[l, "\lambda_{ih}"'] \arrow[r, "\lambda_{ij}"] & \widetilde{U}_j & \widetilde{U}_k \arrow[l, "\lambda_{kj}"'] \arrow[r, "\lambda_{kl}"] & \widetilde{U}_l & \widetilde{U}_m \arrow[l, "\lambda_{ml}"'] \arrow[r, "\lambda_{mn}"] & \widetilde{U}_n \\
& & \curvearrowright & & \curvearrowright & & \\
& & \widetilde{U}_f \arrow[uul, "\lambda_{fi}"] \arrow[uur, "\lambda_{fk}"'] & \curvearrowright & \widetilde{U}_s \arrow[uul, "\lambda_{sk}"] \arrow[uur, "\lambda_{sm}"'] \arrow[dr, phantom, sloped, "\ni"] & & \\
& \tilde{x}_f \arrow[ur, phantom, sloped, "\in"] & & & & \tilde{x}_s & \\
& & & \widetilde{U}_r \arrow[uul, "\lambda_{rf}"] \arrow[uur, "\lambda_{rs}"'] \arrow[d, phantom, sloped, "\ni"] & & & \\
& & & \tilde{x}_r & & &
\end{tikzcd} \tag{4.21}$$

is commutative (and the obvious relations between marked points hold). Now:

$$m\Big([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], m\Big([\lambda_{kj}, \tilde{x}_k, \lambda_{kl}], [\lambda_{ml}, \tilde{x}_m, \lambda_{mn}]\Big)\Big) =$$
$$= m\Big([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], [\lambda_{kj} \circ \lambda_{fk} \circ \lambda_{rf}, \tilde{x}_r, \lambda_{mn} \circ \lambda_{sm} \circ \lambda_{rs}]\Big) =$$
$$= [\lambda_{ih} \circ \lambda_{fi} \circ \lambda_{rf}, \tilde{x}_r, \lambda_{mn} \circ \lambda_{sm} \circ \lambda_{rs}]$$

where:

- in order to compute the first multiplication we have used the commutativity of the square and of the triangle on the right (this can be done without any problem because of (i) of lemma 4.1.5);
- in order to compute the last multiplication, we have chosen the pair of embeddings $1_{\widetilde{U}_r}$ and $\lambda_{rf} \circ \lambda_{fi}$ from $(\widetilde{U}_r, G_r, \pi_r)$ to $(\widetilde{U}_r, G_r, \pi_r)$ and $(\widetilde{U}_i, G_i, \pi_i)$ respectively.

Since diagram (4.21) is symmetric, we obtain the same result if we first compute the multiplication of the first two arrows, and then we multiply them with the third. In other words, we have proved that:

$$m \circ (1_R \times m) = m \circ (m \times 1_R).$$

- with a direct check one can also prove axioms (iv) and (v) of groupoid objects.

□

Until now we have defined the 5 set maps and we have proved the axioms of groupoid objects in (**Sets**); our next purpose is to prove the following result:

Proposition 4.1.7. *We can give to R a suitable topology such that it becomes a complex manifold.*

Differently from the proof due to Dorette Pronk ([Pr], §4.4.2), in order to prove this result I prefer to give an explicit description of a complex manifold atlas on R; this will be also useful later on, in order to prove some properties about the maps s and t. In order to do that, we will use the following lemma:

Lemma 4.1.8. *The equivalence relation $\sim$ is the trivial one whenever we restrict to any open set of $\hat{R}$ of the form $\widetilde{U}_k^{ij}$.*

Proof. Let us suppose we have fixed two points in $\widetilde{U}_k^{ij}$ which are equivalent via $\sim_A$:

$$(\lambda_{ki}, \tilde{x}_k, \lambda_{kj}) \sim_A (\lambda_{ki}, \tilde{x}'_k, \lambda_{kj});$$

then there exist a uniformizing system $(\widetilde{U}_m, G_m, \pi_m)$, a point $\tilde{x}_m \in \widetilde{U}_m$ and embeddings $\lambda_{mk}, \lambda'_{mk}$ both defined from $(\widetilde{U}_m, G_m, \pi_m)$ to $(\widetilde{U}_k, G_k, \pi_k)$ such that the following two diagrams commute:

$$\begin{tikzcd}
& \widetilde{U}_k \arrow[dl, "\lambda_{ki}"'] \arrow[dr, "\lambda_{kj}"] & \\
\widetilde{U}_i \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_m \arrow[u, "\lambda_{mk}"'] \arrow[d, "\lambda'_{mk}"] \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_j \\
& \widetilde{U}_k \arrow[ul, "\lambda_{ki}"] \arrow[ur, "\lambda_{kj}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \tilde{x}_k \arrow[dl, "\lambda_{ki}"'] \arrow[dr, "\lambda_{kj}"] & \\
\tilde{x}_i \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_m \arrow[u, "\lambda_{mk}"'] \arrow[d, "\lambda'_{mk}"] \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_j. \\
& \tilde{x}'_k \arrow[ul, "\lambda_{ki}"] \arrow[ur, "\lambda_{kj}"'] &
\end{tikzcd}$$

In particular, $\lambda_{ki} \circ \lambda_{mk} = \lambda_{ki} \circ \lambda'_{mk}$ and λ_{ki} is injective, so $\lambda_{mk} = \lambda'_{mk}$. Hence $\tilde{x}_k = \lambda_{mk}(\tilde{x}_m) = \lambda'_{mk}(\tilde{x}_m) = \tilde{x}'_k$, so we have proved that:

$$(\lambda_{ki}, \tilde{x}_k, \lambda_{kj}) = (\lambda_{ki}, \tilde{x}'_k, \lambda_{kj}).$$

□

Proof. (of proposition 4.1.7) Let us consider any open set of $\hat{R}$ of the form:

$$A := \widetilde{U}_k^{ij}$$

and let us denote with A^{sat} the saturated of A in $\hat{R}$ with respect to $\sim$. *We claim that also* A^{sat} *is open in* $\hat{R}$. Indeed, let us consider any point in A^{sat}, i.e. a point which is equivalent to a point of $\widetilde{U}_k^{ij}$. By definition of $\sim$, this must be necessarily of the form $(\lambda_{li}, \tilde{x}_l, \lambda_{lj})$, moreover, there must exists a uniformizing system $(\widetilde{U}_m, G_m, \pi_m)$, a point $\tilde{x}_m$ and embeddings $\lambda_{mk}, \lambda_{ml}$ making the following two diagrams commute:

$$\begin{tikzcd}
& \widetilde{U}_k \arrow[ld, "\lambda_{ki}"'] \arrow[rd, "\lambda_{kj}"] & \\
\widetilde{U}_i \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_m \arrow[u, "\lambda_{mk}"'] \arrow[d, "\lambda_{ml}"'] \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_j \\
& \widetilde{U}_l \arrow[lu, "\lambda_{li}"] \arrow[ru, "\lambda_{lj}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \tilde{x}_k \arrow[ld, "\lambda_{ki}"'] \arrow[rd, "\lambda_{kj}"] & \\
\tilde{x}_i \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_m \arrow[u, "\lambda_{mk}"'] \arrow[d, "\lambda_{ml}"'] \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_j. \\
& \tilde{x}_l \arrow[lu, "\lambda_{li}"] \arrow[ru, "\lambda_{lj}"'] &
\end{tikzcd}$$

Now let us consider the set $\widetilde{U}_n := \lambda_{ml}(\widetilde{U}_m) \subseteq \widetilde{U}_l$, which is open (because λ_{ml} is an embedding between open sets of $\mathbb{C}^n$) and which contains the point $\tilde{x}_l$. If we fix any other point $\tilde{x}'_m$ in $\widetilde{U}_m$ we get a diagram similar to the second one. If we call $\tilde{x}'_l$ the corresponding image in $\widetilde{U}_l$, we get that every point of $\widetilde{U}_n$ is equivalent to some point in A. So we have proved that the set of points of the form $(\lambda_{li}, \tilde{x}'_l, \lambda_{lj})$ (with $\tilde{x}'_l$ in $\widetilde{U}_n$) is completely contained in A^{sat}. Moreover, this set is open in $\widetilde{U}_l^{ij}$, so it is open also in $\hat{R}$. Hence *for every point* $(\lambda_{li}, \tilde{x}_l, \lambda_{lj})$ *in* A^{sat} *we have found an open neighborhood of it completely contained in* A^{sat}, *so this set is open in* $\hat{R}$.

Hence the set $q(A^{\mathrm{sat}})$ is open in R by definition of quotient topology; moreover, by definition of saturated, it coincides with $q(A)$. Since this holds for every choice of $A = \widetilde{U}_k^{ij}$ we get that the family:

$$\{(\widetilde{W}_k^{ij} := q(\widetilde{U}_k^{ij}) = q(\widetilde{U}_k^{ij\mathrm{sat}}))\}_{\widetilde{U}_k^{ij} \subseteq \hat{R}}$$

is an open covering of R. Then our aim is to construct from it a complex manifold atlas on R. If we use the previous lemma, we get that $\sim$ is the trivial equivalence relation on every set $\widetilde{U}_k^{ij}$, so $q(\widetilde{U}_k^{ij})$ is homeomorphic to $\widetilde{U}_k^{ij}$ via q (which is invertible if we restrict to this set). Moreover, we recall that by construction $\widetilde{U}_k^{ij}$ is just a copy of $\widetilde{U}_k$, so the set map ϕ_k^{ij} defined from $\widetilde{W}_k^{ij}$ to $\widetilde{U}_k$ as:

$$\phi_k^{ij}([\lambda_{ki},\tilde{x}_k,\lambda_{kj}]) := \tilde{x}_k$$

is an homeomorphism (with codomain an open subset of $\mathbb{C}^n$). So it makes sense to consider the family of charts:

$$\mathcal{F} := \{(\widetilde{W}_k^{ij},\phi_k^{ij})\}_{\widetilde{U}_k^{ij}\subseteq\hat{R}}.$$

Since the domains of these charts cover all R, it remains only to prove the compatibility condition on the intersection of any pair of charts; so let us fix any pair of domains $\widetilde{W}_k^{ij}$ and $\widetilde{W}_l^{i'j'}$ with non-empty intersection and let us fix any point:

$$P = [\lambda_{ki},\tilde{x}_k,\lambda_{kj}] = [\lambda_{li'},\tilde{x}_l,\lambda_{lj'}]$$

in the intersection. By definition of $\sim$, we get that necessarily $i'=i$ and $j'=j$; moreover, there exist a uniformizing system $(\widetilde{U}_m,G_m,\pi_m)$, a point $\tilde{x}_m\in\widetilde{U}_m$ and a pair of embeddings $\lambda_{mk},\lambda_{ml}$ as in (4.5). Now the images of the point P via the coordinate functions ϕ_k^{ij} and ϕ_l^{ij} are respectively:

$$\phi_k^{ij}([\lambda_{ki},\tilde{x}_k,\lambda_{kj}]) = \tilde{x}_k = \lambda_{mk}(\tilde{x}_m) \quad \text{and} \quad \phi_l^{ij}([\lambda_{li},\tilde{x}_l,\lambda_{lj}]) = \tilde{x}_l = \lambda_{ml}(\tilde{x}_m).$$

So if we call ϕ the transition map:

$$\phi_l^{ij}\circ(\phi_k^{ij})^{-1} : \phi_k^{ij}(\widetilde{W}_k^{ij}\cap\widetilde{W}_l^{ij}) \to \phi_l^{ij}(\widetilde{W}_k^{ij}\cap\widetilde{W}_l^{ij}),$$

we get that:

$$\phi(\tilde{x}_k) = \tilde{x}_l = \lambda_{ml}(\tilde{x}_m) = \lambda_{ml}\circ\lambda_{mk}^{-1}(\tilde{x}_k).$$

As before, using diagram (4.5) we get that this is the expression of ϕ not only in the point $\tilde{x}_k$, but also in an open neighborhood of it (not necessarily coinciding with all the domain of ϕ). Hence we have proved that in an open neighborhood of $\tilde{x}_k$ the transition map ϕ coincides with $\lambda_{ml}\circ\lambda_{mk}^{-1}$, which is holomorphic.

Hence ϕ is locally holomorphic, so it is holomorphic on all its domain. Moreover, the previous construction still holds if we swap the roles of $\widetilde{W}_k^{ij}$ and $\widetilde{W}_l^{ij}$, hence also the transition map

$$\psi := \phi_k^{ij} \circ (\phi_l^{ij})^{-1} : \phi_l^{ij}(\widetilde{W}_k^{ij} \cap \widetilde{W}_l^{ij}) \to \phi_k^{ij}(\widetilde{W}_k^{ij} \cap \widetilde{W}_l^{ij})$$

is holomorphic; moreover, one can easily see that this is the inverse of ϕ, hence ϕ is a biholomorphic map. So the compatibility condition is satisfied and the family $\mathcal{F}$ is a complex manifold for R; using the fact that $\sim$ locally is trivial, we get that this manifold has the same complex dimension of $\hat{R}$ (i.e. the dimension of the orbifold atlas $\mathcal{U}$). □

Lemma 4.1.9. *The maps s and t are both étale.*

Proof. Let us prove the statement only for s, for t it is analogous; since the property of being étale is a local one, we can check it by restricting to the domains of suitable charts in source and target. So let us fix any point $[\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]$ in R and the chart $(\widetilde{W}_k^{ij}, \phi_k^{ij})$ around it. We recall that

$$s([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]) = \lambda_{ki}(\tilde{x}_k) \in \widetilde{U}_i \subseteq U$$

where $\widetilde{U}_i$ means a copy of $\widetilde{U}_i$ in the manifold U; so a chart around this point is just $(\widetilde{U}_i, id)$. Hence the map s can be expressed in coordinates as:

$$\tilde{s} := id \circ s \circ (\phi_k^{ij})^{-1} : \widetilde{U}_k \to \widetilde{U}_i$$

and a direct check proves that this map concides with the holomorphic embedding λ_{ki}. So s is a biholomorphism if restricted to $\widetilde{W}_k^{ij}$ in domain and to $\lambda_{ki}(\widetilde{U}_k)$ in codomain. Hence we have proved that s is étale. □

Lemma 4.1.10. *The relative diagonal $(s, t) : R \to U \times U$ is proper.*

Proof. (adapted from [Pr], proposition 4.4.8 and corollary 4.4.9) By definition of product topology, a basis for the topology on $U \times U$ is given by the sets of the form $\widetilde{U}_i \times \widetilde{U}_j$ (for any pair of indexes $i, j \in I$). So let us fix any point $(\tilde{x}_i, \tilde{x}_j) \in \widetilde{U}_i \times \widetilde{U}_j$; if $\pi_i(\tilde{x}_i) \neq \pi_j(\tilde{x}_j)$, then we can use the fact that X is Hausdorff (by definition of orbifold) and we get that there exists two open disjoint neighborhoods D_i and D_j of $\pi_i(\tilde{x}_i)$ and $\pi_j(\tilde{x}_j)$. If we call $\widetilde{D}_i := \pi_i^{-1}(D_i)$ and $\widetilde{D}_j := \pi_j^{-1}(D_j)$, we get that $\widetilde{D}_i \times \widetilde{D}_j$ is an open neighborhood of $(\tilde{x}_i, \tilde{x}_j)$ and its preimage via (s, t) is empty.

Now let us consider the case when $\pi_i(\tilde{x}_i) = \pi_j(\tilde{x}_j)$; in this case we can use property (ii) of orbifold atlases and remark 2.7 in order to find a uniformizing system $(\widetilde{U}_k, G_k, \pi_k) \in \mathcal{U}$, a point $\tilde{x}_k \in \widetilde{U}_k$ and embeddings $\lambda_{ki}, \lambda_{kj}$ such that $\lambda_{ki}(\tilde{x}_k) = \tilde{x}_i$ and $\lambda_{kj}(\tilde{x}_k) = \tilde{x}_j$. Hence:

$$[\lambda_{ki}, \tilde{x}_k, \lambda_{kj}] \in (s, t)^{-1}(\tilde{x}_i, \tilde{x}_j).$$

Now let us fix any positive radius r such that the open ball $\widetilde{A}$ of radius r and centered in $\tilde{x}_k$ is completely contained in $\widetilde{U}_k$ (which is an open set of $\mathbb{C}^n$). Then we can apply lemma 2.5.1 to $\tilde{x}$ and $\widetilde{A}$ and we get in particular an open neighborhood $\widetilde{B}$ of $\tilde{x}_k$ contained in $\widetilde{A}$ and stable under the action of the stabilizer group $(G_k)_{\tilde{x}_k}$. Then let us consider the open sets $\widetilde{W}_i := \lambda_{ki}(\widetilde{B}) \subseteq \widetilde{U}_i$, $\widetilde{W}_i := \lambda_{kj}(\widetilde{B}) \subseteq \widetilde{U}_j$ and the set $\widetilde{W}_j \times \widetilde{W}_j$, which is an open neighborhood of $(\tilde{x}_i, \tilde{x}_j)$ in $U \times U$. Now let us fix any point:

$$[\lambda_{li}, \tilde{y}_l, \lambda_{lj}] \in (s, t)^{-1}(\widetilde{W}_i \times \widetilde{W}_j),$$

so there exists a point $(\tilde{y}_i, \tilde{y}_j) \in \widetilde{W}_i \times \widetilde{W}_j$ such that $\lambda_{li}(\tilde{y}_l) = \tilde{y}_i$ and $\lambda_{lj}(\tilde{y}_l) = \tilde{y}_i$. Moreover, by definition of $\widetilde{W}_i$ and $\widetilde{W}_j$ we have that there exists a pair of points $\tilde{y}_k, \tilde{y}'_k$ in $\widetilde{B}$ such that $\lambda_{ki}(\tilde{y}_k) = \tilde{y}_i$ and $\lambda_{kj}(\tilde{y}'_k) = \tilde{y}_j$. Hence $\pi_k(\tilde{y}_k) = \pi_k(\tilde{y}'_k)$, so by definition of uniformizing system there exists an element $g_k \in G_k$ such that $g_k(\tilde{y}_k) = \tilde{y}'_k$. Hence we get that:

$$[\lambda_{ki}, \tilde{y}_k, \lambda_{kj} \circ g_k] \in (s, t)^{-1}\{(\tilde{y}_i, \tilde{y}_j)\}.$$

The question is: *are there any other points in the preimage of* $(\tilde{y}_i, \tilde{y}_j)$*? How many?* In order to solve this problem, let us suppose that there exists another point $[\lambda_{li}, \tilde{y}_l, \lambda_{lj}]$ in $(s,t)^{-1}\{(\tilde{y}_i, \tilde{y}_j)\}$. In particular, we get that $\pi_l(\tilde{y}_l) = \pi_k(\tilde{y}_k)$, so we can apply property (ii) of orbifold atlas for $\mathcal{U}$ and remark 2.7, and we get that there exists $(\widetilde{U}_m, G_m, \pi_m) \in \mathcal{U}$, a point $\tilde{y}_m \in \widetilde{U}_m$ and embeddings $\lambda_{mk}, \lambda_{ml}$ such that $\lambda_{mk}(\tilde{y}_m) = \tilde{y}_k$ and $\lambda_{ml}(\tilde{y}_m) = \tilde{y}_l$.

Now let us consider the pair of embeddings:

$$\alpha := \lambda_{ki} \circ \lambda_{mk} \quad \text{and} \quad \beta := \lambda_{li} \circ \lambda_{ml}$$

both defined from $(\widetilde{U}_m, G_m, \pi_m)$ to $(\widetilde{U}_i, G_i, \pi_i)$. If we use lemma 2.1.7 we get that there exists a unique $g_i \in G_i$ such that $g_i \circ \alpha = \beta$. Moreover, we get that $\alpha(\tilde{y}_m) = \beta(\tilde{y}_m)$, hence g_i belongs to the stabilizer in G_i at $\tilde{y}_i$, which is isomorphic via $\Lambda_{ki} \circ \Lambda_{mk}$ to the stabilizer in G_m at $\tilde{y}_m$. Then using lemma 2.1.10, we get that there exists a unique element $g_m \in G_m$ such that, if we call $\tilde{\lambda}_{mk} := \lambda_{mk} \circ g_m$, we get that:

$$\tilde{\lambda}_{ml}(\tilde{y}_m) = \tilde{y}_k \quad \text{and} \quad \lambda_{ki} \circ \tilde{\lambda}_{mk} = \lambda_{li} \circ \lambda_{ml}.$$

Now let us consider the pair of embeddings:

$$\gamma := \lambda_{kj} \circ g_k \circ \tilde{\Lambda}_{mk} \quad \text{and} \quad \delta := \lambda_{lj} \circ \lambda_{ml};$$

using again lemma 2.1.7 we get that there exists a unique $g_j \in G_j$ such that $g_j \circ \gamma = \delta$ and, as before, this element belongs to the stabilizer of G_j at $\tilde{y}_j$. So we get that:

$$g_j \circ \lambda_{kj} \circ g_k \circ \tilde{\Lambda}_{mk} = \lambda_{lj} \circ \lambda_{ml} \quad \text{and} \quad g_j \circ \lambda_{kj} \circ g_k(\tilde{y}_k) = g_j(\tilde{y}_j) = \tilde{y}_j.$$

Hence using all the previous data we get that:

$$[\lambda_{li}, \tilde{y}_l, \lambda_{lj}] = [\lambda_{ki}, \tilde{y}_k, g_j \circ \lambda_{kj} \circ g_k].$$

hence we have proved that:

$$(s,t)^{-1}(\tilde{y}_i,\tilde{y}_j) \subseteq \{[[\lambda_{ki},\tilde{y}_k,g_j\circ\lambda_{kj}\circ g_k]]\}_{g_j\in G_j}. \tag{4.22}$$

Hence every set of the previuos form is *finite* (because G_j is so). Moreover, since (4.22) holds for every point $(\tilde{y}_i,\tilde{y}_j)$ in $\widetilde{W}_i\times\widetilde{W}_j$ with non-empty preimage, we have proved that:

$$(s,t)^{-1}(\widetilde{W}_i\times\widetilde{W}_j)\subseteq\coprod_{g_j\in G_j} q(\widetilde{U}_k^{\lambda_{ki},g_j\circ\lambda_{kj}\circ g_k})$$

where $q:\hat{R}\to R$ is the quotient map and the sets $\widetilde{U}_k^{\lambda_{ki},g_j\circ\lambda_{kj}\circ g_k}$ are the connected components of $\hat{R}$ which are simply copies of $\widetilde{U}_k$ and indexed over the pair of embeddings λ_{ki} and $g_j\circ\lambda_{kj}\circ g_k$. Actually, using the previuos construction we get that:

$$(s,t)^{-1}(\widetilde{W}_i\times\widetilde{W}_j)\subseteq\coprod_{g_j\in G_j} q(\widetilde{B}_k^{\lambda_{ki},g_j\circ\lambda_{kj}\circ g_k})$$

where the sets $\widetilde{B}_k^{\lambda_{ki},g_j\circ\lambda_{kj}\circ g_k}$ are just copies of $\widetilde{B}$ in the sets $\widetilde{U}_k^{\lambda_{ki},g_j\circ\lambda_{kj}\circ g_k}$. Let us remark that the sets $\widetilde{B}_k^{\lambda_{ki},g_j\circ\lambda_{kj}\circ g_k}$ are with compact closure in $\hat{R}$ since they are limited (actually $\widetilde{B}\subseteq\widetilde{A}$, which was an open ball with finite radius). Hence also their images via q have compact closure in R.

Now let us fix any compact set K in $U\times U$: we want to prove that $(s,t)^{-1}(K)$ is a compact set in R. Using the first part of the proof, for every point in K with empty preimage in R we define as before the open set $\widetilde{D}_i\times\widetilde{D}_j$ such that $(s,t)^{-1}(A\widetilde{D}_i\times\widetilde{D}_j)=\varnothing$. If we do so for every point in K with empty preimage, we construct an open set $\widetilde{D}$ such that $(s,t)^{-1}(\widetilde{D})=\varnothing$ and $K\smallsetminus\widetilde{D}$ contains only points with nonempty preimage. Now $K\smallsetminus\widetilde{D}$ is closed in K, hence again compact, so there is no loss of generality in assuming that every point $(\tilde{x}_i,\tilde{x}_j)$ in K has nonempty preimage. Hence for every such point we can define the corresponding open neighborhood $\widetilde{W}_i\times\widetilde{W}_j$ (depending on $(\tilde{x}_i,\tilde{x}_j)$) in $U\times U$. Using the fact that K is compact, we can extract from this open covering a finite covering:

$$\{\widetilde{W}_i \times \widetilde{W}_j\}_{i\in I, j\in J}$$

for finite sets of indexes I and J.Hence:

$$(s,t)^{-1}(K) \subseteq \coprod_{i\in I, j\in J} (s,t)^{-1}(\widetilde{W}_i \times \widetilde{W}_j) \subseteq \coprod_{i\in I, j\in J} \coprod_{g_j\in G_j} q(\widetilde{B}_k^{\lambda_{ki}, g_j\circ\lambda_{kj}\circ g_k})$$

so $(s,t)^{-1}(K)$ is contained in a finite union of sets with compact closure, so it is contained in a compact set of R. Moreover, we have already proved that both s and t are holomorphic, hence continuous, so also (s,t) is continuous; since K is closed (because it is compact), we get that also $(s,t)^{-1}(K)$ is closed. Since it is contained in a compact set and it is also closed we get that $(s,t)^{-1}(K)$ is compact. □

Proposition 4.1.11. *$R \overset{s}{\underset{t}{\rightrightarrows}} U$ is an object of (**Grp**).*

Proof. Since we have already proved that $R \overset{s}{\underset{t}{\rightrightarrows}} U$ is a groupoid object in (**Sets**), we have only to prove the additional properties about the five structure maps. In particular, we recall that by definition of (**Grp**) we require that:

(i) the five structure maps are holomorphic, i.e. morphisms in (**Manifolds**);

(ii) the map (s,t) is proper;

(iii) both s and t are étale.

In the previous lemmas we have already proved the last two requests; moreover since s and t are étale, in particular they are holomorphic. Hence it remains only to check (i) for the maps m, i, e.

- In order to prove that $m : R\,{}_t\times_s R \to U$ is holomorphic, it suffices to use the explicit description of the charts on the fiber product given in proposition 3.5.2 (where $m = n$, so the first terms don't appear), to restrict enough the domains of the charts and to observe that a commutative diagram of the form (4.16) holds not only on the pair of points fixed on that construction, but also in an open neighborhood of them in both the first and the second variable (before passing to the quotient via q).

- Let us prove now that $i : R \to R$ is holomorphic; for every point $[\lambda_{ki}, \tilde{x}_k, \lambda_{kj}] \in R$ let us choose the chart $(\widetilde{W}_k^{ij}, \phi_k^{ij})$ around it and the chart $(\widetilde{W}_k^{ji}, \phi_k^{ji})$ around its image $[\lambda_{kj}, \tilde{x}_k, \lambda_{ki}]$ via the map i. Then in coordinates this map is just

$$\phi_k^{ji} \circ i \circ (\phi_k^{ij})^{-1} = id$$

  which is clearly holomorphic.

- Using an analogous argument, one can also prove that $e : U \to R$ is holomorphic.

□

4.2 Morphisms

Now let us pass to morphisms: our aim is to associate to every compatible system (i.e. a morphism in (**Pre-Orb**)) a morphism in (**Grp**).

Definition 4.3. Let $\mathcal{U}$ and $\mathcal{V}$ be orbifold atlases for X and Y respectively, let $f : X \to Y$ be a continuous map between the underlying topological spaces and let $\tilde{f} : \mathcal{U} \to \mathcal{V}$ be a compatible system for f.

Let $R \overset{s}{\underset{t}{\rightrightarrows}} U$ be the groupoid object associated to $\mathcal{U}$ and let $R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'$ be obtained in the same way from the atlas $\mathcal{V}$. We recall that U is the topological space obtained as the disjoint union of the open sets $\widetilde{U}_i$ of the atlas $\mathcal{U}$ and in the same way, $U' = \coprod_{(\widetilde{V}_j, H_j, \phi_j) \in \mathcal{V}} \widetilde{V}_j$.

Moreover, having a compatible system $\tilde{f} : \mathcal{U} \to \mathcal{V}$ gives us local holomorphic liftings $\tilde{f}_{\widetilde{U}_i, \widetilde{V}_i}$ if $\tilde{f}(\widetilde{U}_i, G_i, \pi_i) = (\widetilde{V}_i, G_i, \phi_i)$. From now on, for every point $\tilde{x}_i \in \widetilde{U}_i$ we will always denote with $\tilde{y}_i$ its image in $\widetilde{V}_i$ via the function $\tilde{f}_{\widetilde{U}_i, \widetilde{V}_i} : \widetilde{U}_i \to \widetilde{V}_i$.

Now *we define $\psi : U \to U'$ to be the set map such that*:

$$\psi|_{\widetilde{U}_i} = \tilde{f}_{\widetilde{U}_i, \widetilde{V}_i} : \widetilde{U}_i \to \widetilde{V}_i \subseteq U'.$$

In other words, for every point $(\tilde{x}_i, \widetilde{U}_i) \in U$ we have $\psi(\tilde{x}_i, \widetilde{U}_i) := (\tilde{y}_i, \widetilde{V}_i)$. The map ψ is clearly an holomorphic map between complex manifolds because it is so locally, since the liftings of f are all holomorphic by definition of compatible system.

Let us define also a set map $\Psi : R \to R'$ as follows: let us fix any point $[\lambda_{ki}, \tilde{x}_k, \lambda_{kj}] \in R$ and a representative $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj})$ of it; then we set:

$$\Psi([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]) := [\tilde{f}(\lambda_{ki}), \tilde{f}_{\widetilde{U}_k, \widetilde{V}_k}(\tilde{x}_k), \tilde{f}(\lambda_{kj})] = [\tilde{f}(\lambda_{ki}), \tilde{y}_k, \tilde{f}(\lambda_{kj})].$$

Let us verify that such a map is well defined, i.e. it does not depend on the representative chosen. Let $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj})$ be equivalent to $(\lambda_{li}, \tilde{x}_l, \lambda_{lj})$ as in (4.13). Then, using the fact that $\tilde{f}$ is a functor by definition 2.8 of compatible system, we get that the following diagram in $\mathcal{V}$ is commutative:

$$\begin{tikzcd}
& \widetilde{V}_l \arrow[ld, "\tilde{f}(\lambda_{li})"'] \arrow[d, "\tilde{f}(\lambda_{lk})"] \arrow[rd, "\tilde{f}(\lambda_{lj})"] & \\
\widetilde{V}_i \arrow[r, "\tilde{f}(\lambda_{ki})"'] \arrow[r, phantom, "\curvearrowright", shift left=4ex] & \widetilde{V}_k \arrow[r, "\tilde{f}(\lambda_{kj})"'] \arrow[r, phantom, "\curvearrowright", shift left=4ex] & \widetilde{V}_j
\end{tikzcd}
\qquad
\begin{tikzcd}
& \tilde{y}_l \arrow[ld, "\tilde{f}(\lambda_{li})"'] \arrow[d, "\tilde{f}(\lambda_{lk})"] \arrow[rd, "\tilde{f}(\lambda_{lj})"] & \\
\tilde{y}_i \arrow[r, "\tilde{f}(\lambda_{ki})"'] \arrow[r, phantom, "\curvearrowright", shift left=4ex] & \tilde{y}_k \arrow[r, "\tilde{f}(\lambda_{kj})"'] \arrow[r, phantom, "\curvearrowright", shift left=4ex] & \tilde{y}_j .
\end{tikzcd}$$

Hence $(\tilde{f}(\lambda_{ki}), \tilde{y}_k, \tilde{f}(\lambda_{kj}))$ is equivalent to $(\tilde{f}(\lambda_{li}), \tilde{y}_l, \tilde{f}(\lambda_{lj}))$ in the sense of (4.13). So using remark 4.4 we have proved that the map Ψ is compatible with the relations that generate the equivalence classes of R and R', i.e. it is a well defined map.

Until now we have defined a pair of set maps $U \xrightarrow{\psi} U'$ and $R \xrightarrow{\Psi} R'$. Our aim is to prove that:

Proposition 4.2.1. *These two maps give rise to a morphism* (ψ, Ψ) *of groupoid objects in (**Sets**) from* $R \overset{s}{\underset{t}{\rightrightarrows}} U$ *to* $R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'$.

Proof. We have to verify that all the axioms of definition 3.2 are satisfied. Since both U and R are manifolds, in order to check these properties it suffices to work set theoretically.

- Let us take any point $[\lambda_{ki}, \tilde{x}_k, \lambda_{kj}] \in R$ and let $[\tilde{f}(\lambda_{ki}), \tilde{y}_k, \tilde{f}(\lambda_{kj})]$ be its image under Ψ. Using (2.17) we get:

  $$\tilde{f}_{\widetilde{U}_i, \widetilde{V}_i} \circ \lambda_{ki}(\tilde{x}_k) = \tilde{f}(\lambda_{ki}) \circ \tilde{f}_{\widetilde{U}_k, \widetilde{V}_k}(\tilde{x}_k),$$

  so:

  $$s' \circ \Psi([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]) = s'([\tilde{f}(\lambda_{ki}), \tilde{y}_k, \tilde{f}(\lambda_{kj})]) = (\tilde{f}(\lambda_{ki})(\tilde{y}_k), \widetilde{V}_i) =$$
  $$= (\tilde{f}_{\widetilde{U}_i, \widetilde{V}_i} \circ \lambda_{ki}(\tilde{x}_k), \widetilde{V}_i) = \psi(\lambda_{ki}(\tilde{x}_k), \widetilde{U}_i) = \psi \circ s([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]).$$

  Hence we have proved that $s' \circ \Psi = \psi \circ s$. In a similar way we get that $t' \circ \Psi = \psi \circ t$.

- Moreover, for every $(\tilde{x}_i, \widetilde{U}_i) \in U$ we get:

  $$e' \circ \psi(\tilde{x}_i, \widetilde{U}_i) = e'(\tilde{f}_{\widetilde{U}_i, \widetilde{V}_i}(\tilde{x}_i), \widetilde{V}_i) = [1_{\widetilde{V}_i}, \tilde{f}_{\widetilde{U}_i, \widetilde{V}_i}(\tilde{x}_i), 1_{\widetilde{V}_i}] \overset{*}{=}$$
  $$\overset{*}{=} [\tilde{f}(1_{\widetilde{U}_i}), \tilde{f}_{\widetilde{U}_i, \widetilde{V}_i}(\tilde{x}_i), \tilde{f}(1_{\widetilde{U}_i})] = \Psi([1_{\widetilde{U}_i}, \tilde{x}_i, 1_{\widetilde{U}_i}]) = \Psi \circ e(\tilde{x}_i, \widetilde{U}_i),$$

  i.e. $e' \circ \psi = \Psi \circ e$.

Remark 4.5. *Note that in* $\overset{*}{=}$ *we have used the fact that* $\tilde{f}$ *preserves identities. In all the definitions of compatible system I found in literature there is no mention of this request, i.e. everybody only requires that* $\tilde{f}$ *is "functorial" in the sense that it preserves compositions. In order to make this construction work, I had necessarily to add this extra condition in chapter 1 on the definition of compatible systems.*

- Now we want to verify that the pair (ψ, Ψ) is compatible also with multiplication. Consider any point $([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], [\lambda_{kj}, \tilde{x}_k, \lambda_{kl}]) \in R\,_t \times_s R$. We recall that in order to compute m on this point we have considered a diagram of this form:

$$\begin{tikzcd}
 & \tilde{x}_i & \lambda_{ij}(\tilde{x}_i) = \lambda_{kj}(\tilde{x}_k) & \tilde{x}_k & \\
 & \Cap & \Cap & \Cap & \\
\widetilde{U}_h & \widetilde{U}_i \arrow[l, "\lambda_{ih}"'] \arrow[r, "\lambda_{ij}"] & \widetilde{U}_j & \widetilde{U}_k \arrow[l, "\lambda_{kj}"'] \arrow[r, "\lambda_{kl}"] & \widetilde{U}_l. \\
 & & \curvearrowright & & \\
 & & \widetilde{U}_f \arrow[uul, "\lambda_{fi}"] \arrow[uur, "\lambda_{fk}"'] & & \\
 & & \Cup & & \\
 & & \tilde{x}_f & &
\end{tikzcd}$$

If we apply to it the functor $\tilde{f}$, we get a diagram in the atlas $\mathcal{V}$ as follows:

$$\begin{tikzcd}
 & \tilde{y}_i & \tilde{f}(\lambda_{ij})(\tilde{y}_i) = \tilde{f}(\lambda_{kj})(\tilde{y}_k) & \tilde{y}_k & \\
 & \Cap & \Cap & \Cap & \\
\widetilde{V}_h & \widetilde{V}_i \arrow[l, "\tilde{f}(\lambda_{ih})"'] \arrow[r, "\tilde{f}(\lambda_{ij})"] & \widetilde{V}_j & \widetilde{V}_k \arrow[l, "\tilde{f}(\lambda_{kj})"'] \arrow[r, "\tilde{f}(\lambda_{kl})"] & \widetilde{V}_l. \\
 & & \curvearrowright & & \\
 & & \widetilde{V}_f \arrow[uul, "\tilde{f}(\lambda_{fi})"] \arrow[uur, "\tilde{f}(\lambda_{fk})"'] & & \\
 & & \Cup & & \\
 & & \tilde{y}_f & &
\end{tikzcd}$$

Note that the fact that we have obtained exactly the same relations between the marked points follows from the properties of the compatible system $\tilde{f}$. Hence, using lemma 4.1.5, we can use the last two diagrams in order to compute m and m':

$$
\begin{gathered}
m' \circ (\Psi \times \Psi)([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], [\lambda_{kj}, \tilde{x}_k, \lambda_{kl}]) = \\
= m'([\tilde{f}(\lambda_{ih}), \tilde{y}_i, \tilde{f}(\lambda_{ij})], [\tilde{f}(\lambda_{kj}), \tilde{y}_k, \tilde{f}(\lambda_{kl})]) = \\
= [\tilde{f}(\lambda_{ih}) \circ \tilde{f}(\lambda_{fi}), \tilde{y}_f, \tilde{f}(\lambda_{kl}) \circ \tilde{f}(\lambda_{fk})] = [\tilde{f}(\lambda_{ih} \circ \lambda_{fi}), \tilde{f}_{\tilde{U}_f, \tilde{V}_f}(\tilde{x}_f), \tilde{f}(\lambda_{kl} \circ \lambda_{fk})] = \\
= \Psi([\lambda_{ih} \circ \lambda_{fi}, \tilde{x}_f, \lambda_{kl} \circ \lambda_{fk}]) = \Psi \circ m([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], [\lambda_{kj}, \tilde{x}_k, \lambda_{kl}]).
\end{gathered}
$$

This holds for every point of $R_t \times_s R$, hence we have proved that $m' \circ (\Psi \times \Psi) = \Psi \circ m$.

- The last property of definition 3.2 is easy to prove working set theoretically.

Hence we have proved that $(\psi, \Psi) : (R \overset{s}{\underset{t}{\rightrightarrows}} U) \longrightarrow (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U')$ is a morphisms between groupoid objects in (**Sets**). ☐

Proposition 4.2.2. *The pair (ψ, Ψ) is a morphism of groupoid objects also in (**Grp**).*

Proof. In the previous section we have already proved that its source and target are both groupoid objects in (**Grp**), so we have only to verify that (ψ, Ψ) is a morphism in (**Grp**). Using the definition of this 2-category and the previous proposition, this just means that we have to prove that both ψ and Ψ are morphisms in the category (**Manifolds**), i.e. that they are holomorphic functions.

Moreover, in definition 4.3 we have already proved that ψ is holomorphic, so it remains only to prove that also Ψ is so; in order to do this, it suffices to prove that Ψ locally coincides with a holomorphic function. We recall that in proposition 4.1.7 we described a manifold atlas for R where the charts are of the form $(\widetilde{W}_k^{ij}, \phi_k^{ij})$; analogously, we can use similar charts of the form $(\widetilde{Z}_k^{ij}, \xi_k^{ij})$ on R', where the domain is just equal to $q'(\widetilde{V}_k^{ij})$. Then if we write Ψ in coordinates with respect to the charts $(\widetilde{W}_k^{ij}, \phi_k^{ij})$ and $(\widetilde{Z}_k^{ij}, \xi_k^{ij})$, we get that Ψ coincides with the holomorphic function $\tilde{f}_{\widetilde{U}_k, \widetilde{V}_k}$. So locally Ψ coincides with some holomorphic function, so it is holomorphic on the whole R.

Having proved the previous proposition and the fact that both ψ and Ψ are holomorphic says that (ψ, Ψ) is a morphism between groupoid objects in the category (**Manifolds**). Moreover, we have that (ψ, Ψ) is also a morfism in (**Grp**) by definition of this 2-category, since we get (**Grp**) from the category of groupoids objects over (**Manifolds**) just by restricting the class of objects, and without adding any extra condition on morphisms or 2-morphisms. $\square$

4.3 2-morphisms

Now let us fix two atlases $\mathcal{U}$ and $\mathcal{V}$ for X and Y respectively, a continuous function $f : X \to Y$, two compatible systems $\tilde{f}_1, \tilde{f}_2 : \mathcal{U} \to \mathcal{V}$ for f and a natural transformation $\delta : \tilde{f}_1 \Rightarrow \tilde{f}_2$ as in definition 2.10.

Let us call $R \overset{s}{\underset{t}{\rightrightarrows}} U$ and $R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'$ the groupoid objects associated to the atlases $\mathcal{U}$ and $\mathcal{V}$ respectively; moreover, let us denote with (ψ, Ψ) and (ϕ, Φ) the morphisms of groupoid objects $(R \overset{s}{\underset{t}{\rightrightarrows}} U) \to (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U')$ associated to $\tilde{f}_1$ and $\tilde{f}_2$ respectively by definition 4.3

Our aim now is to associate to δ a natural transformation α in the 2-category (**Grp**) from (ψ, Ψ) to (ϕ, Φ). Hence we have to define a morphism $\alpha : U \to R'$ which satisfies definition 3.4, and this can be done using the following proposition:

Definition-Proposition 4.4. The *set map*:

$$\alpha : U = \coprod_{(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}} \widetilde{U}_i \to R'$$

$$\alpha(\tilde{x}_i, \widetilde{U}_i) := [1_{\widetilde{V}_i^1}, (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1}(\tilde{x}_i), \delta_{\widetilde{U}_i}]$$

is a natural transformation from (ψ, Ψ) to (ϕ, Φ) in (**Grp**).

Proof. First of all, *we claim that α is holomorphic*: indeed for every point $(\tilde{x}_i, \widetilde{U}_i) \in U$ let us choose $\widetilde{U}_i \subseteq U$ as open neighborhood of it and let us restrict α to this set. In this case α has target in the open set:

$$A := q' \left((\widetilde{V}_i^1)^{1_{\widetilde{V}_i^1}, \delta_{\widetilde{U}_i}} \right).$$

As in proposition 4.1.7, this set is biholomorphic to $\widetilde{V}_i^1$. By composing with these biholomorphism, we get that α (restricted to $\widetilde{U}_i \subseteq U$) coincides with the holomorphic map $(\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1}$. So we have found charts in domain and codomain where α is holomorphic around the fixed point $(\tilde{x}_i, \widetilde{U}_i)$; since this holds for every point of U, we have proved that this set map is holomorphic on the whole U. In other words we have proved that α is a morphism in (**Manifolds**).

So in order to prove that α is a natural transformation in (**Grp**) it suffices to verify the axioms of definition 3.4; since R, U, R', U' are all manifolds, it suffices to work set theoretically.

(i) Given any point $(\tilde{x}_i, \widetilde{U}_i) \in U$, we get that:

$$s' \circ \alpha(\tilde{x}_i, \widetilde{U}_i) = s' \Big([1_{\widetilde{V}_i^1}, (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1}(\tilde{x}_i), \delta_{\widetilde{U}_i}] \Big) = \Big((\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1}(\tilde{x}_i), \widetilde{V}_i^1 \Big) = \psi(\tilde{x}_i, \widetilde{U}_i)$$

(ii) and:

$$t' \circ \alpha(\tilde{x}_i, \widetilde{U}_i) = \Big(\delta_{\widetilde{U}_i} \circ (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1}(\tilde{x}_i), \widetilde{V}_i^2 \Big) = \Big((\tilde{f}_2)_{\widetilde{U}_i, \widetilde{V}_i^2}(\tilde{x}_i), \widetilde{V}_i^2 \Big) = \phi(\tilde{x}_i, \widetilde{U}_i).$$

(iii) Let us fix any $[\lambda_{ki}, \tilde{x}_k, \lambda_{kj}] \in R$. For simplicity, let us call $\tilde{x}_i := \lambda_{ki}(\tilde{x}_k) \in \widetilde{U}_i$ and $\tilde{x}_j := \lambda_{kj}(\tilde{x}_k) \in \widetilde{U}_j$. Moreover, as in chapter 2, in this part we adopt the following notation: for every embedding λ_{hl} in $\mathcal{U}$ we set:

$$\lambda^m_{hl} := \tilde{f}_m(\lambda_{hl}) \quad \text{for } m = 1, 2;$$

so λ^m_{hl} will be an embedding from $\widetilde{V}^m_h$ to $\widetilde{V}^m_l$ in the atlas $\mathcal{V}$. Moreover, for every point $\tilde{x}_h \in \widetilde{U}_h$ we define:

$$\tilde{y}^m_h := (\tilde{f}_m)_{\widetilde{U}_h, \widetilde{V}^m_h}(\tilde{x}_h) \in \widetilde{V}^m_h \quad \text{for } m = 1, 2.$$

So we get the following facts:

$$\alpha \circ s([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]) = \alpha(\tilde{x}_i, \widetilde{U}_i) = [1_{\widetilde{V}^1_i}, \tilde{y}^1_i, \delta_{\widetilde{U}_i}]$$
$$\text{and} \quad \Phi([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]) = [\lambda^2_{ki}, \tilde{y}^2_k, \lambda^2_{kj}].$$

Now using these identities we want to compute:

$$m' \circ (\alpha \circ s, \Phi)([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]) = m'\Big([1_{\widetilde{V}^1_i}, \tilde{y}^1_i, \delta_{\widetilde{U}_i}], [\lambda^2_{ki}, \tilde{y}^2_k, \lambda^2_{kj}]\Big).$$

In this case we can use this diagram:

$$\begin{tikzcd}
 & \tilde{y}^1_i \arrow[d, phantom, sloped, "\in"] & \delta_{\widetilde{U}_i}(\tilde{y}^1_i) = \tilde{y}^2_i = \lambda^2_{ki}(\tilde{y}_k) \arrow[d, phantom, sloped, "\in"] & \tilde{y}^2_k \arrow[d, phantom, sloped, "\in"] & \\
\widetilde{V}^1_i & \widetilde{V}^1_i \arrow[l, "1_{\widetilde{V}^1_i}"'] \arrow[r, "\delta_{\widetilde{U}_i}"] & \widetilde{V}^2_i & \widetilde{V}^2_k \arrow[l, "\lambda^2_{ki}"'] \arrow[r, "\lambda^2_{kj}"] & \widetilde{V}^2_j \\
 & & \curvearrowright & & \\
 & & \widetilde{V}^1_k \arrow[uul, "\lambda^1_{ki}"] \arrow[uur, "\delta_{\widetilde{U}_k}"'] & & \\
 & & \tilde{y}^1_k \arrow[u, phantom, sloped, "\in"] & &
\end{tikzcd}$$

where the central part of the diagram is commutative using (2.18). Hence we get that:

$$m' \circ (\alpha \circ s, \Phi)([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]) = [\lambda^1_{ki}, \tilde{y}^1_k, \lambda^2_{kj} \circ \delta_{\widetilde{U}_k}]. \tag{4.23}$$

On the other hand, we get that:

$$\alpha \circ t([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]) = \alpha(\tilde{x}_j, \widetilde{U}_j) = [1_{\widetilde{V}^1_j}, \tilde{y}^1_j, \delta_{\widetilde{U}_j}]$$
$$\text{and} \quad \Psi([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]) = [\lambda^1_{ki}, \tilde{y}^1_k, \lambda^1_{kj}].$$

Hence in order to compute:

$$m' \circ (\Psi, \alpha \circ t)([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]) = m'\Big([\lambda^1_{ki}, \tilde{y}^1_k, \lambda^1_{kj}], [1_{\widetilde{V}^1_j}, \tilde{y}^1_j, \delta_{\widetilde{U}_j}]\Big)$$

we can use this diagram:

$$\begin{tikzcd}
 & \tilde{y}^1_k \arrow[d, phantom, sloped, "\in"] & \tilde{y}^1_j \arrow[d, phantom, sloped, "\in"] & \tilde{y}^1_j \arrow[d, phantom, sloped, "\in"] & \\
\widetilde{V}^1_i & \widetilde{V}^1_k \arrow[l, "\lambda^1_{ki}"'] \arrow[r, "\lambda^1_{kj}"] & \widetilde{V}^1_j & \widetilde{V}^1_j \arrow[l, "1_{\widetilde{V}^1_j}"'] \arrow[r, "\delta_{\widetilde{U}_j}"] & \widetilde{V}^2_j. \\
 & & \curvearrowright & & \\
 & & \widetilde{V}^1_k \arrow[uul, "1_{\widetilde{V}^1_k}"] \arrow[uur, "\lambda^1_{kj}"'] \arrow[d, phantom, sloped, "\ni"] & & \\
 & & \tilde{y}^1_k & &
\end{tikzcd}$$

Hence we get:

$$m' \circ (\Psi, \alpha \circ t)([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]) = [\lambda^1_{ki}, \tilde{y}^1_k, \delta_{\widetilde{U}_j} \circ \lambda^1_{kj}]. \tag{4.24}$$

Using again (2.18) we get that $\delta_{\widetilde{U}_j} \circ \lambda^1_{kj} = \lambda^2_{kj} \circ \delta_{\widetilde{U}_k}$, hence (4.23) and (4.24) are equal. Since this holds for every point $[\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]$ of R, we get that the third axiom of definition 3.4 is satisfied.

So we have proved that $\alpha : (\psi, \Psi) \Rightarrow (\phi, \Phi)$ is an holomorphic map and it satisfies the axioms for a natural transformation between morphisms of groupoid objects over (**Manifolds**); so it is a natural transformation in (**Grp**) by definition of this 2-category. □

4.4 The 2-functor F

Until now we have described:

(a) how to associate to every orbifold atlas $\mathcal{U}$ a groupoid object $R \overset{s}{\underset{t}{\rightrightarrows}} U$, which is an object in (**Grp**);

(b) how to associate to every compatible system $\tilde{f}$ a morphism (ψ, Ψ) of groupoid objects, which is in particular a morphism in (**Grp**);

(c) how to associate to every natural transformation δ between compatible systems a natural tranformation α in (**Grp**).

Proposition 4.4.1. *Whenever we fix a pair of objects $\mathcal{U}, \mathcal{V}$ in (**Pre-Orb**) with associated groupoid objects $R \overset{s}{\underset{t}{\rightrightarrows}} R$ and $R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'$ respectively, we get a functor:*

$$F = F_{\mathcal{U},\mathcal{V}} : (\textbf{\textit{Pre-Orb}})(\mathcal{U}, \mathcal{V}) \to (\textbf{\textit{Grp}})\Big((R \overset{s}{\underset{t}{\rightrightarrows}} U), (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U')\Big)$$

defined by (b) on the level of objects and by (c) on the level of morphisms.

Proof. We recall that in chapter 2 and 3 we have already proved that (**Pre-Orb**) and (**Grp**) are both 2-categories, hence the source and the target of $F_{\mathcal{U},\mathcal{V}}$ are both categories by definition 1.9. So we have only to verify that F preserves compositions and identities.

First of all, let us take any pair of composable morphisms (with respect to $\odot$) in the first category, i.e. two natural transformations $\delta : \tilde{f}_1 \Rightarrow \tilde{f}_2$

and $\sigma : \tilde{f}_2 \Rightarrow \tilde{f}_3$. Let us call (ψ, Ψ), (ϕ, Φ) and (θ, Θ) the morphisms of groupoid objects associated to $\tilde{f}_i$ for $i = 1, 2, 3$ and $\alpha : (\psi, \Psi) \Rightarrow (\phi, \Phi)$, $\beta : (\phi, \Phi) \Rightarrow (\theta, \Theta)$ the natural transformations in (**Grp**) associated to δ and σ respectively using the previous constructions.

Moreover, let us call $\mu := \sigma \odot \delta : \tilde{f}_1 \Rightarrow \tilde{f}_3$ and let us denote with γ the natural transformation associated to it in (**Grp**). Then for every point $(\tilde{x}_i, \widetilde{U}_i) \in U$ we have:

$$F(\sigma) \odot F(\delta)(\tilde{x}_i, \widetilde{U}_i) = \beta \odot \alpha(\tilde{x}_i, \widetilde{U}_i) = m' \circ (\alpha, \beta)(\tilde{x}_i, \widetilde{U}_i) =$$
$$= m'([1_{\widetilde{V}_i^1}, (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1}(\tilde{x}_i), \delta_{\widetilde{U}_i}], [1_{\widetilde{V}_i^2}, (\tilde{f}_2)_{\widetilde{U}_i, \widetilde{V}_i^2}(\tilde{x}_i), \sigma_{\widetilde{U}_i}]) \overset{*}{=}$$
$$\overset{*}{=} [1_{\widetilde{V}_i^1}, (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1}(\tilde{x}_i), \sigma_{\widetilde{U}_i} \circ \delta_{\widetilde{U}_i}] = \gamma(\tilde{x}_i, \widetilde{U}_i) = F(\sigma \odot \delta)(\tilde{x}_i, \widetilde{U}_i);$$

note that in the passage denoted with $\overset{*}{=}$ we used the commutative diagram:

$$\begin{tikzcd}
& (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1}(\tilde{x}_i) \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & (\tilde{f}_2)_{\widetilde{U}_i, \widetilde{V}_i^2}(\tilde{x}_i) \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & (\tilde{f}_2)_{\widetilde{U}_i, \widetilde{V}_i^2}(\tilde{x}_i) \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & \\
\widetilde{V}_i^1 & \widetilde{V}_i^1 \arrow[l, "1_{\widetilde{V}_i^1}"'] \arrow[r, "\delta_{\widetilde{U}_i}"] & \widetilde{V}_i^2 & \widetilde{V}_i^2 \arrow[l, "1_{\widetilde{V}_i^2}"'] \arrow[r, "\sigma_{\widetilde{U}_i}"] & \widetilde{V}_i^3. \\
& & \curvearrowright & & \\
& & \widetilde{V}_i^1 \arrow[uul, "1_{\widetilde{V}_i^1}"] \arrow[uur, "\delta_{\widetilde{U}_i}"'] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & \\
& & (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1}(\tilde{x}_i) & &
\end{tikzcd}$$

Hence we have proved that for every pair (δ, σ) of composable morphisms in (**Pre-Orb**)$(\mathcal{U}, \mathcal{V})$, we have that $F(\sigma) \odot F(\delta) = F(\sigma \odot \delta)$, i.e. F preserves compositions.

Moreover, we recall that in §2.4 for every compatible system $\tilde{f}$ from $\mathcal{U}$ to $\mathcal{V}$ we have defined $i_{\tilde{f}}$ to be a natural transformation from $\tilde{f}$ to itself such that for every uniformizing system $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we have $(i_{\tilde{f}})_{\widetilde{U}_i} = 1_{\widetilde{V}_i}$. Let us call (ψ,Ψ) the morphisms associated to $\tilde{f}$ and α the natural transformation in (**Grp**) associated to $i_{\tilde{f}}$; then for every point $(\tilde{x}_i,\widetilde{U}_i)\in U$ we have:

$$\alpha(\tilde{x}_i,\widetilde{U}_i)=[1_{\widetilde{V}_i},\tilde{f}_{\widetilde{U}_i,\widetilde{V}_i}(\tilde{x}_i),1_{\widetilde{V}_i}]=e'(\psi(\tilde{x}_i,\widetilde{U}_i)).$$

Hence $\alpha=e'\circ\psi=\Psi\circ e=i_{(\psi,\Psi)}$; so F preserves also the identities of (**Pre-Orb**)$(\mathcal{U},\mathcal{V})$. □

Theorem 4.4.2. *The previous data define a 2-functor F from (**Pre-Orb**) to (**Grp**).*

Proof. It suffices to verify axioms (a),(b) and (c) of remark 1.5.

(a) Let us fix any pair of compatible systems $\tilde{f}:\mathcal{U}\to\mathcal{V}$ and $\tilde{g}:\mathcal{V}\to\mathcal{W}$. Then for simplicity, let us call:

$$(R\overset{s}{\underset{t}{\rightrightarrows}}U):=F(\mathcal{U}),\quad (R'\overset{s'}{\underset{t'}{\rightrightarrows}}U'):=F(\mathcal{V}),\quad (R''\overset{s''}{\underset{t''}{\rightrightarrows}}U''):=F(\mathcal{W})$$

$$(\psi,\Psi):=F(\tilde{f}),\quad (\phi,\Phi):=F(\tilde{g}),\quad \tilde{h}:=\tilde{g}\circ\tilde{f}\quad\text{and}\quad(\theta,\Theta):=F(\tilde{h}).$$

So we want to prove that:

$$\theta=\phi\circ\psi\quad\text{and}\quad\Theta=\Phi\circ\Psi. \tag{4.25}$$

Note that using remark 3.2, it suffices to prove only the second equality. Indeed, if this is proved, we get that:

$$\theta=s''\circ\Theta\circ e=(s''\circ\Phi)\circ(\Psi\circ e)=$$
$$=\phi\circ s'\circ e'\circ\psi=\phi\circ 1_{U'}\circ\psi=\phi\circ\psi.$$

Now in order to prove the second equality of (4.25) it suffices to work set theoretically; so let us fix any point $[\lambda_{ki},\tilde{x}_k,\lambda_{kj}]\in R$. Then we have:

$$\Phi\circ\Psi([\lambda_{ki},\tilde{x}_k,\lambda_{kj}])=\Phi([\tilde{f}(\lambda_{ki}),\tilde{f}_{\widetilde{U}_k,\widetilde{V}_k}(\tilde{x}_k),\tilde{f}(\lambda_{kj})])=$$

$$= [\tilde{g} \circ \tilde{f}(\lambda_{ki}), \tilde{g}_{\widetilde{V}_k, \widetilde{W}_k} \circ \tilde{f}_{\widetilde{U}_k, \widetilde{V}_k}(\tilde{x}_k), \tilde{g} \circ \tilde{f}(\lambda_{kj})] = [\tilde{h}(\lambda_{ki}), \tilde{h}_{\widetilde{U}_k, \widetilde{W}_k}(\tilde{x}_k), \tilde{h}(\lambda_{kj})] =$$

$$= \Theta([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]);$$

so (4.25) is proved.

(b) Let us fix a diagram of compatible systems and natural transformations in (**Pre-Orb**) of the form:

$$\begin{tikzcd}
\mathcal{U} \arrow[r, bend left, "\tilde{f}_1"] \arrow[r, bend right, "\tilde{f}_2"'] \arrow[r, phantom, "\Downarrow \delta"] & \mathcal{V} \arrow[r, bend left, "\tilde{g}_1"] \arrow[r, bend right, "\tilde{g}_2"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathcal{W}.
\end{tikzcd}$$

For simplicity, let us use the notations of (a) on the level of objects and let us call:

$$F(\tilde{f}_i) =: (\psi_i, \Psi_i), \quad F(\tilde{g}_i) =: (\phi_i, \Phi_i) \quad \text{for}\, i = 1, 2,$$

$$F(\delta) =: \alpha : U \to R' \quad F(\eta) := \beta : U' \to R'' \quad \text{and} \quad F(\eta * \delta) := \gamma : U \to R''.$$

By definition of $*$ in (**Pre-Orb**), for every uniformizing system $(\widetilde{U}_i, G_i, \pi_i)$ in $\mathcal{U}$ we have:

$$(\eta * \delta)_{\widetilde{U}_i} := \eta_{\widetilde{V}_i^2} \circ \tilde{g}_1(\delta_{\widetilde{U}_i}) : \widetilde{W}_i^{11} \to \widetilde{W}_i^{22}$$

where we use the notations of chapter 2, section 3. So for every point $(\tilde{x}_i, \widetilde{U}_i) \in U$ we have:

$$\gamma(\tilde{x}_i, \widetilde{U}_i) = [1_{\widetilde{W}_i^{11}}, (\tilde{g}_1 \circ \tilde{f}_1)_{\widetilde{U}_i, \widetilde{W}_i^{11}}(\tilde{x}_i), \eta_{\widetilde{V}_i^2} \circ \tilde{g}_1(\delta_{\widetilde{U}_i})]. \tag{4.26}$$

On the other hand,

$$\begin{aligned}
\Big(F(\eta) * F(\delta)\Big)(\tilde{x}_i, \widetilde{U}_i) = \big(\beta * \alpha\big)(\tilde{x}_i, \widetilde{U}_i) = m''\Big(\Phi_1 \circ \alpha(\tilde{x}_i, \widetilde{U}_i), \beta \circ \psi_2(\tilde{x}_i, \widetilde{U}_i)\Big) = \\
= m''\Big(\Phi_1 \left([1_{\widetilde{V}^1_i,}, (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}^1_i}(\tilde{x}_i), \delta_{\widetilde{U}_i}]\right), \beta\left((\tilde{f}_2)_{\widetilde{U}_i, \widetilde{V}^2_i}(\tilde{x}_i), \widetilde{V}^2_i\right)\Big) = \\
= m''\Big(\Big[1_{\widetilde{W}^{11}_i}, (\tilde{g}_1)_{\widetilde{V}^1_i, \widetilde{W}^{11}_i} \circ (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}^1_i}(\tilde{x}_i), \tilde{g}_1(\delta_{\widetilde{U}_i})\Big], \\
\Big[1_{\widetilde{W}^{21}_i}, (\tilde{g}_1)_{\widetilde{V}^2_i, \widetilde{W}^{21}_i} \circ (\tilde{f}_2)_{\widetilde{U}_i, \widetilde{V}^2_i}(\tilde{x}_i), \eta_{\widetilde{V}^2_i}\Big]\Big) \overset{*}{=} \\
\overset{*}{=} \Big[1_{\widetilde{W}^{11}_i}, (\tilde{g}_1)_{\widetilde{V}^1_i, \widetilde{W}^{11}_i} \circ (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}^1_i}(\tilde{x}_i), \eta_{\widetilde{V}^2_i} \circ \tilde{g}_1(\delta_{\widetilde{U}_i})\Big]
\end{aligned} \tag{4.27}$$

where in $\overset{*}{=}$ we used the following commutative diagram:

$$\begin{tikzcd}
& (\tilde{g}_1)_{\widetilde{V}^1_i, \widetilde{W}^{11}_i} \circ (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}^1_i}(\tilde{x}_i) & \tilde{q} & (\tilde{g}_1)_{\widetilde{V}^2_i, \widetilde{W}^{21}_i} \circ (\tilde{f}_2)_{\widetilde{U}_i, \widetilde{V}^2_i}(\tilde{x}_i) & \\[-15pt]
& \Cap & \Cap & \Cap & \\[-15pt]
\widetilde{W}^{11}_i & \widetilde{W}^{11}_i \arrow[l, "1_{\widetilde{W}^{11}_i}"'] \arrow[r, "\tilde{g}_1\left(\delta_{\widetilde{U}_i}\right)"] & \widetilde{W}^{21}_i & \widetilde{W}^{21}_i \arrow[l, "1_{\widetilde{W}^{21}_i}"'] \arrow[r, "\eta_{\widetilde{V}^2_i}"] & \widetilde{W}^{22}_i \\
& & \widetilde{W}^{11}_i \arrow[ul, "1_{\widetilde{W}^{11}_i}"] \arrow[ur, "\tilde{g}_1\left(\delta_{\widetilde{U}_i}\right)"'] \arrow[u, phantom, "\curvearrowright"] & & \\[-15pt]
& & \Cup & & \\[-15pt]
& & (\tilde{g}_1)_{\widetilde{V}^1_i, \widetilde{W}^{11}_i} \circ (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}^1_i}(\tilde{x}_i) & &
\end{tikzcd}$$

.

with $\tilde{q} := (\tilde{g}_1)_{\widetilde{V}^2_i, \widetilde{W}^{21}_i} \circ (\tilde{f}_2)_{\widetilde{U}_i, \widetilde{V}^2_i}(\tilde{x}_i)$

By comparing (4.26) with (4.27), we get that:

$$F(\eta) * F(\delta) = F(\eta * \delta).$$

(c) Let us fix any orbifold $\mathcal{U}$ with associated groupoid object $R \overset{s}{\underset{t}{\rightrightarrows}} U$ and let us call $(\psi, \Psi) := F(1_{\mathcal{U}})$. Then we have:

$$\psi = 1_U \quad \text{and} \quad \Psi = 1_R$$

hence $F(1_{\mathcal{U}}) = 1_{F(\mathcal{U})}$. Morever, $F(i_{\mathcal{U}}) = e : U \to R$, i.e. $F(i_{\mathcal{U}}) = i_{F(\mathcal{U})}$.

Hence we have completely proved that F is a 2-functor from (**Pre-Orb**) to (**Grp**). $\square$

> “Questo ve l’ho detto tre volte,
> e perciò è vero.”
>
> Lewis Carroll
> “The Hunting of the Snark”

Chapter 5

Unsolved problems

5.1 Morita equivalences

Definition 5.1. ([M],§2.4) A homomorphism :

$$(\psi,\Psi):(R\overset{s}{\underset{t}{\rightrightarrows}}U)\rightarrow(R'\overset{s'}{\underset{t'}{\rightrightarrows}}U')$$

between Lie groupoids is called a *Morita equivalence* if the following 2 conditions hold:

(i) let us consider the fiber product:

$$\begin{tikzcd}
R' \times_{U'} U \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "\psi"] \\
R' \arrow[r, "s'"'] & U'
\end{tikzcd}$$

since $R'\overset{s'}{\underset{t'}{\rightrightarrows}}U'$ is a Lie groupoid, we get that the map s' is a submersion, so we can apply proposition 3.5.2 and we get that the fiber product has a natural structure of manifold and that also π_2 is a submersion. *Then we require that the set map*:

$$t \circ \pi_1 : R' \times_{U'} U \to U'$$

is a surjective submersion. This request makes sense because both source and target of this map are complex manifolds;

(ii) *we require also that the square*:

$$\begin{tikzcd}
R \arrow[r, "\Psi"] \arrow[d, "(s,t)"'] & R' \arrow[d, "(s',t')"] \\
U \times U \arrow[r, "(\psi\times\phi)"'] & U' \times U'
\end{tikzcd} \tag{5.1}$$

is cartesian in (**Manifolds**). Note that the square is always commutative because of the first and the second diagram of definition 3.2.

Definition 5.2. Two groupoid objects $R_i \underset{t_i}{\overset{s_i}{\rightrightarrows}} U_i$ (for $i = 1, 2$) in (**Manifolds**) are said to be *Morita equivalent* (or *weak equivalent*) if there exists a third groupoid object $R_3 \underset{t_3}{\overset{s_3}{\rightrightarrows}} U_3$ and two Morita equivalences:

$$(R_1 \underset{t_1}{\overset{s_1}{\rightrightarrows}} U_1) \overset{(\psi,\Psi)}{\longleftarrow} (R_3 \underset{t_3}{\overset{s_3}{\rightrightarrows}} U_3) \overset{(\phi,\Phi)}{\longrightarrow} (R_2 \underset{t_2}{\overset{s_2}{\rightrightarrows}} U_2).$$

This is actually an equivalence relation, see for example [MM], chapter 5 for the proof.

Remark 5.1. (i) Let us fix any point $u' \in U'$; the first condition of Morita equivalence requires in particular that $t \circ \pi_1$ is surjective; so there exists a (not necessarily unique) point $(r', u) \in R' \times_{U'} U$ such that:

$$t(r') = t \circ \pi_1(r', u) = u'. \tag{5.2}$$

Since $(r', u) \in R' \times_{U'} U$, we have that:

$$s'(r') = \phi(u). \tag{5.3}$$

In other words, for every point $u' \in U'$ there exists a point $u \in U$ and a point $r' \in R'$ such that (5.2) and (5.3) hold. Now we recall that in lemma 3.1.2 we showed that to every groupoid $R \underset{t}{\overset{s}{\rightrightarrows}} U$ we can associate a category $\mathscr{R}$ where the objects are the points of U and the morphisms are the points of R (and so are all isomorphisms in this category). Moreover, in lemma 3.2.1 we described how to associate to every morphism (ψ, Ψ) from a groupoid object to another a functor $\widetilde{\Psi}$ between the corresponding categories, given on the level of objects by ψ and on the level of morphisms by Ψ. Then (5.2) and (5.3) (together with the fact that r' is invertible if considered as a morphism in $\mathscr{R}$) implies that *the funtor* $\tilde{\Psi}$ *is essentially surjective.*

Note that we don't have used at all the fact that $t \circ \pi_1$ is a submersion so this condition is *not* equivalent to (i).

(ii) Let us fix any pair of points $u, v \in U$; as in lemma 3.1.2 let us define the sets:

$$\mathscr{R}(u,v) := \{r \in R \text{ s.t. } s(r) = u, t(r) = v\}$$
$$\text{and} \quad \mathscr{R}'(\psi(u), \psi(v)) := \{r' \in R' \text{ s.t. } s'(r') = \psi(u), t'(r') = \psi(v)\}$$

and let us consider the diagram:

$$\begin{tikzcd}
\mathscr{R}'(\psi(u),\psi(v)) \arrow[dr, dashed, "\gamma"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right, "{(u,v)}"'] & & \\
& R \arrow[r, "\Psi"] \arrow[d, "{(s,t)}"'] \arrow[dr, phantom, "\square"] & R' \arrow[d, "{(s',t')}"] \\
& U \times U \arrow[r, "(\psi\times\psi)"'] & U' \times U'.
\end{tikzcd}$$

(5.4)

where i is just the inclusion map and (u, v) is the constant map that to every point of $R'(\psi(u), \psi(v))$ associates the point $(u, v) \in U \times U$. It is easy to see that the external diagram is commutative; moreover request (ii) says that the internal square is cartesian, so using the **UP** of fiber products in (**Manifolds**) we get that there exists a unique holomorphic map γ such that:

$$i = \Psi \circ \gamma \quad \text{and} \quad (u, v) = (s, t) \circ \gamma. \tag{5.5}$$

Now let us fix any point $r' \in R'$ such that $s'(r') = \psi(u)$ and $t'(r') = \psi(v)$ (i.e. a point of $\mathscr{R}'(\psi(u), \psi(v))$ and let us call $r := \gamma(r')$. Then, using (5.5), we get that $s(r) = u$ and $t(r) = v$, i.e. $r \in \mathscr{R}(u, v)$. So we can consider γ as a set map:

$$\gamma : \mathscr{R}'(\psi(u), \psi(v)) \to \mathscr{R}(u, v).$$

Moreover, using these notations and the first part of (5.5) we get that $r' = \Psi \circ \gamma(r') = \Psi(r)$. Since this holds for every point $r' \in \mathscr{R}'(\psi(u), \psi(v))$ and for every pair of points $u, v \in U$, we get that:

$$\mathscr{R}'(\psi(u), \psi(v)) = \Psi(\mathscr{R}(u, v)) \quad \forall u, v \in U$$

i.e. the functor $\tilde{\Psi}$ defined in (i) is *full* .

Now let us fix any $r \in \mathscr{R}(u, v)$ and let $r' := \Psi(r)$ and $\bar{r} := \gamma(r')$. We want to prove that $r = \bar{r}$; indeed by definition of fiber product in (**Sets**) we get that $\bar{r}$ is the *unique* point in $\mathscr{R}(u, v)$ such that:

$$(s, t)(\bar{r}) = (u, v) \quad \text{and} \quad \Psi(\bar{r}) = r' = \Psi(r).$$

Since also r has this property, by uniqueness we get that $r = \bar{r}$, i.e. $\gamma(\Psi(r)) = r$; since this holds for every point in $\mathscr{R}(u, v)$, we get that the functor $\tilde{\Psi}$ is *faithful*. Using also the previuos part, we get that $\tilde{\Psi}$ is *fully faithful.*

So we have proved the following result:

Proposition 5.1.1. *Whenever the morphism* (ψ, Ψ) *is a Morita equivalence, we get that the functor* $\tilde{\Psi}$ *associated to it by lemma 3.2.1 is essentially surjective and fully faithfull, i.e. it is an* equivalence of categories.

However, the converse may not be true.

Now let us fix any orbifold atlas $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i\in I}$ on a topological space X and let us denote with $\mathcal{U}' = \{(\widetilde{U}_{i'}, G_{i'}, \pi_{i'})\}_{i\in I'}$ the maximal atlas associated to $\mathcal{U}$ as in definition 2.17. Then by definition we have that $\mathcal{U}$ is a subcategory of $\mathcal{U}'$, so we can consider a compatible system $\tilde{id} : \mathcal{U} \rightarrow \mathcal{U}'$ over the identity of X as follows:

- as a functor, it is just the inclusion on the level of objects and morphisms (i.e: uniformizing systems and embeddings);
- for every uniformizing system $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we set $\tilde{id}_{\widetilde{U}_i,\widetilde{U}_i} := 1_{\widetilde{U}_i}$.

It is easy to see that all the axioms of definition 2.8 are satisfied; as an useful notation, we will write an index as i if it belongs to the set I (and so also to I' if we consider $\mathcal{U}$ as a subcategory of $\mathcal{U}'$) and with i' if it belongs to I' and we don't know whether $(\widetilde{U}_{i'}, G_{i'}, \pi_{i'})$ belongs to $\mathcal{U}$ or not.

Definition 5.3. In the following pages we will use the following objects and morphisms obtained by applying the 2-functor F described in the previous chapter:

- $R \underset{t}{\overset{s}{\rightrightarrows}} U$ is the groupoid object associated to the orbifold atlas $\mathcal{U}$;
- $R' \underset{t'}{\overset{s'}{\rightrightarrows}} U'$ is the groupoid object associated to the orbifold atlas $\mathcal{U}'$;
- $(\psi, \Psi) : (R \underset{t}{\overset{s}{\rightrightarrows}} U) \rightarrow (R' \underset{t'}{\overset{s'}{\rightrightarrows}} U')$ is the morphism between groupoid objects associated to the compatible system $\tilde{id}$.

Our aim is to prove the following result:

Proposition 5.1.2. *The morphism* (ψ, Ψ) *is a Morita equivalence.*

Proof. We have to verify the axioms of the previous definition, so let us first focus our attention on the map $t \circ \pi_1$ defined on the fiber product (which is a manifold, as already stated in the definition of Morita equivalence):

$$\begin{tikzcd}
R' \times_{U'} U \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "\psi"] \\
R' \arrow[r, "s'"'] & U'.
\end{tikzcd}$$

Set-theoretically (and up to bijections), we have that:

$$\begin{aligned}
R' \times_{U'} U = \{(r', u) \in R' \times U \text{ s.t. } s'(r') = \psi(u)\} = \\
= \left\{\left([\lambda_{k'i'}, \tilde{x}_{k'}, \lambda_{k'j'}], (\tilde{x}_i, \widetilde{U}_i)\right) \text{ s.t. } (\lambda_{k'i'}(\tilde{x}_{k'}), \widetilde{U}_{i'}) = (\tilde{x}_i, \widetilde{U}_i)\right\} = \\
= \left\{\left([\lambda_{k'i}, \tilde{x}_{k'}, \lambda_{k'j'}], (\tilde{x}_i, \widetilde{U}_i)\right) \text{ s.t. } \lambda_{k'i}(\tilde{x}_{k'}) = \tilde{x}_i\right\}.
\end{aligned}$$

Now for every point $\left([\lambda_{k'i}, \tilde{x}_{k'}, \lambda_{k'j'}], (\tilde{x}_i, \widetilde{U}_i)\right)$ in this set we get that:

$$t \circ \pi_1\left([\lambda_{k'i}, \tilde{x}_{k'}, \lambda_{k'j'}], (\tilde{x}_i, \widetilde{U}_i)\right) = t\left([\lambda_{k'i}, \tilde{x}_{k'}, \lambda_{k'j'}]\right) = \lambda_{k'j'}(\tilde{x}_{k'}). \tag{5.6}$$

Our first aim is to prove axiom (i) of definition 5.1; in particular, let us first prove that $t \circ \pi_1$ *is surjective*, so let us fix any point $(\tilde{x}_{j'}, \widetilde{U}_{j'}) \in U'$ and let us prove that it is in the image of this map.

Let us consider the point $\pi_{j'}(\tilde{x}_{j'}) \in X$; by hypothesis $\mathcal{U}$ is an orbifold atlas for X, so there exists a uniformizing system $(\widetilde{U}_i, G_i, \pi_i)$ in $\mathcal{U}$ for an open neighborhood of this point in X. Now $\mathcal{U}'$ contains both $(\widetilde{U}_i, G_i, \pi_i)$ and $(\widetilde{U}_{j'}, G_{j'}, \pi_{j'})$, so by axiom (ii) of orbifold atlases (and remark 2.7) there exists a uniformizing system $(\widetilde{U}_{k'}, G_{k'}, \pi_{k'})$ in $\mathcal{U}'$, a point $\tilde{x}_{k'}$ in $\widetilde{U}_{k'}$ and embeddings:

$$(\widetilde{U}_i, G_i, \pi_i) \overset{\lambda_{k'i}}{\longleftarrow} (\widetilde{U}_{k'}, G_{k'}, \pi_{k'}) \overset{\lambda_{k'j'}}{\longrightarrow} (\widetilde{U}_{j'}, G_{j'}, \pi_{j'})$$

such that $\lambda_{k'j'}(\tilde{x}_{k'}) = \tilde{x}_{j'}$. Then if we call $\tilde{x}_i := \lambda_{k'i}(\tilde{x}_{k'})$ we get that the point:

$$\Big([\lambda_{k'i}, \tilde{x}_{k'}, \lambda_{k'j'}], (\tilde{x}_i, \widetilde{U}_i)\Big)$$

belongs to the fiber product $R' \times_{U'} U$. Moreover, $t \circ \pi_1$ applied to this point is exactly the point $(\tilde{x}_{j'}, \widetilde{U}_{j'})$ we have fixed. Hence we have proved that $t \circ \pi_1$ is surjective.

Now let us prove also that this map is a submersion; since this is a local property, it suffices to check it when we work in coordinates. So let us fix any point as in (5.6); we use the explicit construction of the charts on the fiber product of manifolds described in proposition 3.5.2 adapted to our specific case in order to get a chart around our fixed point. We will not verify any condition about the good definitions of our sets an set maps since they are exactly the same of the above mentioned proposition.

We define A to be the analogous of the set $A_{x,z}$ described in the proof of proposition 3.5.2; in our case A will be an open neighborhood of the point we have fixed in the fiber product and on it the coordinate function will be a map $\phi : A \to \phi(A)$ with $\phi(A)$ open subset of $\mathbb{C}^r$ (with $r = \dim(R') + \dim(U) - \dim(U') = \dim(U)$). A direct check proves that on every point $\tilde{y}_i \in \phi(A)$ the inverse of the coordinate map will have the following expression:

$$\phi^{-1}(\tilde{y}_i) = \Big([\lambda_{k'i}, \lambda_{k'i}^{-1}(\tilde{y}_i), \lambda_{k'j'}], (\tilde{y}_i, \widetilde{U}_i)\Big).$$

Now the image of our fixed point via $t \circ \pi_1$ is just $(\lambda_{k'j'}(\tilde{x}_{k'}), \widetilde{U}_{j'})$; if we recall that U' is the disjoint union of open sets of $\mathbb{C}^n$, we get that we can choose a chart around this point in U' given by $(\widetilde{U}_{j'}, 1_{\widetilde{U}_{j'}})$.

Hence if we write down $t \circ \pi_1$ in coordinates with respect to the charts we fixed, we get the map:

$$h := 1_{\widetilde{U}_{\hat{j}}} \circ t \circ \pi_1 \circ \phi^{-1};$$

for every point $\tilde{y}_i \in \phi(C)$ we get that:

$$h(\tilde{y}_i) = t \circ \pi_1 \Big([\lambda_{k'i}, \lambda_{k'i}^{-1}(\tilde{y}_i), \lambda_{k'j'}], (\tilde{y}_i, \widetilde{U}_i) \Big) =$$

$$= t \Big([\lambda_{k'i}, \lambda_{k'i}^{-1}(\tilde{y}_i), \lambda_{k'j'}] \Big) = \lambda_{k'j'} \circ \lambda_{k'i}^{-1}(\tilde{y}_i).$$

Since both $\lambda_{k'j'}$ and $\lambda_{k'i}$ are biholomorphic maps (if restricted in target), we get that $t \circ \pi_1$ in coordinates locally coincides with a biholomorphic map, hence it is clearly a submersion.

Hence condition (i) is proved. Now let us pass to condition (ii). We already said in the definition of Morita equivalence that the square (5.1) is a commutative diagram in (**Manifolds**), so it remains only to check the **UP** of fiber products. So let us consider any other complex manifold T together with a pair of holomorphic maps $a : T \to U \times U$ and $b : T \to R'$ such that:

$$(\psi \times \psi) \circ a = (s', t') \circ b; \tag{5.7}$$

then *we want to prove that there exists a unique holomorphic map* $\gamma : T \to R$ *which makes the following diagram commute*:

$$\begin{tikzcd}
T \arrow[rrd, bend left, "b"] \arrow[rd, dashed, "\gamma"] \arrow[rdd, bend right, "a"'] \arrow[rrd, phantom, "\curvearrowright"] \arrow[rdd, phantom, "\curvearrowright"] & & \\
& R \arrow[r, "\Psi"] \arrow[d, "{(s,t)}"'] \arrow[rd, phantom, "\curvearrowright"] & R' \arrow[d, "{(s',t')}"] \\
& U \times U \arrow[r, "(\psi\times\psi)"'] & U' \times U'.
\end{tikzcd}$$

In order to define the map γ, let us work set-theoretically so let us fix a point $t \in T$ and let us use the following notations:

$$a(t) =: \Big((\tilde{x}_i, \widetilde{U}_i), (\tilde{x}_j, \widetilde{U}_j) \Big) \quad \text{and} \quad b(t) =: [\lambda_{k'i'}, \tilde{x}_{k'}, \lambda_{k'i'}].$$

Let us recall that ψ is just the inclusion map of U in U', so using (5.7) we get that:

$$\Big((\tilde{x}_i, \widetilde{U}_i), (\tilde{x}_j, \widetilde{U}_j) \Big) = (\psi, \psi) \circ a(t) = (s', t') \circ b(t) =$$
$$= (s', t') \Big([\lambda_{k'i'}, \tilde{x}_{k'}, \lambda_{k'i'}] \Big) = \Big((\lambda_{k'i'}(\tilde{x}_{k'}), \widetilde{U}_{j'}), (\lambda_{k'j'}(\tilde{x}_{k'}), \widetilde{U}_{j'}) \Big);$$

hence:

$$i' = i, \quad j' = j, \quad \tilde{x}_i = \lambda_{k'i}(\tilde{x}_{k'}) \quad \text{and} \quad \tilde{x}_j = \lambda_{k'j}(\tilde{x}_{k'}).$$

Now we notice that:

$$\pi_i(\tilde{x}_i) = \pi_i(\lambda_{k'i}(\tilde{x}_{k'})) = \pi_{k'}(\tilde{x}_{k'}) = \pi_j(\lambda_{k'j}(\tilde{x}_{k'})) = \pi_j(\tilde{x}_j);$$

so if we apply property (ii) of orbifold atlases for $\mathcal{U}$ and remark 2.7 we get that there exists a uniformizing system $(\widetilde{U}_l, G_l, \pi_l) \in \mathcal{U}$, a point $\tilde{x}_l \in \widetilde{U}_l$ and embeddings between uniformizing systems:

$$(\widetilde{U}_i, G_i, \pi_i) \overset{\lambda_{li}}{\longleftarrow} (\widetilde{U}_l, G_l, \pi_l) \overset{\lambda_{lj}}{\longrightarrow} (\widetilde{U}_j, G_j, \pi_j)$$

such that $\lambda_{li}(\tilde{x}_l) = \tilde{x}_i$ and $\lambda_{lj}(\tilde{x}_l) = \tilde{x}_j$.

Now we recall that $\mathcal{U} \subseteq \mathcal{U}'$, so we can consider the pair of uniformizing systems $(\widetilde{U}_{k'}, G_{k'}, \pi_{k'})$ and $(\widetilde{U}_l, G_l, \pi_l)$ in the atlas $\mathcal{U}'$: since $\pi_{k'}(\tilde{x}_{k'}) = \pi_l(\tilde{x}_l)$, we can apply again condition (ii) of orbifold atlases in order to prove that there exists a uniformizing system $(\widetilde{U}_{m'}, G_{m'}, \pi_{m'}) \in \mathcal{U}'$, a point $\tilde{x}_{m'} \in \widetilde{U}_{m'}$ and a pair of embeddings $\lambda_{m'k'}$, $\lambda_{m'l}$ such that:

$$\lambda_{m'k'}(\tilde{x}_{m'}) = \tilde{x}_{k'} \quad \text{and} \quad \lambda_{m'l}(\tilde{x}_{m'}) = \tilde{x}_l.$$

Now let us consider the pair of embeddings:

$$\mu := \lambda_{li} \circ \lambda_{m'l} \quad \text{and} \quad \delta := \lambda_{k'i} \circ \lambda_{m'k'}$$

both defined from $(\widetilde{U}_{m'}, G_{m'}, \pi_{m'})$ to $(\widetilde{U}_i, G_i, \pi_i)$. Using lemma 2.1.7 we get that there exists a unique $g \in G_i$ such that $g \circ \mu = \delta$; so if we call $\tilde{\lambda}_{li} := g \circ \lambda_{ij}$ we get that:

$$\tilde{\lambda}_{li} \circ \lambda_{m'l} = \lambda_{k'i} \circ \lambda_{m'k'}. \tag{5.8}$$

Moreover, by construction we know that:

$$\lambda_{li} \circ \lambda_{m'l}(\tilde{x}_{m'}) = \tilde{x}_i = \lambda_{k'i} \circ \lambda_{m'k'}(\tilde{x}_{m'})$$

hence g must belong to the stabilizer of $\tilde{x}_i$ in G_i, so

$$\tilde{\lambda}_{li}(\tilde{x}_l) = \tilde{x}_i. \tag{5.9}$$

In the same way we can replace λ_{lj} with an embedding $\tilde{\lambda}_{lj}$ such that

$$\tilde{\lambda}_{lj} \circ \lambda_{m'l} = \lambda_{k'j} \circ \lambda_{m'k'} \quad \text{and} \quad \tilde{\lambda}_{lj}(\tilde{x}_l) = \tilde{x}_j \tag{5.10}$$

Using together (5.8), (5.9) and (5.10) we get that the following 2 diagrams are commutative:

$$\begin{tikzcd}
 & & \widetilde{U}_{k'} \arrow[dll, "\lambda_{k'i}"'] \arrow[drr, "\lambda_{k'j}"] & & \\
\widetilde{U}_i & \curvearrowright & \widetilde{U}_{m'} \arrow[u, "\lambda_{m'k'}"'] \arrow[d, "\lambda_{m'l}"] & \curvearrowright & \widetilde{U}_j \\
 & & \widetilde{U}_l \arrow[ull, "\tilde{\lambda}_{li}"] \arrow[urr, "\tilde{\lambda}_{lj}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \tilde{x}_{k'} \arrow[dll, "\lambda_{k'i}"'] \arrow[drr, "\lambda_{k'j}"] & & \\
\tilde{x}_i & \curvearrowright & \tilde{x}_{m'} \arrow[u, "\lambda_{m'k'}"'] \arrow[d, "\lambda_{m'l}"] & \curvearrowright & \tilde{x}_j; \\
 & & \tilde{x}_l \arrow[ull, "\tilde{\lambda}_{li}"] \arrow[urr, "\tilde{\lambda}_{lj}"'] & &
\end{tikzcd}$$

so we have proved that:

$$[\tilde{\lambda}_{li}, \tilde{x}_i, \tilde{\lambda}_{lj}] = [\lambda_{k'i}, \tilde{x}_{k'}, \lambda_{k'j}]$$

if we consider both them as points of R' (clearly this relation is meaningless if we work in R). *But the first of these two points can also be considered as a point of* R, so we would like to define:

$$\gamma(t) := [\tilde{\lambda}_{li}, \tilde{x}_i, \tilde{\lambda}_{lj}];$$

indeed in this way it is easy to prove that:

$$(s,t) \circ \gamma(t) = a(t) \quad \text{and} \quad \Psi \circ \gamma(t) = b(t).$$

However we have to solve the following problem: in the previous construction we have found a point $(\tilde{\lambda}_{li}, \tilde{x}_i, \tilde{\lambda}_{lj})$ of R which was equivalent in R' to $b(t)$, but the previous construction does not ensure uniqueness of such a point. *If we find another point of* R *which is equivalent to* $b(t)$ *in* R'*, what is its relationship in* R *with the previos one? In other words, the class* $[\tilde{\lambda}_{li}, \tilde{x}_i, \tilde{\lambda}_{lj}]$ *is unique?*

So let us suppose we have chosen any pair of points $(\lambda_{li}, \tilde{x}_i, \lambda_{lj})$ and $(\lambda_{ni}, \tilde{x}_n, \lambda_{nj})$ in $\psi(R) \subseteq R'$ both equivalent to $b(t)$ in R'. Then we can apply the lemma that follows this proof in order to prove that:

$$(\lambda_{li}, \tilde{x}_i, \lambda_{lj}) \sim (\lambda_{ni}, \tilde{x}_n, \lambda_{nj}) \text{ in } R.$$

(We apply the lemma to the orbifold atlas $\mathcal{U}'$, where we have the equivalence between the two points. In the central part we just put any uniformizing system together with a pair of embeddings as in property (ii) of orbifold atlases applied to $\mathcal{U}$, since both the top and the bottom part of the diagram are not only in $\mathcal{U}'$, but also in $\mathcal{U}$.)

Hence the set map $\gamma : T \to R$ *is well defined.* Note that since we require that $\Psi \circ \gamma = b$ we get easily that this map is also *unique.* So we have proved that the diagram of axiom (ii) is cartesian in (**Sets**). Moreover, R is a complex manifold and both (s,t) and Ψ are holomorphic maps. So in order to prove that this diagram is cartesian also in (**Manifolds**) it suffices to prove that whenever the maps a and b are holomorphic, then also γ is so.

So let us fix as before a point $t \in T$; until now we have proved that there exists a *unique* point $[\lambda_{li}, \tilde{x}_l, \lambda_{lj}]$ in R such that $b(t) = [\lambda_{li}, \tilde{x}_l, \lambda_{lj}] = \Psi([\lambda_{li}, \tilde{x}_l, \lambda_{lj}])$. In particular, $b(t)$ belongs to the open set $B := q'(\widetilde{U}_l^{ij}) \subseteq R'$ and we have proved in the previous chapter that we can define a chart $(B, \phi_l'^{ij})$ on the manifold R' where the map $\phi_l'^{ij}$ is an homeomorphism from B to $\widetilde{U}_l \subseteq \mathbb{C}^n$, given by:

$$\phi_l'^{ij}([\lambda_{li}, \tilde{y}_l, \lambda_{lj}]) := \tilde{y}_l$$

Let us call $\widetilde{B} := b^{-1}(A)$; this set is open since b is continuos, so eventually by restricting to a smaller set we can assume that there exists a chart on the manifold T of the form $(\widetilde{B}, \xi)$ where ξ is an homeomorphism from $\widetilde{B}$ to an open set $\xi(\widetilde{B}) \subseteq \mathbb{C}^m$. Since by hypothesis b is an holomorphic map between complex manifolds, we get that the composite:

$$\phi_l'^{ij} \circ b \circ \xi^{-1} : \xi(\widetilde{B}) \to \widetilde{U}_l$$

is holomorphic. But we recall that $\gamma(t)$ has values in $q(\widetilde{U}_l^{ij})$, also homeomorphic to $\widetilde{U}_l$ via the map ϕ_l^{ij}, which has the same fomrmal expression of $\phi_l'^{ij}$, so the previous one is also the local expression in coordinates of the map γ; so γ is holomorphic.

Hence also property (ii) of definition 5.1 is satisfied, so the morphism (ψ, Ψ) is a Morita equivalence. □

Lemma 5.1.3. *Let us fix an atlas $\mathcal{V}$, and let us suppose we have two equivalent points $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj})$ and $(\lambda_{li}, \tilde{x}_l, \lambda_{lj})$ with respect to this atlas (i.e: we can find a commutative pair of diagrams of the form 4.4 where also the central* *part belongs to* $\mathcal{V}$*. Moreover, let us suppose we have a uniformizing system* $(\widetilde{U}_n, G_n, \pi_n) \in \mathcal{V}$ *such that* $\pi_n(\widetilde{U}_n)$ *contains* $\pi_k(\tilde{x}_k)$ *and a pair of embeddings* $\tilde{\lambda}_{nk}, \tilde{\lambda}_{nl}$*. Then there exists a point* $\tilde{x}_n \in \widetilde{U}_n$ *and embeddings* $\lambda_{nk}, \lambda_{nl}$ *which make the following diagrams commute:*

$$\begin{tikzcd}
 & \widetilde{U}_k \arrow[dl, "\lambda_{ki}"'] \arrow[dr, "\lambda_{kj}"] & \\
\widetilde{U}_i \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_n \arrow[u, "\lambda_{nk}"'] \arrow[d, "\lambda_{nl}"] \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_j \\
 & \widetilde{U}_l \arrow[ul, "\lambda_{li}"] \arrow[ur, "\lambda_{lj}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \tilde{x}_k \arrow[dl, "\lambda_{ki}"'] \arrow[dr, "\lambda_{kj}"] & \\
\tilde{x}_i \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_n \arrow[u, "\lambda_{nk}"'] \arrow[d, "\lambda_{nl}"] \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_j. \\
 & \tilde{x}_l \arrow[ul, "\lambda_{li}"] \arrow[ur, "\lambda_{lj}"'] &
\end{tikzcd}$$

In other words, whenever we have a pair of commutative diagrams of the form (4.4), then any other diagram of the same form (but with different "center") which is not commutative can be modified only in the vertical arrows in order to be commutative.

Proof. By hypothesis, $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj}) \sim (\lambda_{li}, \tilde{x}_l, \lambda_{lj})$, so there exists a uniformizing system $(\widetilde{U}_m, G_m, \pi_m)$ in $\mathcal{V}$, a point $\tilde{x}_m \in \widetilde{U}_m$ and embeddings $\lambda_{mk}, \lambda_{ml}$ making diagram (4.4) commutative. In particular, we have that $\pi_m(\tilde{x}_m) = \pi_k(\lambda_{mk}(\tilde{x}_m)) = \pi_k(\tilde{x}_k)$ and by hypothesis $\pi_k(\tilde{x}_k) \in \pi_n(\widetilde{U}_n)$.

So by definition of orbifold atlas applied to $\mathcal{V}$, there exist an open neighborhood $U_p \subseteq U_m \cap U_n$ of this point, a uniformizing system $(\widetilde{U}_p, G_p, \pi_p) \in \mathcal{V}$ for it and embeddings $\lambda_{pm}, \lambda_{pn}$. Moreover we can use remark 2.7, so without loss of generality we can assume that there exists $\tilde{x}_p \in \widetilde{U}_p$ such that $\lambda_{pm}(\tilde{x}_p) = \tilde{x}_m$. Now let us consider the pair of embeddings:

$$\alpha := \tilde{\lambda}_{nk} \circ \lambda_{pn} \quad \text{and} \quad \beta := \lambda_{mk} \circ \lambda_{pm}$$

both defined from $(\widetilde{U}_p, G_p, \pi_p)$ to $(\widetilde{U}_k, G_k, \pi_k)$; then using lemma 2.1.7 we get that there exists a unique $g \in G_k$ such that $g \circ \alpha = \beta$. So if we define $\lambda_{nk} := g \circ \tilde{\lambda}_{nk}$ we have that $\lambda_{nk} \circ \lambda_{pn} = \lambda_{mk} \circ \lambda_{pm}$. Moreover, by construction, $\tilde{x}_k = \lambda_{mk}(\tilde{x}_m) = \lambda_{mk} \circ \lambda_{pm}(\tilde{x}_p)$, so if we define $\tilde{x}_n := \lambda_{pn}(\tilde{x}_p) \in \widetilde{U}_n$, we have that:

$$\lambda_{nk}(\tilde{x}_n) = \tilde{x}_k.$$

In the same way, we can define the embedding λ_{nl} such that $\lambda_{ml} \circ \lambda_{pm} = \lambda_{nl} \circ \lambda_{pn}$; now using these results and diagram (4.4) we get commutative diagrams:

$$\begin{tikzcd}
\widetilde{U}_n \arrow[r, "\lambda_{nk}"] \arrow[dr, phantom, "\curvearrowright"] & \widetilde{U}_k \arrow[dr, "\lambda_{kj}"] & \\
\widetilde{U}_p \arrow[u, "\lambda_{pn}"] \arrow[r, "\lambda_{pm}"] \arrow[d, swap, "\lambda_{pn}"] \arrow[dr, phantom, "\curvearrowright"] & \widetilde{U}_m \arrow[u, swap, "\lambda_{mk}"] \arrow[d, "\lambda_{ml}"] \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_j \\
\widetilde{U}_n \arrow[r, swap, "\lambda_{nl}"] & \widetilde{U}_l \arrow[ur, swap, "\lambda_{lj}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{x}_n \arrow[r, "\lambda_{nk}"] \arrow[dr, phantom, "\curvearrowright"] & \tilde{x}_k \arrow[dr, "\lambda_{kj}"] & \\
\tilde{x}_p \arrow[u, "\lambda_{pn}"] \arrow[r, "\lambda_{pm}"] \arrow[d, swap, "\lambda_{pn}"] \arrow[dr, phantom, "\curvearrowright"] & \tilde{x}_m \arrow[u, swap, "\lambda_{mk}"] \arrow[d, "\lambda_{ml}"] \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_j \\
\tilde{x}_n \arrow[r, dashed, swap, "\lambda_{nl}"] & \tilde{x}_l \arrow[ur, swap, "\lambda_{lj}"] &
\end{tikzcd}
\tag{5.11}$$

Note that we have dashed the last arrow because a priori we don't know whether $\lambda_{nl}(\tilde{x}_n)$ is equal to $\tilde{x}_l$ or not. Using the diagram on the left, we get that:

$$\lambda_{kj} \circ \lambda_{nk} \circ \lambda_{pn} = \lambda_{lj} \circ \lambda_{nl} \circ \lambda_{pn}$$

so $\lambda_{kj} \circ \lambda_{nk}$ and $\lambda_{lj} \circ \lambda_{nl}$ coincide on $\lambda_{pn}(\widetilde{U}_p)$, which is open in $\widetilde{U}_n$ because λ_{pn} is an embedding between open sets of the same complex dimension. Since we are working with holomorphic functions on *connected* domains, we get that:

$$\lambda_{kj} \circ \lambda_{nk} = \lambda_{lj} \circ \lambda_{nl}. \tag{5.12}$$

Hence:

$$\lambda_{lj}(\tilde{x}_l) = \tilde{x}_j = \lambda_{kj}(\tilde{x}_k) = \lambda_{kj} \circ \lambda_{nk}(\tilde{x}_n) = \lambda_{lj} \circ \lambda_{nl}(\tilde{x}_n)$$

and λ_{lj} is an embedding, hence it is injective, so we have that $\lambda_{nl}(\tilde{x}_n) = \tilde{x}_j$, so the previous diagram is commutative in the dashed part.

Using a pair of diagrams similar to the previous ones, we can also prove that:

$$\lambda_{ki} \circ \lambda_{nk} = \lambda_{li} \circ \lambda_{nl}. \tag{5.13}$$

Now equations (5.12) and (5.13) together with diagram (5.11) prove the statement, so we are done. □

Proposition 5.1.4. *Suppose we have fixed two* equivalent *orbifold atlases* $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *on a topological space* X *and let us call* $R_i \underset{t_i}{\overset{s_i}{\rightrightarrows}} U_i$ *(for* $i = 1, 2$*) the groupoid objects associated to them by the 2-functor* F *described in the previous chapter. Then these two groupoid objects are Morita equivalent.*

Proof. It suffices to consider the unique maximal atlas $\mathcal{U}'$ associated to both them and to apply the previous proposition twice. □

Hence I think that *if* the 2-category (**Orb**) can be constructed, then the 2-functor $\tilde{F}$ induced on this new 2-category has codomain in the 2-category where the objects are *classes of Morita equivalent groupoid objects* in (**Manifolds**) with étale source and target and proper relative diagonal. This last 2-category is not too hard to describe explicitly (see [M] and [Pr]) also on the level of morphisms and 2-morphisms, so the only problem to solve is to define what are the morphisms and the 2-morphisms in (**Orb**).

Clearly the morphisms will have to be equivalence classes of compatible systems between equivalence classes of orbifold atlases and will have to correspond to equivalence classes of morphisms between equivalence classes of Morita equivalent groupoid objects, but until now I have not found any more explicit description of them, so also the next step (the definition of 2-morphisms) remains undone.

The main problem about morphisms arises when we try to compare 2 compatible systems (over the same continuous map between topological spaces) with different (but equivalent) orbifold atlases as source or target. The naif idea is just to induce in a "canonical" way another pair of compatible systems between the corresponding maximal atlases in source and target, and then compare them as functors and collection of holomorphic liftings. The problem is that I have no idea of how this can be made.

5.2 Essential surjectivity

The aim of this work was to prove that differential geometers and algebraic geometers actually mean the same thing when they talk about "orbifolds". The fact that we managed to describe the 2-functor F allows us to be quite confident that this is the case. However, also if we suppose that the problems described in the previous section can be solved, there is another thing to take into account: the fact that a 2-functor between 2-categories $\mathscr{A}$ and $\mathscr{B}$ exists does not allow us to think that $\mathscr{A}$ and $\mathscr{B}$ are *different descriptions of the same geometric objects*. So we would like to prove, for example, that the 2-functor F (or at least $\tilde{F}$, that we have not defined completely), is *essentially surjective* on the level of objects and an *equivalence of categories* on the level of morphisms and 2-morphisms (i.e: whenever we fix any pair of objects and we consider the functor as datum (2) of definition 1.12). In this thesis these questions weren't explored for reasons of space and time.

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