AN APPENDIX TO "SOME INSIGHTS ON BICATEGORIES OF FRACTIONS - II"

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Contents

Introduction

In our paper [T2] we considered a setup as follows:

- we fix a bicategory $\mathscr{A}$ and a class of 1-morphisms $\mathbf{W}_{\mathscr{A}}$, such that the pair $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ satisfies conditions (BF1) – (BF5) for a (right) bicalculus of fractions, as described in [Pr]; in other terms, we are able to construct a bicategory of fractions $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$;
- we fix another pair $(\mathscr{B}, \mathbf{W}_{\mathscr{B}})$ admitting a (right) bicalculus of fractions; in [T2, § 2.1] we described the (right) saturation $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ of $\mathbf{W}_{\mathscr{B}}$ and we proved that also the pair $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ admits a (right) bicalculus of fractions;
- we fix any pseudofunctor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, such that each morphism of $\mathbf{W}_{\mathscr{A}}$ is sent by $\mathcal{F}$ to a morphism of $\mathbf{W}_{\mathscr{B}}$.

Given that, in [T2, Corollary 3.3] we described a pseudofunctor

$$\mathscr{N} : \mathscr{A}\Big[\mathbf{W}_{\mathscr{A}}^{-1}\Big] \longrightarrow \mathscr{B}\Big[\mathbf{W}_{\mathscr{B}}^{-1}\Big]$$

induced by $\mathcal{F}$. The construction of $\mathscr{N}$ was mainly a consequence of the construction of a similar pseudofunctor $\mathcal{M}$ in [T2, Proposition 3.1]; such a construction was done following the lines of [Pr, pagg. 265–266], where actually it was never proved explicitly that the resulting pseudofunctor ($\widetilde{\mathcal{F}}$ in the notations of [Pr]) is actually a pseudofunctor. Actually in that paper there are no explicit descriptions for the associators and the unitors for the induced pseudofunctor. Therefore, in order to make things more rigorous, we give here a complete proof of the fact that $\mathscr{N}$ is a pseudofunctor (adding identities and unitors, this will also be a proof of the fact that $\mathcal{M}$ is a pseudofunctor).

Date: November 21, 2014.
2010 *Mathematics Subject Classification.* 18A05, 18A30, 22A22.
Key words and phrases. bicategories of fractions, bicalculus of fractions, induced pseudofunctors.

For simplicity of exposition, we will only consider the case when $\mathbf{W}_{\mathscr{B}} = \mathbf{W}_{\mathscr{B},\mathrm{sat}}$ (the case when $\mathbf{W}_{\mathscr{B}} \subsetneq \mathbf{W}_{\mathscr{B},\mathrm{sat}}$ is similar, but requires some more computations from time to time).

This proof is completely self-contained. It does not involves hard techniques, but it consists most of the time of completely straightforward computations. For this reasons, we decided not to put it in the mentioned paper [T2], but to leave it only on our website (http://matteotommasini.altervista.org) for any reader interested in the details. Any updated version will be also posted there.

In the initial part of this paper we will briefly recall some basic notions and notations about bicategories of fractions and we will define associators and commutators for $\mathcal{N}$ (the definition of $\mathcal{N}$ on objects, morphisms and 2-morphisms is the same as the one given in [T2]). The core of the paper will be the (very long, but most of the time straightforward) proof that all the coherence axioms of a pseudofunctor are satisfied by $\mathcal{N}$.

1. Notations

$\mathscr{A}$ will be any bicategory; for any triple of morphisms $f_{\mathscr{A}} : A_{\mathscr{A}} \to B_{\mathscr{A}}$, $g_{\mathscr{A}} : B_{\mathscr{A}} \to C_{\mathscr{A}}$, $h_{\mathscr{A}} : C_{\mathscr{A}} \to D_{\mathscr{A}}$ in $\mathscr{A}$, the associator for that triple will be denoted by

$$\theta_{h_{\mathscr{A}},g_{\mathscr{A}},f_{\mathscr{A}}} : h_{\mathscr{A}} \circ (g_{\mathscr{A}} \circ f_{\mathscr{A}}) \Longrightarrow (h_{\mathscr{A}} \circ g_{\mathscr{A}}) \circ f_{\mathscr{A}}.$$

Moreover, for any morphism $f_{\mathscr{A}}$ as above, we denote its right unitor by

$$\pi_{f_{\mathscr{A}}} : f_{\mathscr{A}} \circ \mathrm{id}_{A_{\mathscr{A}}} \Longrightarrow f_{\mathscr{A}}. \tag{1.1}$$

If there is no ambiguity, we will simply write $\theta_{\bullet}$ and $\pi_{\bullet}$ for any 2-morphism as above.

$\mathbf{W}_{\mathscr{A}}$ will be any class of 1-morphisms in $\mathscr{A}$, such that the pair $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ satisfies conditions (BF1) – (BF5) (see [T1] or the original paper [Pr], with the only difference in axiom (BF1) due to [T2, Remark 1.8]). The pair $(\mathscr{B}, \mathbf{W}_{\mathscr{B}})$ is any other pair satisfying the same technical conditions, and $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ is the (right) saturated of $\mathbf{W}_{\mathscr{B}}$ (see [T2, Definition 2.1]). $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is an pseudofunctor. For any pair $f_{\mathscr{A}}, g_{\mathscr{A}}$ as above, the 2-morphism

$$\psi^{\mathcal{F}}_{g_{\mathscr{A}},f_{\mathscr{A}}} : \mathcal{F}_1(g_{\mathscr{A}} \circ f_{\mathscr{A}}) \Longrightarrow \mathcal{F}_1(g_{\mathscr{A}}) \circ \mathcal{F}_1(f_{\mathscr{A}}) \tag{1.2}$$

is the associator for $\mathcal{F}$ relative to the pair $(g_{\mathscr{A}}, f_{\mathscr{A}})$; for any object $A_{\mathscr{A}}$, the 2-morphism

$$\sigma^{\mathcal{F}}_{A_{\mathscr{A}}} : \mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}}) \Longrightarrow \mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})} \tag{1.3}$$

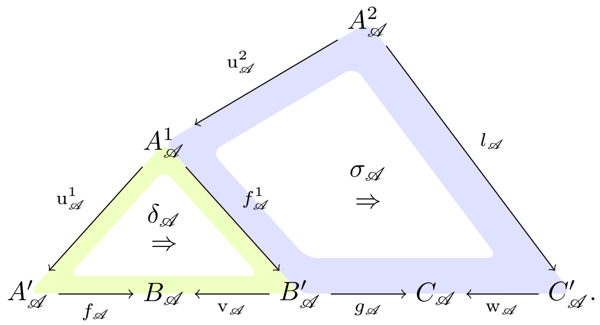
is the unitor for $\mathcal{F}$ relative to $A_{\mathscr{A}}$. We assume full knowledge of the construction of any bicategory of fractions $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ as described by Pronk in [Pr]. Apart from that, the only results that we will use all the time are those mentioned or proved in our papers [T1] and [T2], in particular, the set of choices $\mathrm{C}(\mathbf{W})$ depending on axiom (BF3). We recall that according to [Pr] there is a second set of choices $\mathrm{D}(\mathbf{W})$ (depending on axiom (BF4)) needed for the construction of $\mathscr{C}\left[\mathbf{W}^{-1}\right]$, but such choices are actually not necessary, as we proved in [T1, Theorem 0.5]. We denote by $\Theta^{\mathscr{A},\mathbf{W}_{\mathscr{A}}}_{\bullet}$, respectively $\Theta^{\mathscr{B},\mathbf{W}_{\mathscr{B},\mathrm{sat}}}_{\bullet}$ the associators of the bicategories $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$ and of $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$ respectively.

In all this paper we assume without further mention that $\mathcal{F}$ sends any element of $\mathbf{W}_{\mathscr{A}}$ to some element $\mathbf{W}_{\mathscr{B}} = \mathbf{W}_{\mathscr{B},\mathrm{sat}}$.

Let us fix any triple of 1-morphisms in $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$ as follows:

$$\begin{aligned} \underline{f}_{\mathscr{A}} &= \Big(A_{\mathscr{A}} \xleftarrow{\mathrm{u}_{\mathscr{A}}} A'_{\mathscr{A}} \xrightarrow{f_{\mathscr{A}}} B_{\mathscr{A}}\Big), \\ \underline{g}_{\mathscr{A}} &= \Big(B_{\mathscr{A}} \xleftarrow{\mathrm{v}_{\mathscr{A}}} B'_{\mathscr{A}} \xrightarrow{g_{\mathscr{A}}} C_{\mathscr{A}}\Big), \\ \underline{h}_{\mathscr{A}} &= \Big(C_{\mathscr{A}} \xleftarrow{\mathrm{w}_{\mathscr{A}}} C'_{\mathscr{A}} \xrightarrow{h_{\mathscr{A}}} D_{\mathscr{A}}\Big). \end{aligned} \tag{1.4}$$

If we want to compute the composition $\underline{h}_{\mathscr{A}} \circ (\underline{g}_{\mathscr{A}} \circ \underline{f}_{\mathscr{A}})$, then we need to use the fixed choices $\mathrm{C}(\mathbf{W}_{\mathscr{A}})$ in order to get data as in the upper parts of the following 2-commutative polygons (starting from the smaller one), with $\mathrm{u}^1_{\mathscr{A}}$ and $\mathrm{u}^2_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\delta_{\mathscr{A}}$ and $\sigma_{\mathscr{A}}$ invertible (here $\sigma_{\mathscr{A}}$ is defined from $(g_{\mathscr{A}} \circ f^1_{\mathscr{A}}) \circ \mathrm{u}^2_{\mathscr{A}}$ to $\mathrm{w}_{\mathscr{A}} \circ l_{\mathscr{A}}$):

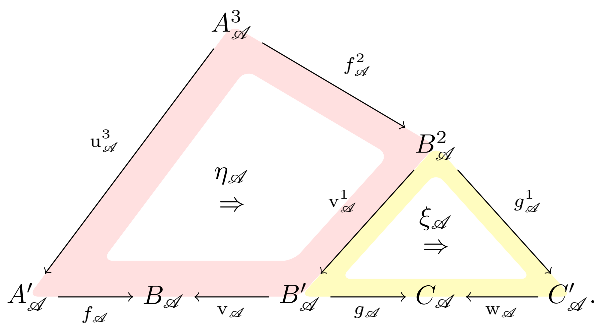


(1.5)

Then according to [Pr, § 2.2] we have

$$\underline{h}_{\mathscr{A}} \circ \Big(\underline{g}_{\mathscr{A}} \circ \underline{f}_{\mathscr{A}}\Big) = \Big(A_{\mathscr{A}} \xleftarrow{(\mathrm{u}_{\mathscr{A}} \circ \mathrm{u}^1_{\mathscr{A}}) \circ \mathrm{u}^2_{\mathscr{A}}} A^2_{\mathscr{A}} \xrightarrow{h_{\mathscr{A}} \circ l_{\mathscr{A}}} D_{\mathscr{A}}\Big). \tag{1.6}$$

On the other hand, if we want to compute the composition $(\underline{h}_{\mathscr{A}} \circ \underline{g}_{\mathscr{A}}) \circ \underline{f}_{\mathscr{A}}$, then we need to use the fixed choices $\mathrm{C}(\mathbf{W}_{\mathscr{A}})$ in order to get data as in the upper parts of the following 2-commutative polygons (starting from the smaller one), with $\mathrm{v}^1_{\mathscr{A}}$ and $\mathrm{u}^3_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\xi_{\mathscr{A}}$ and $\eta_{\mathscr{A}}$ invertible (here $\eta_{\mathscr{A}}$ is defined from $f_{\mathscr{A}} \circ \mathrm{u}^3_{\mathscr{A}}$ to $(\mathrm{v}_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}}) \circ f^2_{\mathscr{A}}$):



(1.7)

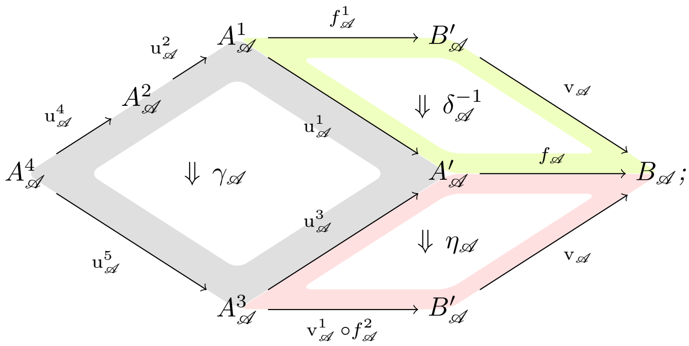
Then we have:

$$\left(\underline{h}_{\mathscr{A}} \circ \underline{g}_{\mathscr{A}}\right) \circ \underline{f}_{\mathscr{A}} = \left(A_{\mathscr{A}} \xleftarrow{\mathrm{u}_{\mathscr{A}} \circ \mathrm{u}^3_{\mathscr{A}}} A^3_{\mathscr{A}} \xrightarrow{(h_{\mathscr{A}} \circ g^1_{\mathscr{A}}) \circ f^2_{\mathscr{A}}} D_{\mathscr{A}}\right). \tag{1.8}$$

Then from [T1, Proposition 0.1] (for $\mathscr{A}$ instead of $\mathscr{C}$) we have

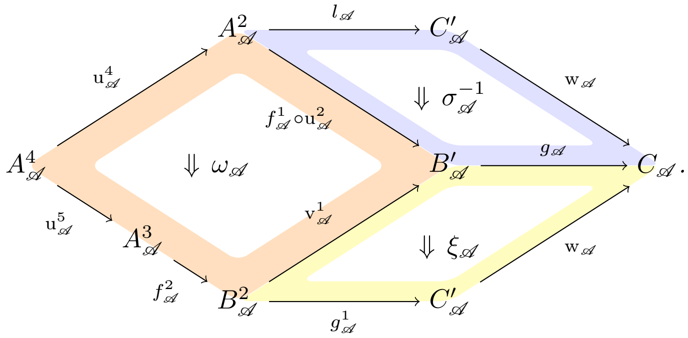
Proposition 1.1. *Let us fix any pair* $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$, *satisfying conditions* (BF), *any triple of morphisms as in* (1.4) *and let us suppose that choices* $\mathrm{C}(\mathbf{W}_{\mathscr{A}})$ *give data as in the upper parts of diagrams* (1.5) *and* (1.7). *Then let us* choose *any set of data as follows:*

(*F1*) *an object* $A^4_{\mathscr{A}}$, *a morphism* $\mathrm{u}^4_{\mathscr{A}} : A^4_{\mathscr{A}} \rightarrow A^2_{\mathscr{A}}$ *in* $\mathbf{W}_{\mathscr{A}}$, *a morphism* $\mathrm{u}^5_{\mathscr{A}} : A^4_{\mathscr{A}} \rightarrow A^3_{\mathscr{A}}$ *and an invertible* 2*-morphism* $\gamma_{\mathscr{A}} : \mathrm{u}^1_{\mathscr{A}} \circ (\mathrm{u}^2_{\mathscr{A}} \circ \mathrm{u}^4_{\mathscr{A}}) \Rightarrow \mathrm{u}^3_{\mathscr{A}} \circ \mathrm{u}^5_{\mathscr{A}}$;
(*F2*) *an invertible* 2*-morphism* $\omega_{\mathscr{A}} : f^1_{\mathscr{A}} \circ (\mathrm{u}^2_{\mathscr{A}} \circ \mathrm{u}^4_{\mathscr{A}}) \Longrightarrow \mathrm{v}^1_{\mathscr{A}} \circ (f^2_{\mathscr{A}} \circ \mathrm{u}^5_{\mathscr{A}})$, *such that* $i_{\mathrm{v}_{\mathscr{A}}} * \omega_{\mathscr{A}}$ *coincides with the following composition* (*associators of* $\mathscr{A}$ *omitted*)*:*



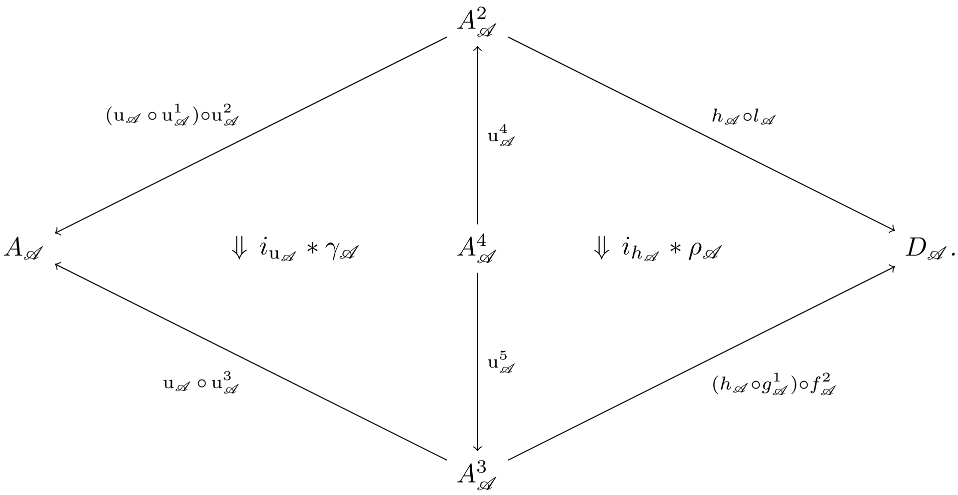
(1.9)

(*F3*) *an invertible* 2*-morphism* $\rho_{\mathscr{A}} : l_{\mathscr{A}} \circ \mathrm{u}^4_{\mathscr{A}} \Longrightarrow g^1_{\mathscr{A}} \circ (f^2_{\mathscr{A}} \circ \mathrm{u}^5_{\mathscr{A}})$, *such that* $i_{\mathrm{w}_{\mathscr{A}}} * \rho_{\mathscr{A}}$ *coincides with the following composition* (*associators of* $\mathscr{A}$ *omitted*)*:*



(1.10)

Then the associator $\Theta^{\mathscr{A}, \mathbf{W}_{\mathscr{A}}}_{\underline{h}_{\mathscr{A}}, \underline{g}_{\mathscr{A}}, \underline{f}_{\mathscr{A}}}$ *from* (1.6) *to* (1.8) *is given by the class of the following diagram* (*associators of* $\mathscr{A}$ *omitted*)*:*



(1.11)

Moreover, a set of choices as in (F1) – (F3) always exists (see [T1, Remark 2.1]*).*

2. The definition of $\mathscr{N}$

We recall that in the present paper we have fixed any pseudofunctor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ that maps any morphism of $\mathbf{W}_{\mathscr{A}}$ to some morphism of $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$.

For each object $A_{\mathscr{A}}$, we define $\mathscr{N}_0(A_{\mathscr{A}}) := \mathcal{F}_0(A_{\mathscr{A}})$; for each morphism $\underline{f}_{\mathscr{A}} := (A'_{\mathscr{A}}, \mathrm{w}_{\mathscr{A}}, f_{\mathscr{A}}) : A_{\mathscr{A}} \to B_{\mathscr{A}}$ in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$, we set

$$\mathscr{N}_1(\underline{f}_{\mathscr{A}}) := \Big(\mathcal{F}_0(A'_{\mathscr{A}}), \mathcal{F}_1(\mathrm{w}_{\mathscr{A}}), \mathcal{F}_1(f_{\mathscr{A}})\Big).$$

Given any pair of morphisms $(A^m_{\mathscr{A}}, \mathrm{w}^m_{\mathscr{A}}, f^m_{\mathscr{A}}) : A_{\mathscr{A}} \to B_{\mathscr{A}}$ in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$ for $m = 1, 2$ and any 2-morphism

$$\Big[A^3_{\mathscr{A}}, \mathrm{v}^1_{\mathscr{A}}, \mathrm{v}^2_{\mathscr{A}}, \alpha_{\mathscr{A}}, \beta_{\mathscr{A}}\Big] : \Big(A^1_{\mathscr{A}}, \mathrm{w}^1_{\mathscr{A}}, f^1_{\mathscr{A}}\Big) \Longrightarrow \Big(A^2_{\mathscr{A}}, \mathrm{w}^2_{\mathscr{A}}, f^2_{\mathscr{A}}\Big)$$

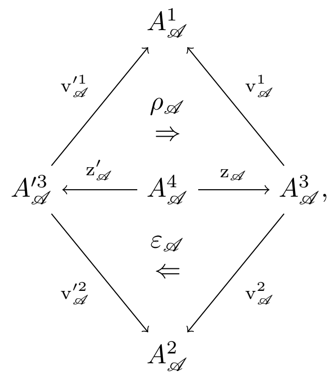
in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$, we set

$$\begin{gathered}\mathscr{N}_2\Big(\Big[A^3_{\mathscr{A}}, \mathrm{v}^1_{\mathscr{A}}, \mathrm{v}^2_{\mathscr{A}}, \alpha_{\mathscr{A}}, \beta_{\mathscr{A}}\Big]\Big) := \Big[\mathcal{F}_0(A^3_{\mathscr{A}}), \mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}}), \mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}}), \qquad (2.1)\\ \psi^{\mathcal{F}}_{\mathrm{w}^2_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}}} \odot \mathcal{F}_2(\alpha_{\mathscr{A}}) \odot \Big(\psi^{\mathcal{F}}_{\mathrm{w}^1_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}}}\Big)^{-1}, \psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}}} \odot \mathcal{F}_2(\beta_{\mathscr{A}}) \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}}}\Big)^{-1}\Big].\end{gathered}$$

First of all, we have to show that $\mathscr{N}_2$ is well-defined. So let us suppose that we have

$$\Big[A^3_{\mathscr{A}}, \mathrm{v}^1_{\mathscr{A}}, \mathrm{v}^2_{\mathscr{A}}, \alpha_{\mathscr{A}}, \beta_{\mathscr{A}}\Big] = \Big[A'^3_{\mathscr{A}}, \mathrm{v}'^1_{\mathscr{A}}, \mathrm{v}'^2_{\mathscr{A}}, \alpha'_{\mathscr{A}}, \beta'_{\mathscr{A}}\Big].$$

By definition of 2-morphisms in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$ (see [Pr, § 2.3]), this implies that there is a set of data $(A^4_{\mathscr{A}}, \mathrm{z}_{\mathscr{A}}, \mathrm{z}'_{\mathscr{A}}, \rho_{\mathscr{A}}, \varepsilon_{\mathscr{A}})$ in $\mathscr{A}$ as in the following diagram



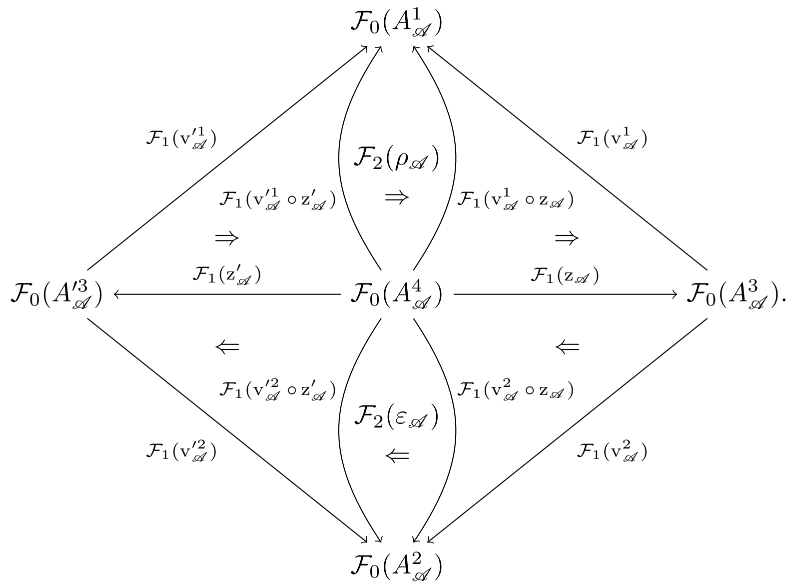
such that $(\mathrm{w}^1_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}}) \circ \mathrm{z}_{\mathscr{A}}$ belongs to $\mathbf{W}_{\mathscr{A}}$, $\rho_{\mathscr{A}}$ and $\varepsilon_{\mathscr{A}}$ are both invertible,

$$\left(i_{\mathrm{w}^2_{\mathscr{A}}} * \varepsilon_{\mathscr{A}}\right) \odot \left(\theta_{\mathrm{w}^2_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}},\mathrm{z}_{\mathscr{A}}}\right)^{-1} \odot \left(\alpha_{\mathscr{A}} * i_{\mathrm{z}_{\mathscr{A}}}\right) \odot \theta_{\mathrm{w}^1_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}},\mathrm{z}_{\mathscr{A}}} \odot \left(i_{\mathrm{w}^1_{\mathscr{A}}} * \rho_{\mathscr{A}}\right) = \\ = \left(\theta_{\mathrm{w}^2_{\mathscr{A}},\mathrm{v}'^2_{\mathscr{A}},\mathrm{z}'_{\mathscr{A}}}\right)^{-1} \odot \left(\alpha'_{\mathscr{A}} * i_{\mathrm{z}'_{\mathscr{A}}}\right) \odot \theta_{\mathrm{w}^1_{\mathscr{A}},\mathrm{v}'^1_{\mathscr{A}},\mathrm{z}'_{\mathscr{A}}}$$

and

$$\left(i_{f^2_{\mathscr{A}}} * \varepsilon_{\mathscr{A}}\right) \odot \left(\theta_{f^2_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}},\mathrm{z}_{\mathscr{A}}}\right)^{-1} \odot \left(\beta_{\mathscr{A}} * i_{\mathrm{z}_{\mathscr{A}}}\right) \odot \theta_{f^1_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}},\mathrm{z}_{\mathscr{A}}} \odot \left(i_{f^1_{\mathscr{A}}} * \rho_{\mathscr{A}}\right) = \\ = \left(\theta_{f^2_{\mathscr{A}},\mathrm{v}'^2_{\mathscr{A}},\mathrm{z}'_{\mathscr{A}}}\right)^{-1} \odot \left(\beta'_{\mathscr{A}} * i_{\mathrm{z}'_{\mathscr{A}}}\right) \odot \theta_{f^1_{\mathscr{A}},\mathrm{v}'^1_{\mathscr{A}},\mathrm{z}'_{\mathscr{A}}}.$$

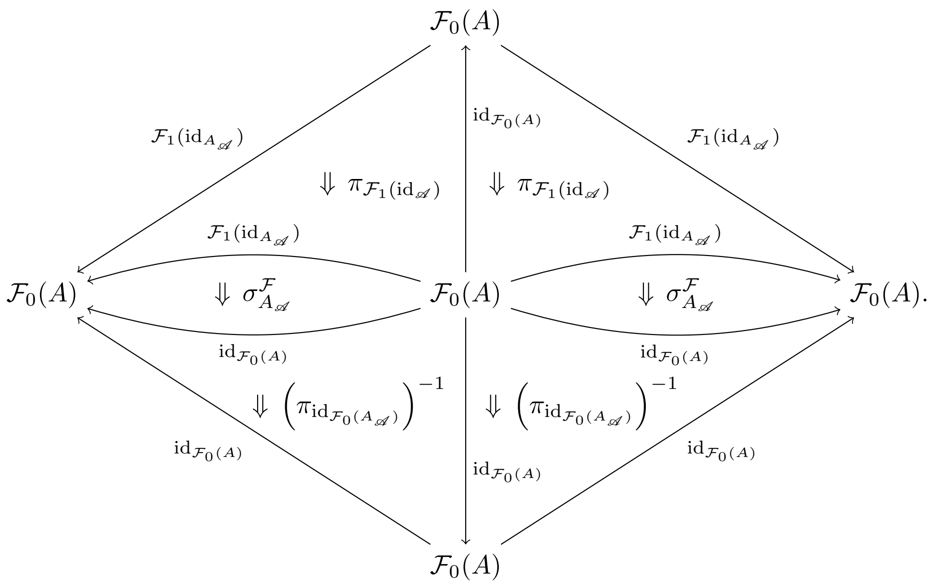
Then the following set of data proves that $\mathcal{N}$ is well-defined on 2-morphisms; the 2-morphisms without a name below are all 2-morphisms of the form $\psi^{\mathcal{F}}_{-,-}$ (see (1.2)) or inverses of such 2-morphisms:



The aim of this paper is to prove that the triple $(\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)$ induces a pseudofunctor $\mathcal{N} : \mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right] \to \mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$. In order to do that, first of all we have to define the unitors for $\mathcal{N}$. So let us fix any object $A_{\mathscr{A}}$; the identity for it in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$ is the triple $(A_{\mathscr{A}}, \mathrm{id}_{A_{\mathscr{A}}}, \mathrm{id}_{A_{\mathscr{A}}})$ and analogously for the identity of $\mathcal{N}_0(A_{\mathscr{A}}) = \mathcal{F}_0(A_{\mathscr{A}})$ in $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$. Then we define the unitor of $\mathcal{N}$ relative to $A_{\mathscr{A}}$

$$\Sigma^{\mathcal{N}}_{A_{\mathscr{A}}} : \mathcal{N}_1(\mathrm{id}_{A_{\mathscr{A}}}) = \Big(\mathcal{F}_0(A_{\mathscr{A}}), \mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}}), \mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}})\Big) \Longrightarrow$$
$$\Longrightarrow \Big(\mathcal{F}_0(A_{\mathscr{A}}), \mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})}, \mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})}\Big) = \mathrm{id}_{\mathcal{N}_0(A_{\mathscr{A}})}$$

as the class of the following data; here $\sigma^{\mathcal{F}}_{\bullet}$ is as in (1.3) and the unitors $\pi_{\bullet}$ are the unitors in $\mathscr{B}$:

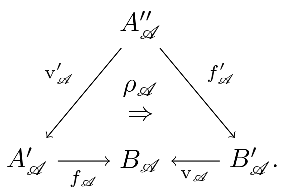


We omit the easy proof that actually this gives a set of unitors for $\mathcal{N}$.

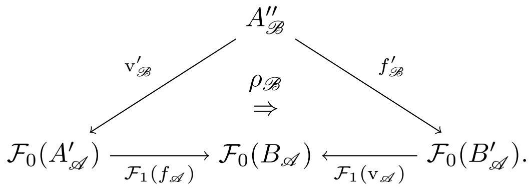
Now we want to compute a set of associators for $\mathcal{N}$. So let us fix any pair of composable 2-morphisms from $A_{\mathscr{A}}$ to $B_{\mathscr{A}}$ and from $B_{\mathscr{A}}$ to $C_{\mathscr{A}}$ in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$ as follows

$$\underline{f}_{\mathscr{A}} := \Big(A_{\mathscr{A}} \xleftarrow{\mathrm{w}_{\mathscr{A}}} A'_{\mathscr{A}} \xrightarrow{f_{\mathscr{A}}} B_{\mathscr{A}}\Big) \quad \text{and} \quad \underline{g}_{\mathscr{A}} := \Big(B_{\mathscr{A}} \xleftarrow{\mathrm{v}_{\mathscr{A}}} B'_{\mathscr{A}} \xrightarrow{g_{\mathscr{A}}} C_{\mathscr{A}}\Big)$$

(with both $\mathrm{w}_{\mathscr{A}}$ and $\mathrm{v}_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$) and let us suppose that choices $\mathrm{C}(\mathbf{W}_{\mathscr{A}})$ give data as in the upper part of the following diagram, with $\mathrm{v}'_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\rho_{\mathscr{A}}$ invertible:



Moreover, let us suppose that choices $\mathrm{C}(\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ give data as in the upper part of the following diagram, with $\mathrm{v}'_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\rho_{\mathscr{B}}$ invertible:



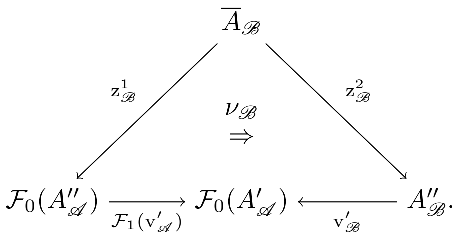
Then we have

$$\mathcal{N}_1\left(\underline{g}_{\mathscr{A}}\circ\underline{f}_{\mathscr{A}}\right):=\left(\mathcal{F}_0(A_{\mathscr{A}})\xleftarrow{\mathcal{F}_1(\mathrm{w}_{\mathscr{A}}\circ\mathrm{v}'_{\mathscr{A}})}\mathcal{F}_0(A''_{\mathscr{A}})\xrightarrow{\mathcal{F}_1(g_{\mathscr{A}}\circ f'_{\mathscr{A}})}\mathcal{F}_0(C_{\mathscr{A}})\right)$$

and

$$\mathcal{N}_1\left(\underline{g}_{\mathscr{A}}\right)\circ\mathcal{N}_1\left(\underline{f}_{\mathscr{A}}\right):=\left(\mathcal{F}_0(A_{\mathscr{A}})\xleftarrow{\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})\circ\mathrm{v}'_{\mathscr{B}}}A''_{\mathscr{B}}\xrightarrow{\mathcal{F}_1(g_{\mathscr{A}})\circ f'_{\mathscr{B}}}\mathcal{F}_0(C_{\mathscr{A}})\right).$$

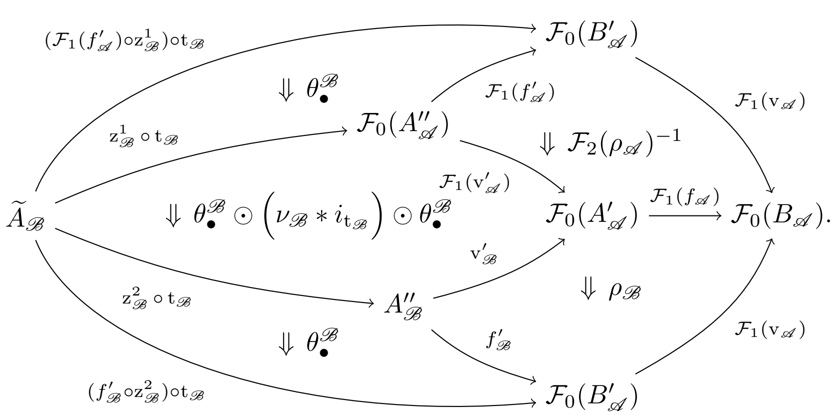
Using (BF3) for $(\mathscr{B},\mathbf{W}_{\mathscr{B},\mathrm{sat}})$, there are data as in the upper part of the following diagram, with $\mathrm{z}^1_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\nu_{\mathscr{B}}$ invertible:



Moreover, using (BF4a) and (BF4b) for $(\mathscr{B},\mathbf{W}_{\mathscr{B},\mathrm{sat}})$, there is an object $\widetilde{A}_{\mathscr{B}}$, a morphism $\mathrm{t}_{\mathscr{B}}:\widetilde{A}_{\mathscr{B}}\to\overline{A}_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and an invertible 2-morphism

$$\beta_{\mathscr{B}}:\,(\mathcal{F}_1(f'_{\mathscr{A}})\circ\mathrm{z}^1_{\mathscr{B}})\circ\mathrm{t}_{\mathscr{B}}\Longrightarrow(f'_{\mathscr{B}}\circ\mathrm{z}^2_{\mathscr{B}})\circ\mathrm{t}_{\mathscr{B}},$$

such that $i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}*\beta_{\mathscr{B}}$ is equal to the following composition (where all the 2-morphisms $\theta_\bullet$ are associators or inverses of associators in $\mathscr{B}$):

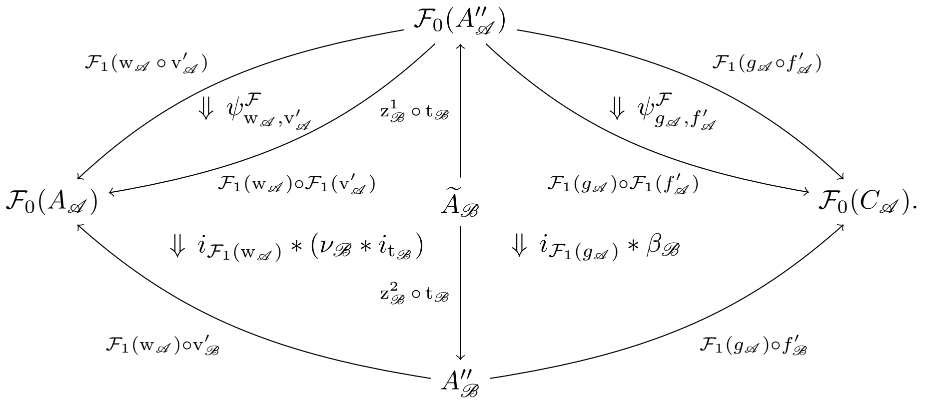


Then we define the associator

$$\Psi^{\mathcal{N}}_{\underline{g}_{\mathscr{A}},\underline{f}_{\mathscr{A}}}:\,\left(\mathcal{F}_0(A''_{\mathscr{A}}),\mathcal{F}_1(\mathrm{w}_{\mathscr{A}}\circ\mathrm{v}'_{\mathscr{A}}),\mathcal{F}_1(g_{\mathscr{A}}\circ f'_{\mathscr{A}})\right)\Longrightarrow$$

$$\Longrightarrow \Big(A''_{\mathscr{B}}, \mathcal{F}_1(\mathrm{w}_{\mathscr{A}}) \circ \mathrm{v}'_{\mathscr{B}}, \mathcal{F}_1(g_{\mathscr{A}}) \circ f'_{\mathscr{B}}\Big) \tag{2.2}$$

as the class of the following diagram, where for simplicity we have omitted all the necessary associators $\theta_{\bullet}$ for $\mathscr{B}$:



(2.3)

Since $\beta_{\mathscr{B}}$ is invertible, so is the class of (2.3) in $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$. Then it is easy to prove that

Lemma 2.1. *The equivalence class* $\Psi^{\mathcal{N}}_{\underline{g}_{\mathscr{A}},\underline{f}_{\mathscr{A}}}$ *of* (2.3) *does not depend on the* 2 *choices done using axioms* (BF3) *and* (BF4) *above.*

Now we have to prove that all the coherence axioms for a pseudofunctor are satisfied by the set of data $\mathcal{N} := (\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \Sigma^{\mathcal{N}}_{\bullet}, \Psi^{\mathcal{N}}_{\bullet})$. Namely, we have to prove that:

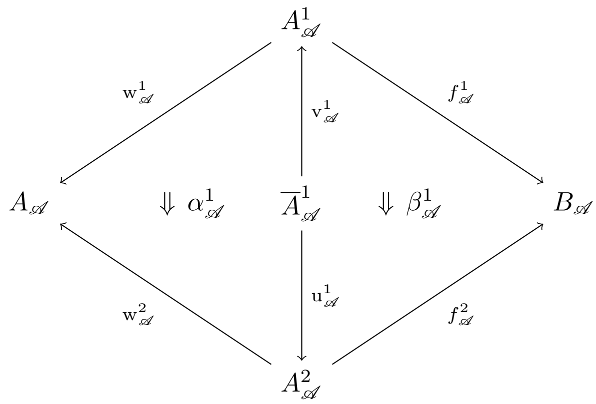
- $\mathcal{N}$ preserves vertical compositions;
- it is compatible with associators of $\mathscr{A}$ and $\mathscr{B}$;
- it is compatible with horizontal compositions.

This will be the aim of the remaining part of this paper. **For simplicity of exposition, from now we will restrict to the special case when $\mathscr{A}$ and $\mathscr{B}$ are both 2-categories (instead of bicategories) and $\mathcal{F}$ is a strict pseudofunctor (i.e. it preserves compositions and identities).** The interested reader can easily fill out the missing details for the general case.

3. $\mathcal{N}$ PRESERVES VERTICAL COMPOSITIONS

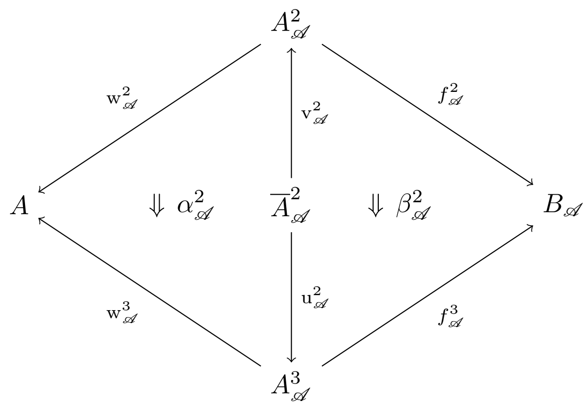
Lemma 3.1. *Let us fix any pair of objects* $A_{\mathscr{A}}, B_{\mathscr{A}}$*, any triple of morphisms* $\underline{f}^m_{\mathscr{A}} = (A^m_{\mathscr{A}}, \mathrm{w}^m_{\mathscr{A}}, f^m_{\mathscr{A}}) : A_{\mathscr{A}} \to B_{\mathscr{A}}$ *for* $m = 1, 2, 3$*, any* 2*-morphism* $\Gamma^1_{\mathscr{A}} : \underline{f}^1_{\mathscr{A}} \Rightarrow \underline{f}^2_{\mathscr{A}}$ *and any* 2*-morphism* $\Gamma^2_{\mathscr{A}} : \underline{f}^2_{\mathscr{A}} \Rightarrow \underline{f}^3_{\mathscr{A}}$ *in* $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$*. Then* $\mathcal{N}_2(\Delta_{\mathscr{A}} \odot \Gamma_{\mathscr{A}}) = \mathcal{N}_2(\Delta_{\mathscr{A}}) \odot \mathcal{N}_2(\Gamma_{\mathscr{A}})$*.*

Proof. Let us fix any representative



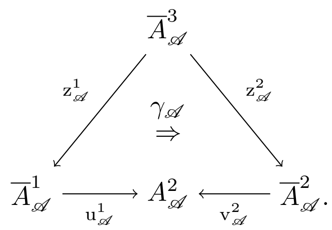
(3.1)

for $\Gamma^1_{\mathscr{A}}$ and any representative



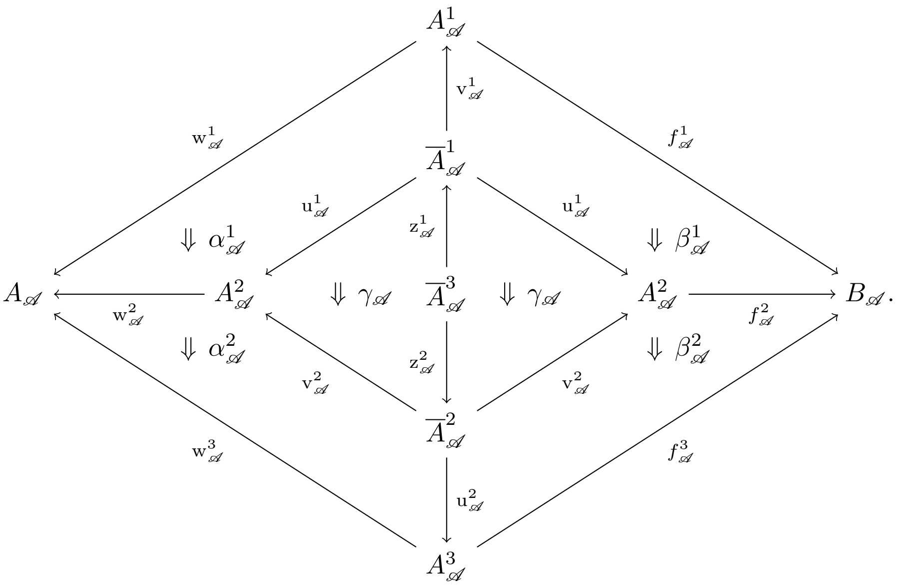
(3.2)

for $\Gamma^2_{\mathscr{A}}$. Following the definition of vertical composition given in [Pr, pag. 258], we use choices $\mathrm{C}(\mathbf{W}_{\mathscr{A}})$ in order to get data as in the upper part of the following diagram, with $\mathrm{z}^1_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\gamma_{\mathscr{A}}$ invertible:

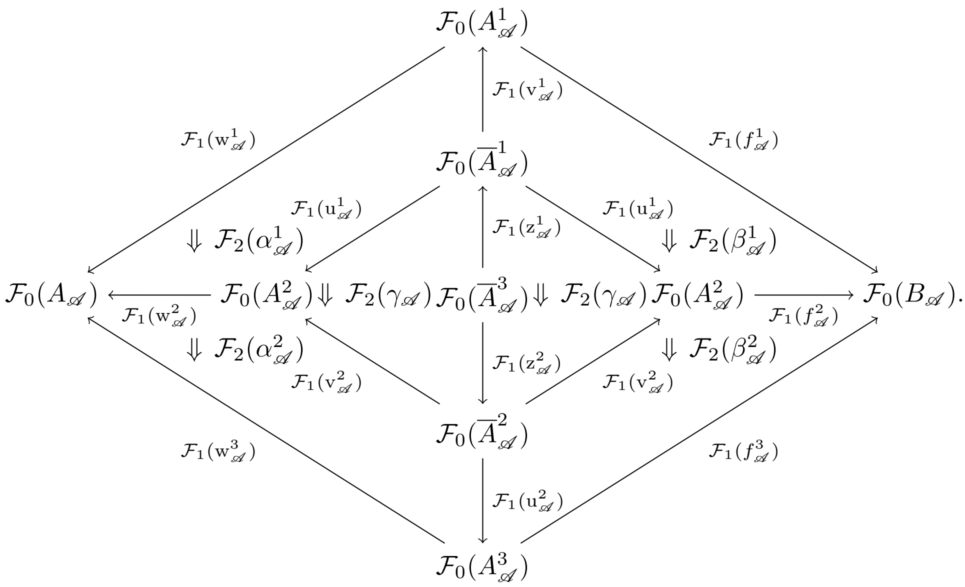


(3.3)

Then $\Gamma^2_{\mathscr{A}} \odot \Gamma^1_{\mathscr{A}}$ is represented by the data in the following diagram:

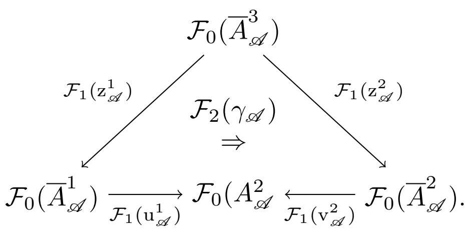


Since we are assuming that $\mathcal{F}$ is a strict pseudofunctor, then $\mathcal{N}_2(\Gamma^2_{\mathscr{A}} \odot \Gamma^1_{\mathscr{A}})$ is represented by the following diagram:



(3.4)

Now let us consider the following diagram:



(3.5)

By hypothesis, $\mathcal{F}_1(\mathrm{z}^1_{\mathscr{A}})$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$; moreover $\mathcal{F}_2(\gamma_{\mathscr{A}})$ is invertible because $\gamma_{\mathscr{A}}$ is so. Therefore, by [T1, Proposition 0.2] for $(\mathscr{C},\mathbf{W}) := (\mathscr{B},\mathbf{W}_{\mathscr{B},\mathrm{sat}})$, using diagram (3.5) we conclude that diagram (3.4) represents $\mathcal{N}_2(\Gamma^2_{\mathscr{A}}) \odot \mathcal{N}_2(\Gamma^1_{\mathscr{A}})$, so we are done. □

4. $\mathcal{N}$ is compatible with associators

To say that $\mathcal{N}$ is compatible with associators is equivalent to proving the following:

Lemma 4.1. *Given any triple of morphisms $\underline{f}_{\mathscr{A}}, \underline{g}_{\mathscr{A}}, \underline{h}_{\mathscr{A}}$ as in* (1.4)*, the composition of the following* 2*-morphisms*

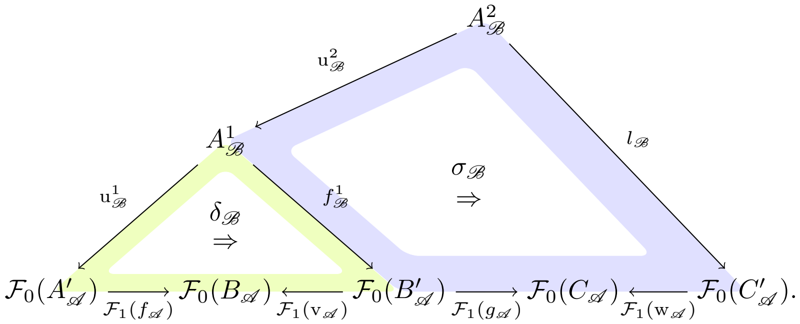
$$
\mathcal{F}_0(A_{\mathscr{A}}) \longrightarrow \mathcal{F}_0(D_{\mathscr{A}}):
\quad
\begin{array}{c}
F^6_{\mathscr{B}} := \mathcal{N}_1(\underline{h}_{\mathscr{A}}) \circ \big(\mathcal{N}_1(\underline{g}_{\mathscr{A}}) \circ \mathcal{N}_1(\underline{f}_{\mathscr{A}})\big) \\
\Downarrow\, G^5_{\mathscr{B}} := i_{\mathcal{N}_1(\underline{h}_{\mathscr{A}})} * \left(\Psi^{\mathcal{N}}_{\underline{g}_{\mathscr{A}},\underline{f}_{\mathscr{A}}}\right)^{-1} \\
F^5_{\mathscr{B}} := \mathcal{N}_1(\underline{h}_{\mathscr{A}}) \circ \mathcal{N}_1(\underline{g}_{\mathscr{A}} \circ \underline{f}_{\mathscr{A}}) \\
\Downarrow\, G^4_{\mathscr{B}} := \left(\Psi^{\mathcal{N}}_{\underline{h}_{\mathscr{A}},\underline{g}_{\mathscr{A}}\circ \underline{f}_{\mathscr{A}}}\right)^{-1} \\
F^4_{\mathscr{B}} := \mathcal{N}_1\big(\underline{h}_{\mathscr{A}} \circ (\underline{g}_{\mathscr{A}} \circ \underline{f}_{\mathscr{A}})\big) \\
\Downarrow\, G^3_{\mathscr{B}} := \mathcal{N}_2\left(\Theta^{\mathscr{A},\mathbf{W}_{\mathscr{A}}}_{\underline{h}_{\mathscr{A}},\underline{g}_{\mathscr{A}},\underline{f}_{\mathscr{A}}}\right) \\
F^3_{\mathscr{B}} := \mathcal{N}_1\big((\underline{h}_{\mathscr{A}} \circ \underline{g}_{\mathscr{A}}) \circ \underline{f}_{\mathscr{A}}\big) \\
\Downarrow\, G^2_{\mathscr{B}} := \Psi^{\mathcal{N}}_{\underline{h}_{\mathscr{A}}\circ \underline{g}_{\mathscr{A}},\underline{f}_{\mathscr{B}}} \\
F^2_{\mathscr{B}} := \mathcal{N}_1(\underline{h}_{\mathscr{A}} \circ \underline{g}_{\mathscr{A}}) \circ \mathcal{N}_1(\underline{f}_{\mathscr{A}}) \\
\Downarrow\, G^1_{\mathscr{B}} := \Psi^{\mathcal{N}}_{\underline{h}_{\mathscr{A}},\underline{g}_{\mathscr{A}}} * i_{\mathcal{N}_1(\underline{f}_{\mathscr{A}})} \\
F^1_{\mathscr{B}} := \big(\mathcal{N}_1(\underline{h}_{\mathscr{A}}) \circ \mathcal{N}_1(\underline{g}_{\mathscr{A}})\big) \circ \mathcal{N}_1(\underline{f}_{\mathscr{A}})
\end{array}
\tag{4.1}
$$

coincides with the associator $\Theta^{\mathscr{B},\mathbf{W}_{\mathscr{B},\mathrm{sat}}}_{\mathcal{N}_1(\underline{h}_{\mathscr{A}}),\mathcal{N}_1(\underline{g}_{\mathscr{A}}),\mathcal{N}_1(\underline{f}_{\mathscr{A}})}$.

Here the 2-morphisms $\Psi^{\mathcal{N}}_{\bullet}$ are defined as in (2.2), the associator $\Theta^{\mathscr{A},\mathbf{W}_{\mathscr{A}}}_{\bullet}$ is computed as in Proposition 1.1 and the associator $\Theta^{\mathscr{B},\mathbf{W}_{\mathscr{B},\mathrm{sat}}}_{\bullet}$ can be obtained analogously.

Proof. We already assume all the notations of (1.5) and (1.7), so that identities (1.6) and (1.8) hold. First of all, we compute the morphisms $F^1_{\mathscr{B}}, \cdots, F^6_{\mathscr{B}}$ mentioned above.

Let us suppose that choices $\mathrm{C}(\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ give data as in the upper parts of the following 2 polygons (starting from the smaller one), with $\mathrm{u}^1_{\mathscr{B}}$ and $\mathrm{u}^2_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\delta_{\mathscr{B}}$ and $\sigma_{\mathscr{B}}$ invertible:

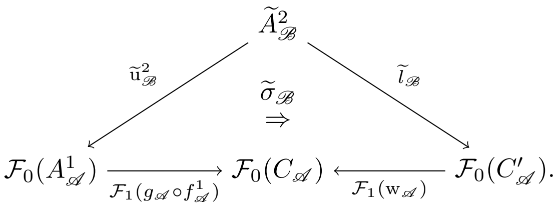


(4.2)

Then we have

$$F^6_{\mathscr{B}} := \Big(\mathcal{F}_0(A_{\mathscr{A}}) \xleftarrow{\mathcal{F}_1(\mathrm{u}_{\mathscr{A}})\circ \mathrm{u}^1_{\mathscr{B}}\circ \mathrm{u}^2_{\mathscr{B}}} A^2_{\mathscr{B}} \xrightarrow{\mathcal{F}_1(h_{\mathscr{A}})\circ l_{\mathscr{B}}} \mathcal{F}_0(D_{\mathscr{A}})\Big).$$

Moreover, let us suppose that choices $\mathrm{C}(\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ give data as in the upper part of the following diagram, with $\widetilde{\mathrm{u}}^2_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\widetilde{\sigma}_{\mathscr{B}}$ invertible:



Then we have

$$F^5_{\mathscr{B}} := \Big(\mathcal{F}_0(A_{\mathscr{A}}) \xleftarrow{\mathcal{F}_1(\mathrm{u}_{\mathscr{A}}\circ \mathrm{u}^1_{\mathscr{A}})\circ \widetilde{\mathrm{u}}^2_{\mathscr{B}}} \widetilde{A}^2_{\mathscr{B}} \xrightarrow{\mathcal{F}_1(h_{\mathscr{A}})\circ \widetilde{l}_{\mathscr{B}}} \mathcal{F}_0(D_{\mathscr{A}})\Big).$$

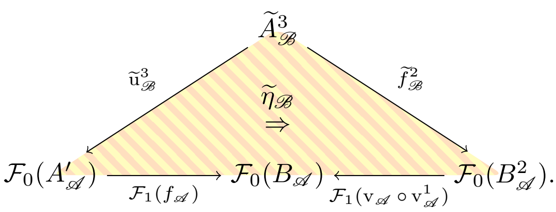
Using (1.6) and (1.8), we have

$$F^4_{\mathscr{B}} := \Big(\mathcal{F}_0(A_{\mathscr{A}}) \xleftarrow{\mathcal{F}_1(\mathrm{u}_{\mathscr{A}}\circ \mathrm{u}^1_{\mathscr{A}}\circ \mathrm{u}^2_{\mathscr{A}})} \mathcal{F}_0(A^2_{\mathscr{A}}) \xrightarrow{\mathcal{F}_1(h_{\mathscr{A}}\circ l_{\mathscr{A}})} \mathcal{F}_0(D_{\mathscr{A}})\Big)$$

and

$$F^3_{\mathscr{B}} := \Big(\mathcal{F}_0(A_{\mathscr{A}}) \xleftarrow{\mathcal{F}_1(\mathrm{u}_{\mathscr{A}}\circ \mathrm{u}^3_{\mathscr{A}})} \mathcal{F}_0(A^3_{\mathscr{A}}) \xrightarrow{\mathcal{F}_1(h_{\mathscr{A}}\circ g^1_{\mathscr{A}}\circ f^2_{\mathscr{A}})} \mathcal{F}_0(D_{\mathscr{A}})\Big).$$

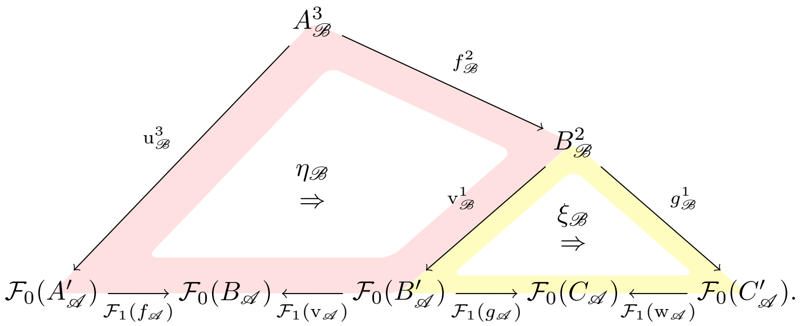
Let us suppose that choices $\mathrm{C}(\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ give data as in the upper part of the following diagram, with $\widetilde{\mathrm{u}}^3_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\widetilde{\eta}_{\mathscr{B}}$ invertible:



Then we have

$$F^2_{\mathscr{B}} := \Big(\mathcal{F}_0(A_{\mathscr{A}}) \xleftarrow{\mathcal{F}_1(\mathrm{u}_{\mathscr{A}})\circ \widetilde{\mathrm{u}}^3_{\mathscr{B}}} \widetilde{A}^3_{\mathscr{B}} \xrightarrow{\mathcal{F}_1(h_{\mathscr{A}}\circ g^1_{\mathscr{A}})\circ \widetilde{f}^2_{\mathscr{B}}} \mathcal{F}_0(D_{\mathscr{A}})\Big).$$

Lastly, let us suppose that choices $\mathrm{C}(\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ give data as in the upper parts of the following 2 polygons (starting from the smaller one), with $\mathrm{v}^1_{\mathscr{B}}$ and $\mathrm{u}^3_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\xi_{\mathscr{B}}$ and $\eta_{\mathscr{B}}$ invertible:

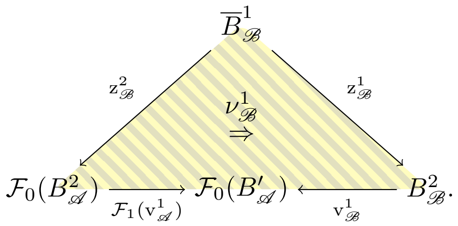


(4.3)

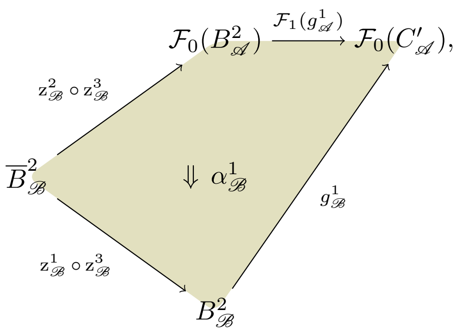
Then

$$F^1_{\mathscr{B}} := \Big(\mathcal{F}_0(A_{\mathscr{A}}) \xleftarrow{\mathcal{F}_1(\mathrm{u}_{\mathscr{A}})\circ \mathrm{u}^3_{\mathscr{B}}} A^3_{\mathscr{B}} \xrightarrow{\mathcal{F}_1(h_{\mathscr{A}})\circ g^1_{\mathscr{B}}\circ f^2_{\mathscr{B}}} \mathcal{F}_0(D_{\mathscr{A}})\Big).$$

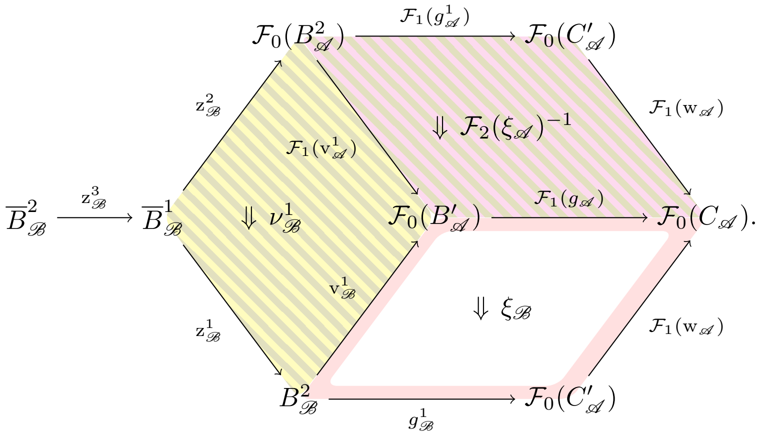
Now we compute the 2-morphism $G^1_{\mathscr{B}}$ appearing in (4.1). We use (BF3) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get data as in the upper part of the following diagram, with $\nu^1_{\mathscr{B}}$ invertible and $\mathrm{z}^2_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$:



We use axiom (BF4a) and (BF4b) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get an object $\overline{B}^2_{\mathscr{B}}$, a morphism $\mathrm{z}^3_{\mathscr{B}} : \overline{B}^2_{\mathscr{B}} \to \overline{B}^1_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and an invertible 2-morphism

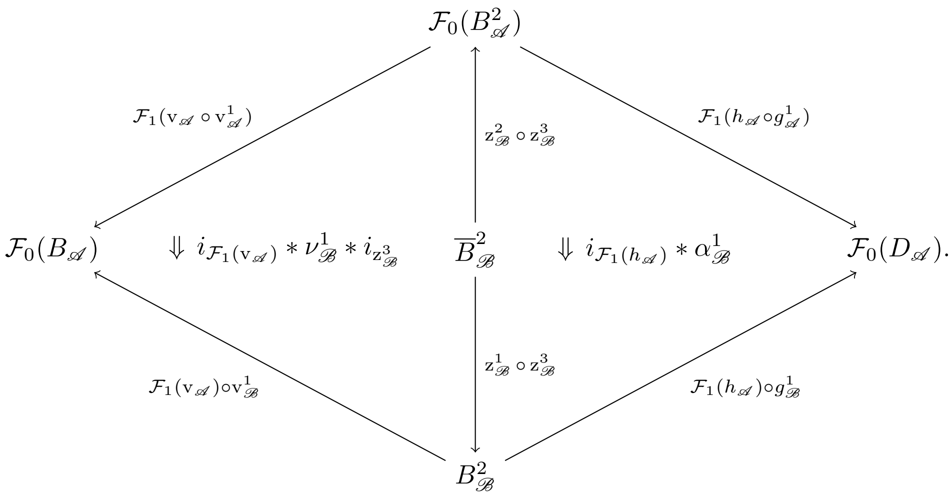


such that $i_{\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})} * \alpha^1_{\mathscr{B}}$ coincides with the following composition



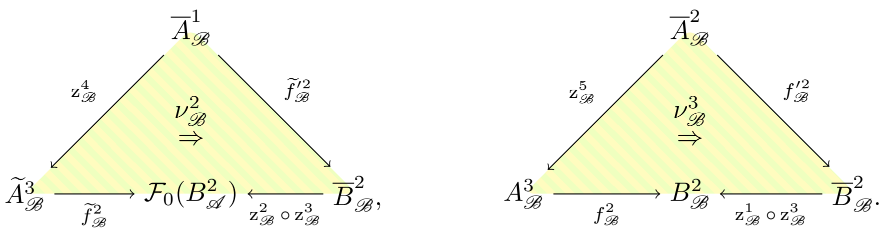
(4.4)

Then by Lemma 2.1 (and using the fact that we are assuming that $\mathcal{F}$ is strict), $\Psi^{\mathcal{N}}_{\underline{h}_{\mathscr{A}},\underline{g}_{\mathscr{A}}}$ is represented by the following diagram:

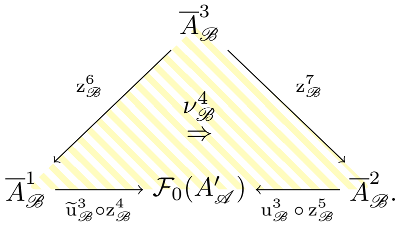


(4.5)

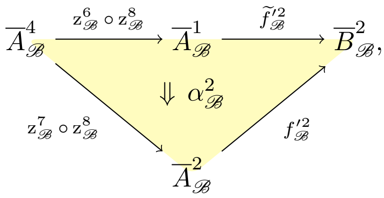
Now we use (BF3) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get data as in upper parts of the following 2 diagrams, with $\mathrm{z}^4_{\mathscr{B}}$ and $\mathrm{z}^5_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\nu^2_{\mathscr{B}}$ and $\nu^3_{\mathscr{B}}$ invertible:



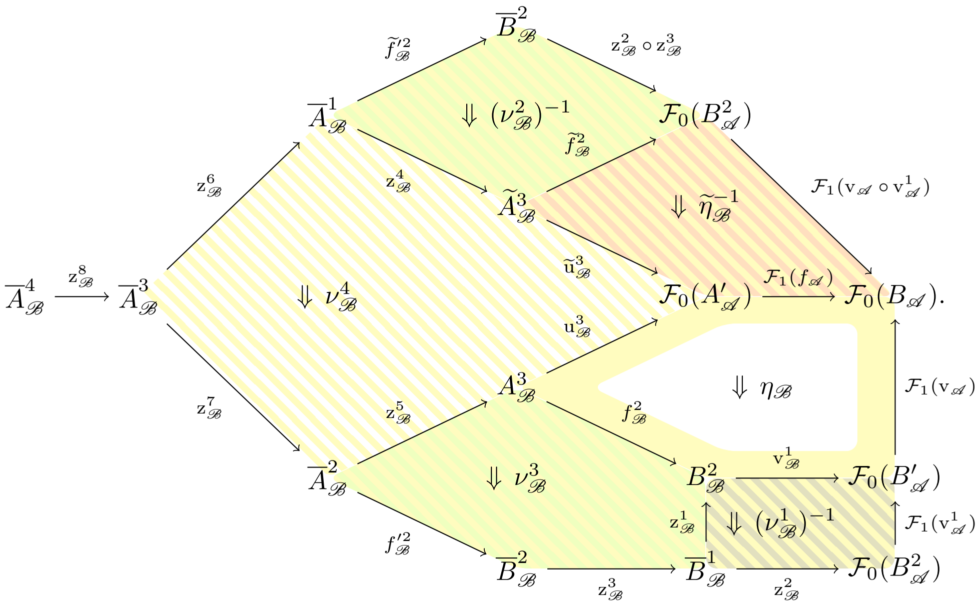
Then we use again (BF3) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get data as in the upper part of the following diagram, with $\mathrm{z}^6_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\nu^4_{\mathscr{B}}$ invertible:



Now we use axioms (BF4a) and (BF4b) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get an object $\overline{A}^4_{\mathscr{B}}$, a morphism $\mathrm{z}^8_{\mathscr{B}} : \overline{A}^4_{\mathscr{B}} \to \overline{A}^3_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and an invertible 2-morphism

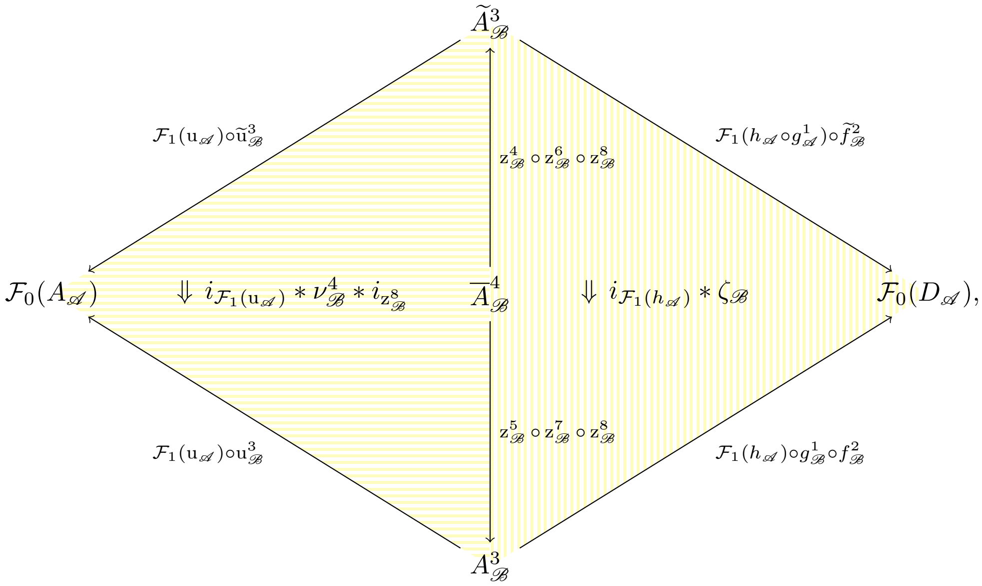


such that $i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}}) \circ \mathrm{z}^2_{\mathscr{B}} \circ \mathrm{z}^3_{\mathscr{B}}} * \alpha^2_{\mathscr{B}}$ coincides with the following composition



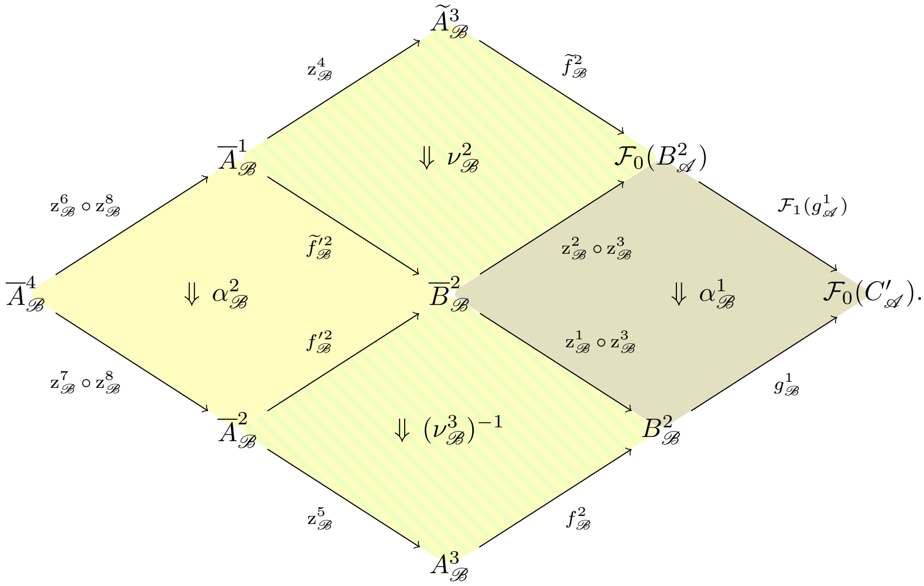
(4.6)

Therefore, using (4.5) and [T1, Proposition 0.3], we conclude that $G^1_{\mathscr{B}}$ coincides with the class of the following diagram:

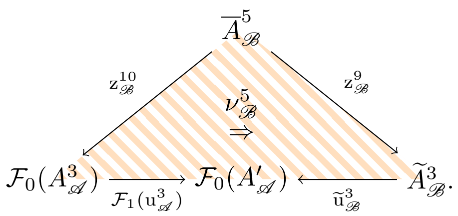


(4.7)

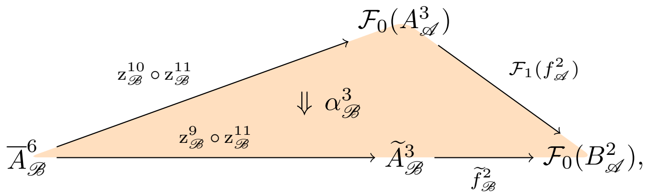
where $\zeta_{\mathscr{B}}$ is the following composition:



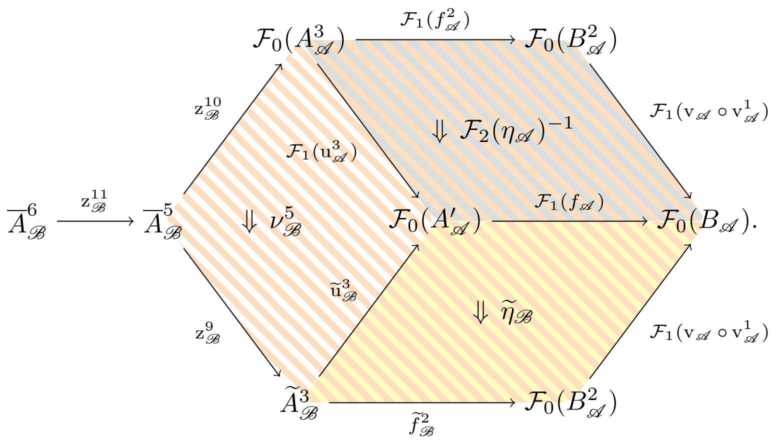
Now we are going to compute $G^2_{\mathscr{B}}$. Using (BF3) for $(\mathscr{B},\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ we obtain data as in the upper part of the following diagram, with $\mathrm{z}^{10}_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\nu^5_{\mathscr{B}}$ invertible:



Using (BF4a) and (BF4b) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$, there are an object $\overline{A}^6_{\mathscr{B}}$, a morphism $\mathrm{z}^{11}_{\mathscr{B}} : \overline{A}^6_{\mathscr{B}} \to \overline{A}^5_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and an invertible 2-morphism

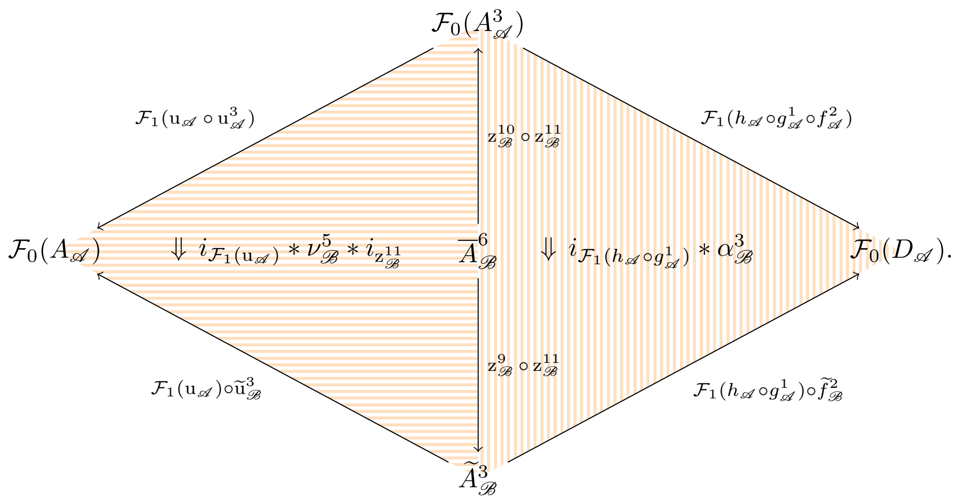


such that $i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}})} * \alpha^3_{\mathscr{B}}$ coincides with the following composition:



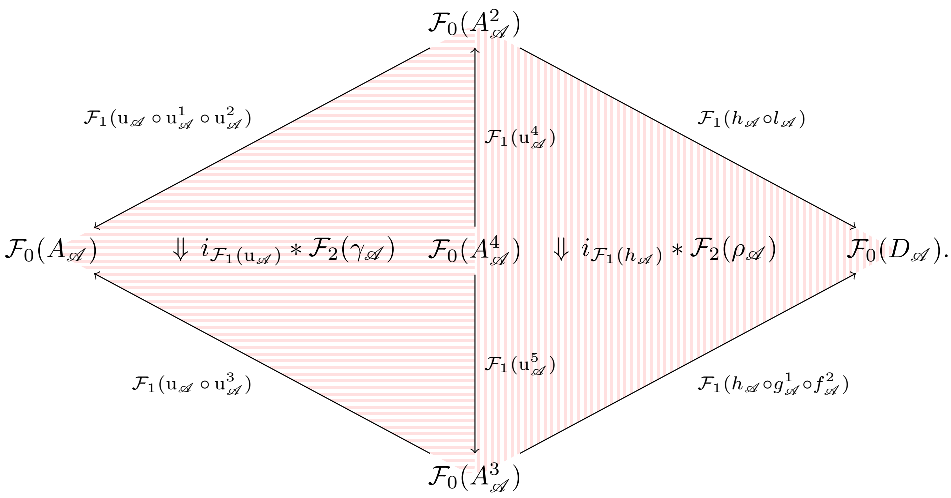
(4.8)

So using Lemma 2.1 for the pair $(\underline{f}_{\mathscr{A}}, \underline{h}_{\mathscr{A}} \circ \underline{g}_{\mathscr{A}})$, we get that $G^2_{\mathscr{B}}$ is represented by the data in the following diagram:



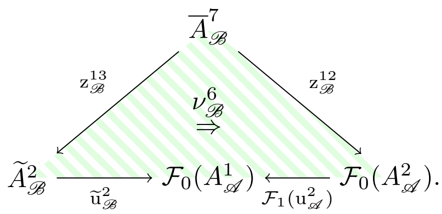
(4.9)

Now we compute $G^3_{\mathscr{B}}$: using Proposition 1.1 and the fact that $\mathcal{F}$ is strict, there is a set of data $(A^4_{\mathscr{A}}, \mathrm{u}^4_{\mathscr{A}}, \mathrm{u}^5_{\mathscr{A}}, \gamma_{\mathscr{A}}, \omega_{\mathscr{A}}, \rho_{\mathscr{A}})$ as in (F1) - (F3), such that $\Theta_{\underline{h}_{\mathscr{A}}, \underline{g}_{\mathscr{A}}, \underline{f}_{\mathscr{A}}}$ is represented by diagram (1.11), so $G^3_{\mathscr{B}}$ is represented by the following diagram:



(4.10)

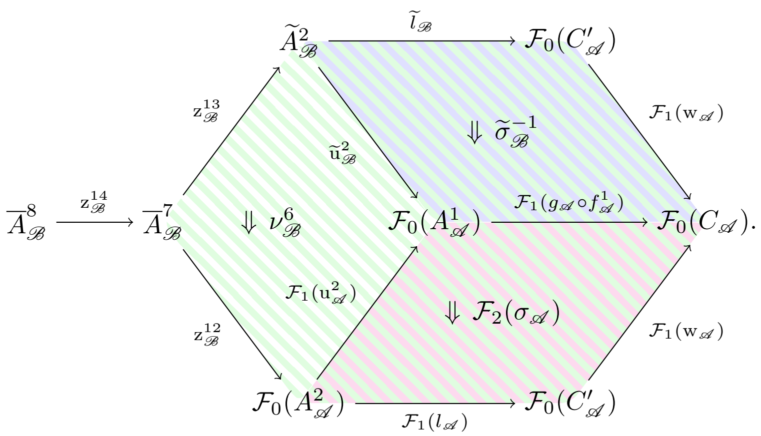
Now we compute $G^4_{\mathscr{B}}$. In order to do that, we use (BF3) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get data as in upper part of the following diagram, with $\mathrm{z}^{13}_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\nu^6_{\mathscr{B}}$ invertible:



Using (BF4a) and (BF4b) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$, we get an object $\overline{A}^8_{\mathscr{B}}$, a morphism $\mathrm{z}^{14}_{\mathscr{B}} : \overline{A}^8_{\mathscr{B}} \to \overline{A}^7_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and an invertible 2-morphism

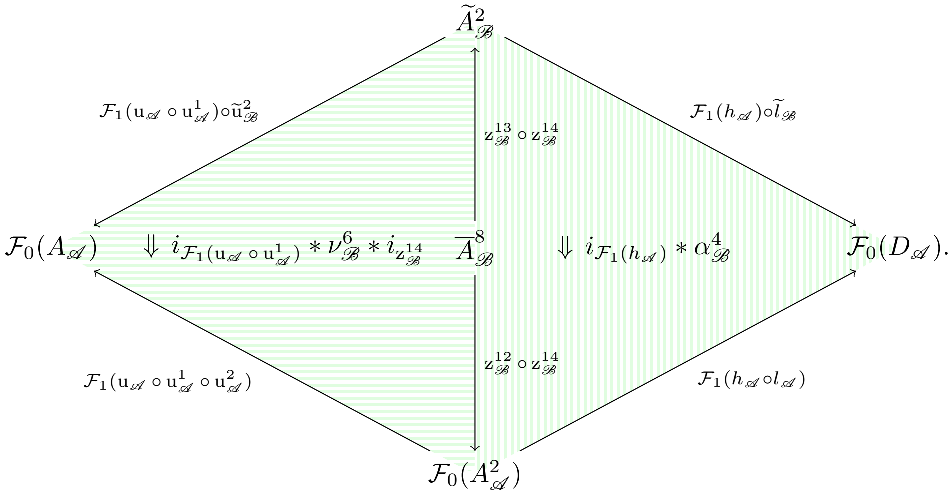
$$\alpha^4_{\mathscr{B}} : \widetilde{l}_{\mathscr{B}} \circ \mathrm{z}^{13}_{\mathscr{B}} \circ \mathrm{z}^{14}_{\mathscr{B}} \Longrightarrow \mathcal{F}_1(l_{\mathscr{A}}) \circ \mathrm{z}^{12}_{\mathscr{B}} \circ \mathrm{z}^{14}_{\mathscr{B}},$$

such that $i_{\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})} * \alpha^4_{\mathscr{B}}$ coincides with the following composition:



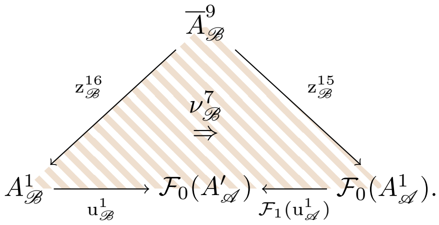
(4.11)

Then $G^4_{\mathscr{B}}$ is represented by the following diagram:



(4.12)

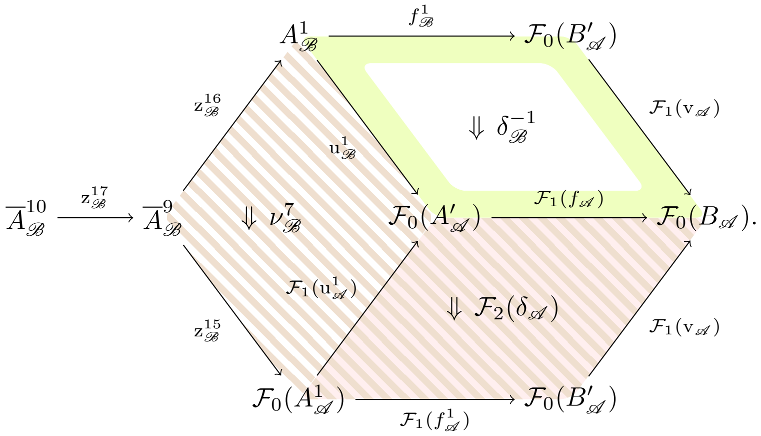
Lastly, we compute $G^5_{\mathscr{B}}$. In order to do that, we use (BF3) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get data as in the upper part of the diagram below, with $\mathrm{z}^{16}_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\nu^7_{\mathscr{B}}$ invertible:



Then we use (BF4a) and (BF4b) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get an object $\overline{A}^{10}_{\mathscr{B}}$, a morphism $\mathrm{z}^{17}_{\mathscr{B}} : \overline{A}^{10}_{\mathscr{B}} \to \overline{A}^9_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and an invertible 2-morphism

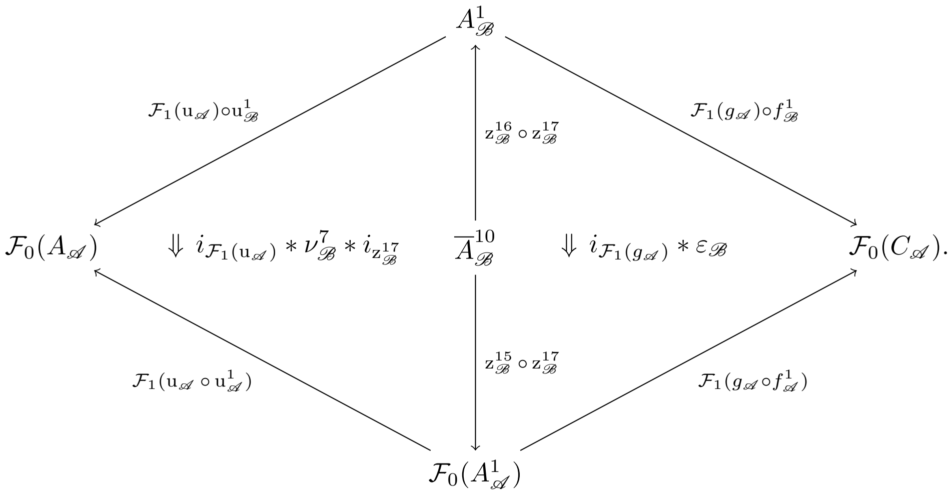
$$\varepsilon_{\mathscr{B}} :\ f^1_{\mathscr{B}} \circ \mathrm{z}^{16}_{\mathscr{B}} \circ \mathrm{z}^{17}_{\mathscr{B}} \Longrightarrow \mathcal{F}_1(f^1_{\mathscr{A}}) \circ \mathrm{z}^{15}_{\mathscr{B}} \circ \mathrm{z}^{17}_{\mathscr{B}},$$

such that $i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})} * \varepsilon_{\mathscr{B}}$ coincides with the following composition:

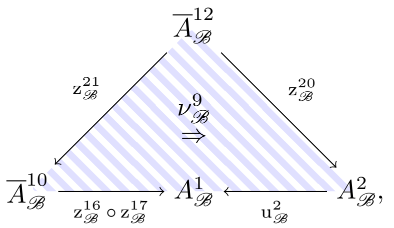


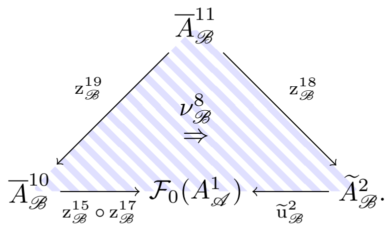
(4.13)

Now by construction both $\mathrm{u}^1_{\mathscr{B}}$ and $\mathrm{z}^{16}_{\mathscr{B}}$ belong to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$, so by (BF2) and (BF5) applied to $(\nu^7_{\mathscr{B}})^{-1}$ we conclude that $\mathcal{F}_1(\mathrm{u}^1_{\mathscr{A}})\circ\mathrm{z}^{15}_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$. Moreover, by construction also $\mathcal{F}_1(\mathrm{u}^1_{\mathscr{A}})$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$. Therefore, by [T2, Proposition 2.11(ii)] we conclude that also $\mathrm{z}^{15}_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$. So by Lemma 2.1 we can use $(\nu^7_{\mathscr{B}})^{-1}$ and $\varepsilon^{-1}_{\mathscr{B}}$ in order to compute $\Psi^{\mathcal{N}}_{\underline{g}_{\mathscr{A}},\underline{f}_{\mathscr{A}}}$. Taking the inverse, we get that $\left(\Psi^{\mathcal{N}}_{\underline{g}_{\mathscr{A}},\underline{f}_{\mathscr{A}}}\right)^{-1}$ is represented by the following diagram:

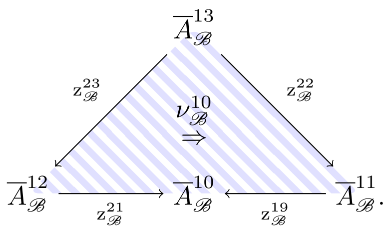


Following [T1, Proposition 0.4], we compute $G^5_{\mathscr{B}}$ as follows. We use (BF3) for $(\mathscr{B},\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get data as in the upper parts of the following 2 diagrams, with $\mathrm{z}^{21}_{\mathscr{B}}$ and $\mathrm{z}^{19}_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\nu^9_{\mathscr{B}}$ and $\nu^8_{\mathscr{B}}$ invertible:





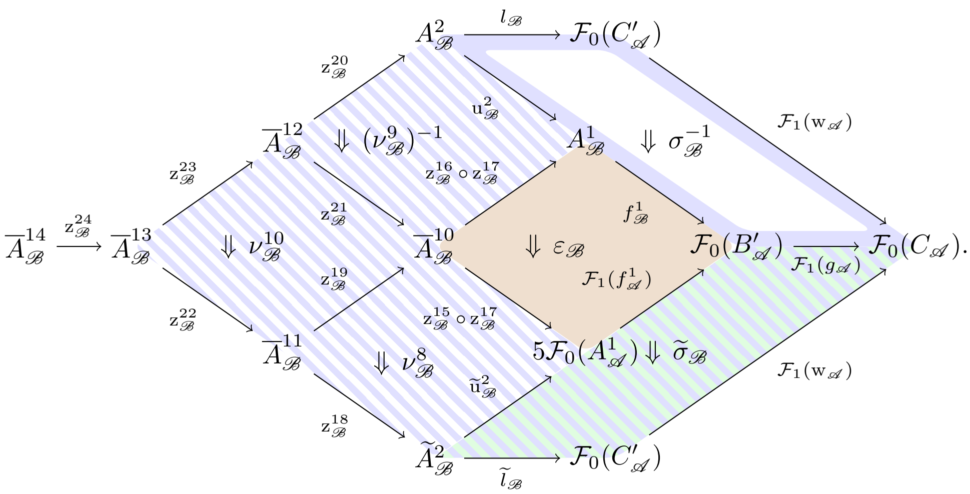
Now we use again (BF3) for $(\mathscr{B},\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get data as in the upper part of the following diagram, with $\mathrm{z}^{23}_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\nu^{10}_{\mathscr{B}}$ invertible:



Moreover, we use (BF4a) and (BF4b) for $(\mathscr{B},\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get an object $\overline{A}^{14}_{\mathscr{B}}$, a morphism $\mathrm{z}^{24}_{\mathscr{B}}:\overline{A}^{14}_{\mathscr{B}}\to\overline{A}^{13}_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and an invertible 2-morphism

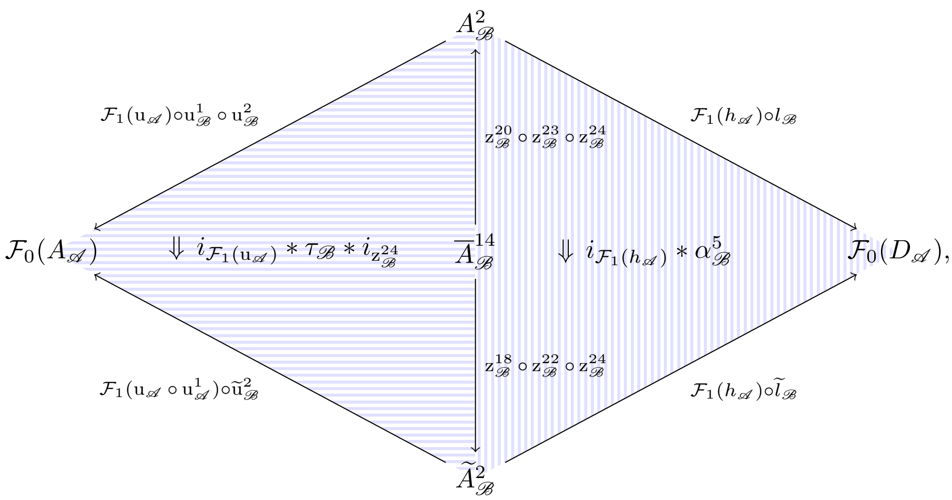
$$\alpha^5_{\mathscr{B}}:\ l_{\mathscr{B}}\circ\mathrm{z}^{20}_{\mathscr{B}}\circ\mathrm{z}^{23}_{\mathscr{B}}\circ\mathrm{z}^{24}_{\mathscr{B}}\Longrightarrow\widetilde{l}_{\mathscr{B}}\circ\mathrm{z}^{18}_{\mathscr{B}}\circ\mathrm{z}^{22}_{\mathscr{B}}\circ\mathrm{z}^{24}_{\mathscr{B}},$$

such that $i_{\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})}*\alpha^5_{\mathscr{B}}$ coincides with the following composition:



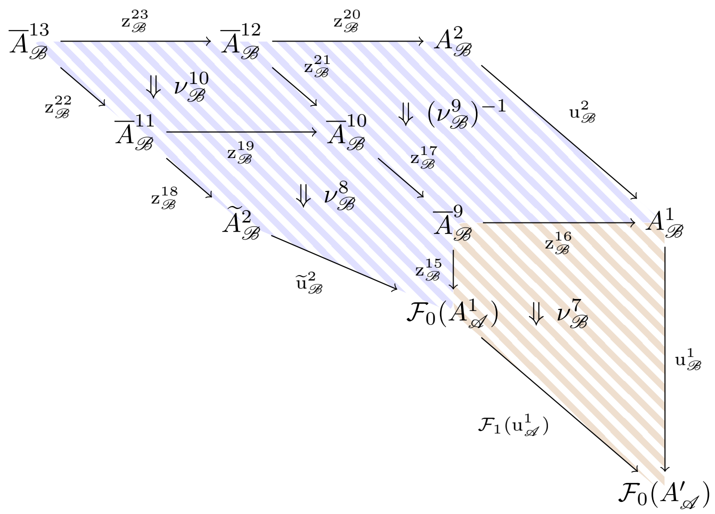
(4.14)

Then by [T1, Proposition 0.4] $G^5_{\mathscr{B}}$ is represented by the following diagram

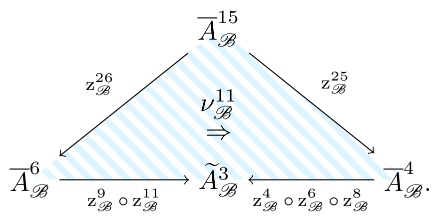


(4.15)

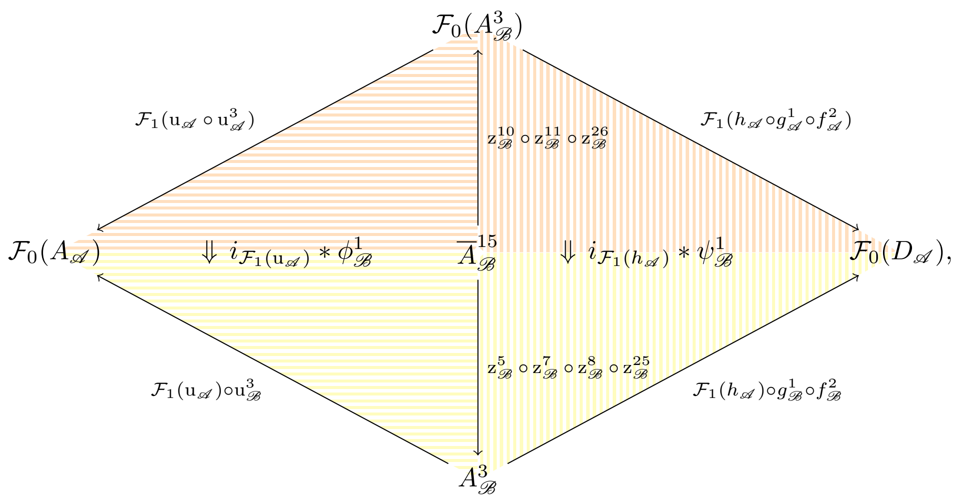
where $\tau_{\mathscr{B}}$ is the following composition:



Now we have to compose vertically all the 2-morphisms $G^1_{\mathscr{B}}, \cdots, G^5_{\mathscr{B}}$ obtained so far. We start by computing $G^1_{\mathscr{B}} \odot G^2_{\mathscr{B}}$ using (4.7) and (4.9). First of all, we use (BF3) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get data as in the upper part of the following diagram, with $z^{26}_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\nu^{11}_{\mathscr{B}}$ invertible:

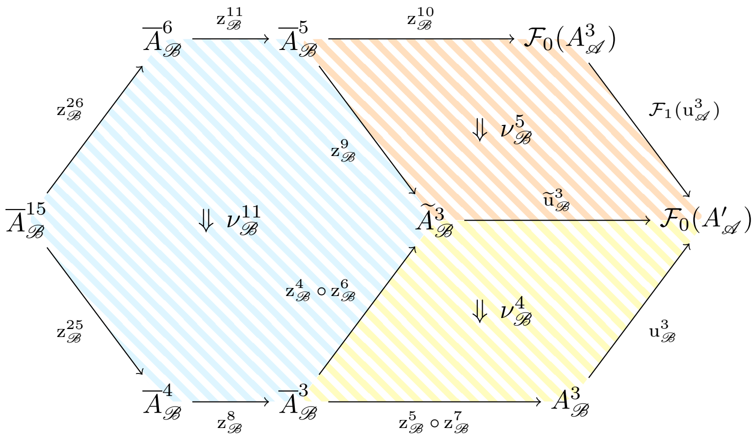


Then by [T1, Proposition 0.2], $G^1_{\mathscr{B}} \odot G^2_{\mathscr{B}}$ is represented by the following diagram

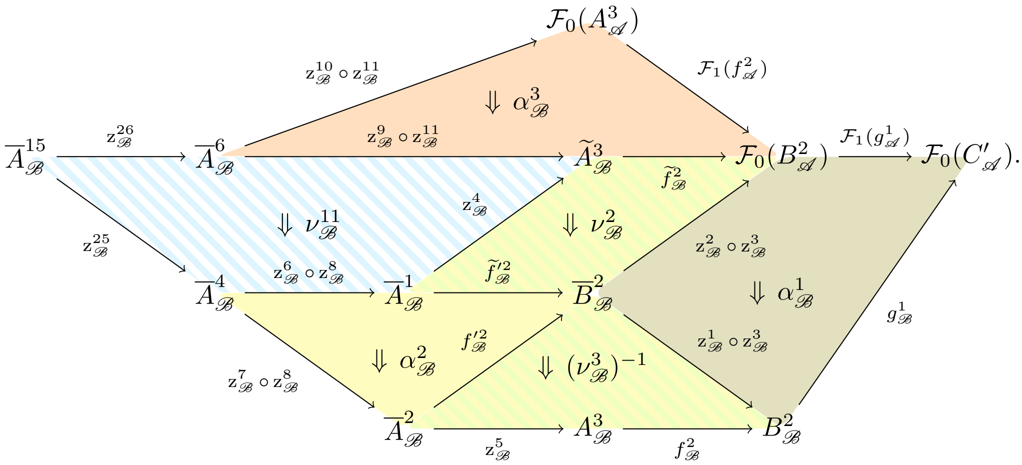


(4.16)

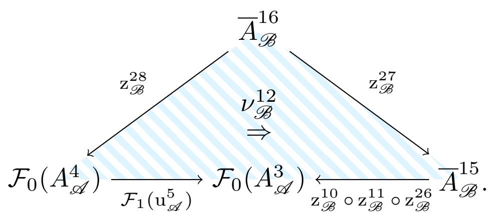
where $\phi^1_{\mathscr{B}}$ is the following composition:



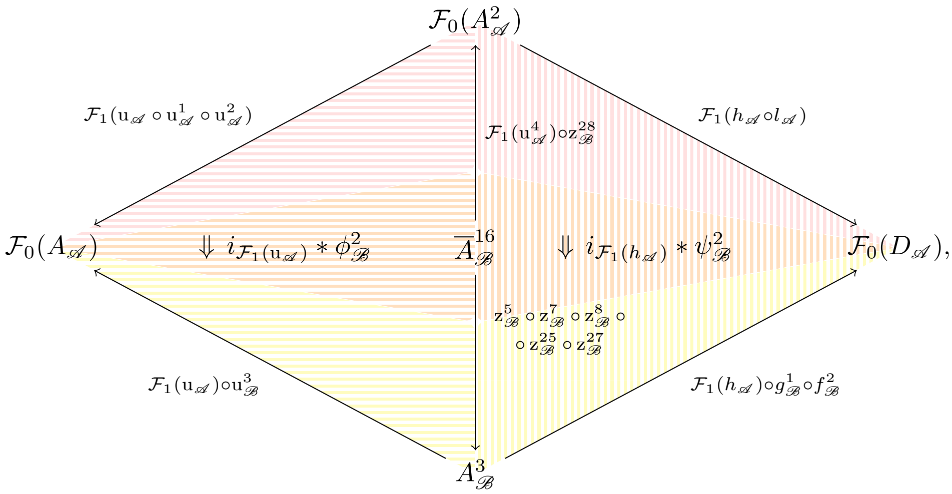
and $\psi^1_{\mathscr{B}}$ is the following composition



Now we compute $G^1_{\mathscr{B}} \odot G^2_{\mathscr{B}} \odot G^3_{\mathscr{B}}$ using (4.16) and (4.10). As a preliminary step, we use (BF3) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get data as in the upper part of the following diagram, with $z^{28}_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\nu^{12}_{\mathscr{B}}$ invertible:

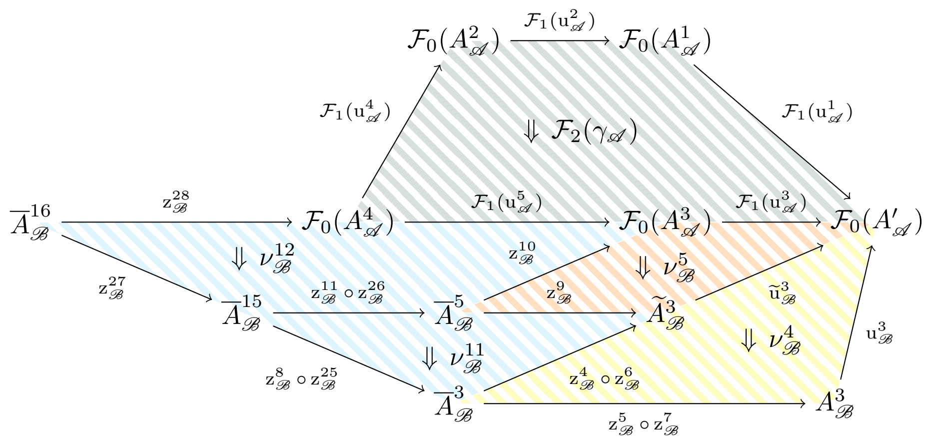


Then by [T1, Proposition 0.2] $G^1_{\mathscr{B}} \odot G^2_{\mathscr{B}} \odot G^3_{\mathscr{B}}$ is represented by the following diagram:

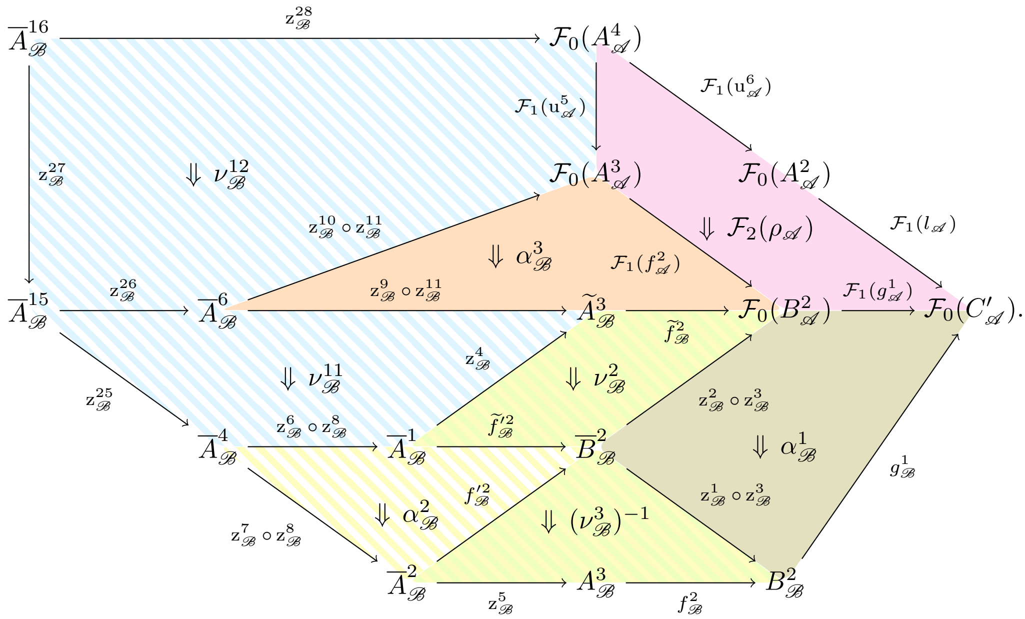


(4.17)

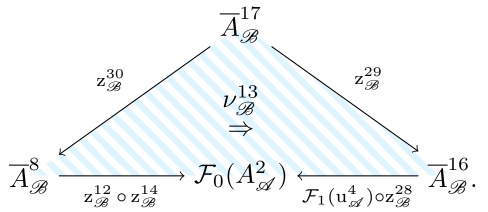
where $\phi^2_{\mathscr{B}}$ is the following composition:



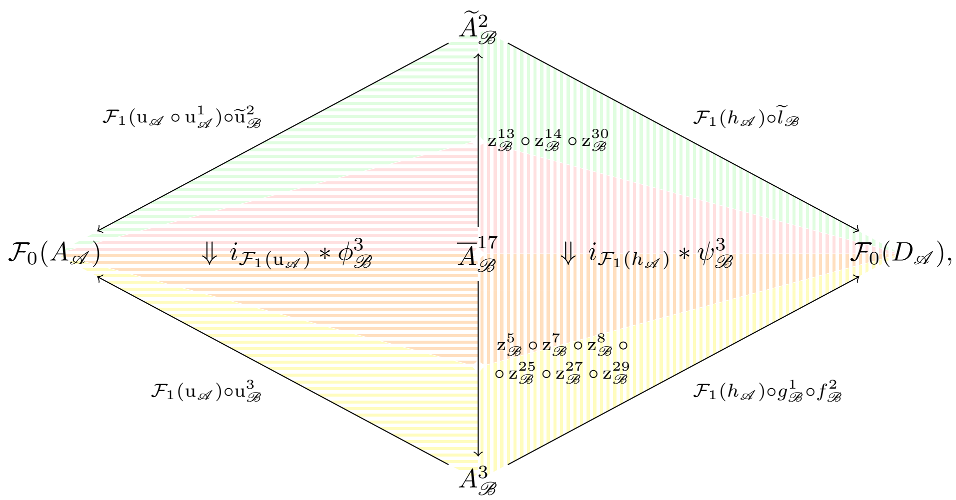
and $\psi^2_{\mathscr{B}}$ is the following composition:



Now we use (BF3) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get data as in the upper part of the following diagram, with $z^{30}_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\nu^{13}_{\mathscr{B}}$ invertible:

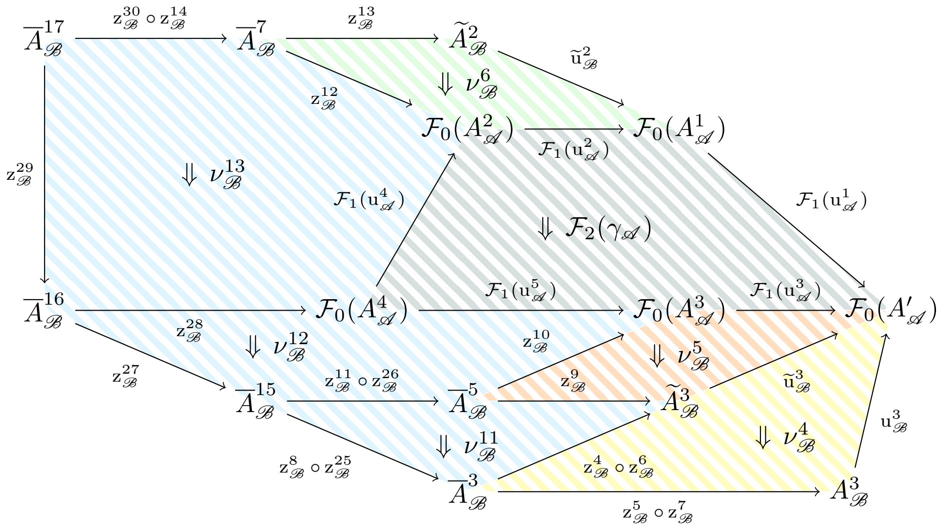


Then using (4.17), (4.12) and [T1, Proposition 0.2], we get that $G^1_{\mathscr{B}} \odot G^2_{\mathscr{B}} \odot G^3_{\mathscr{B}} \odot G^4_{\mathscr{B}}$ is represented by the following diagram

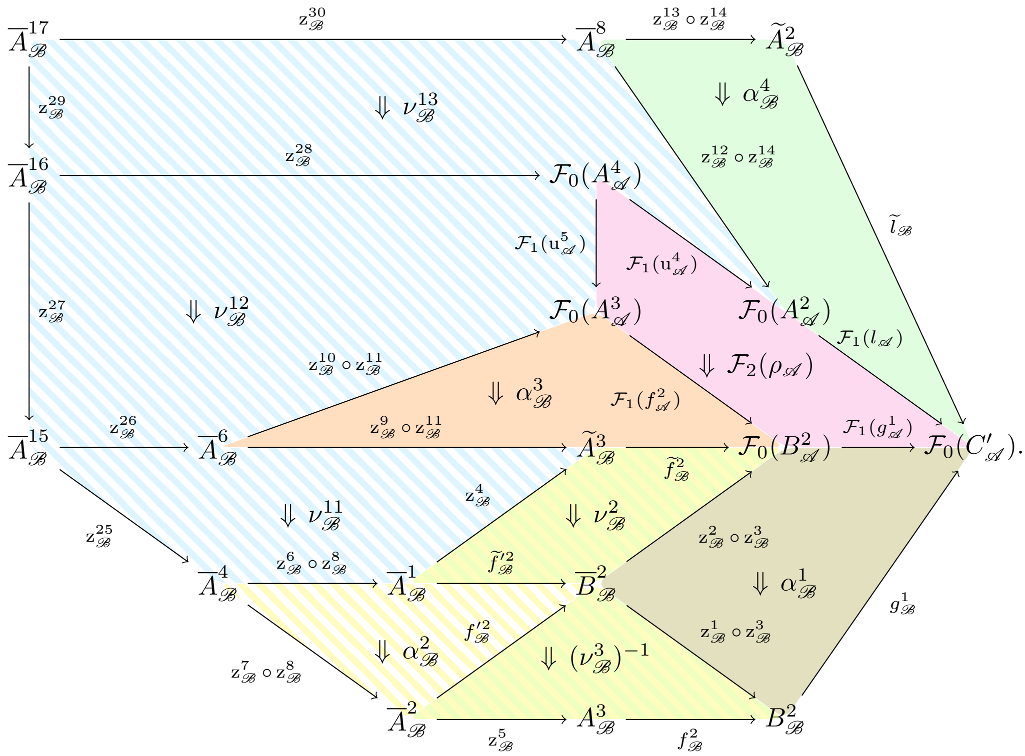


(4.18)

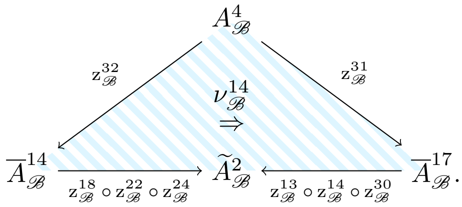
where $\phi^3_{\mathscr{B}}$ is the following composition:



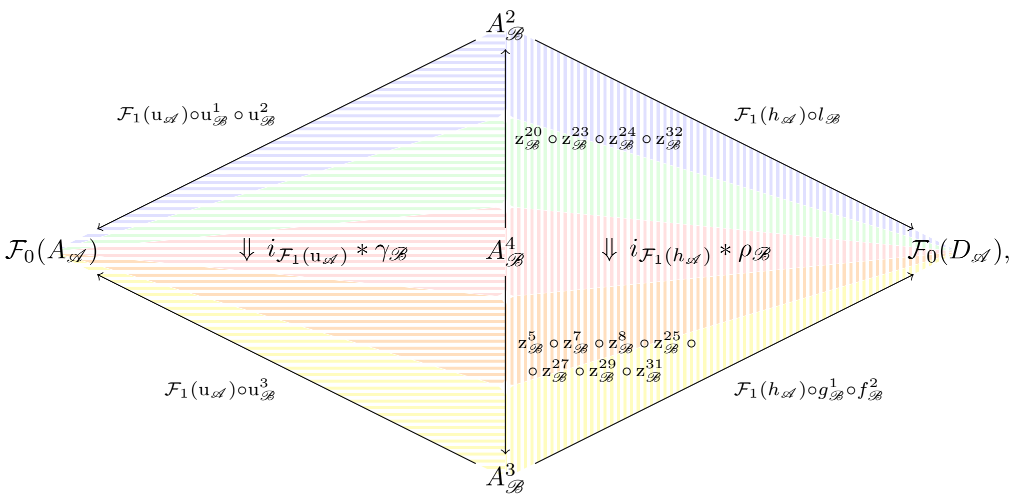
and $\psi^3_{\mathscr{B}}$ is the following composition:



Lastly, we use choices (BF3) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get data as in the upper part of the following diagram, with $\mathrm{z}^{32}_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\nu^{14}_{\mathscr{B}}$ invertible:

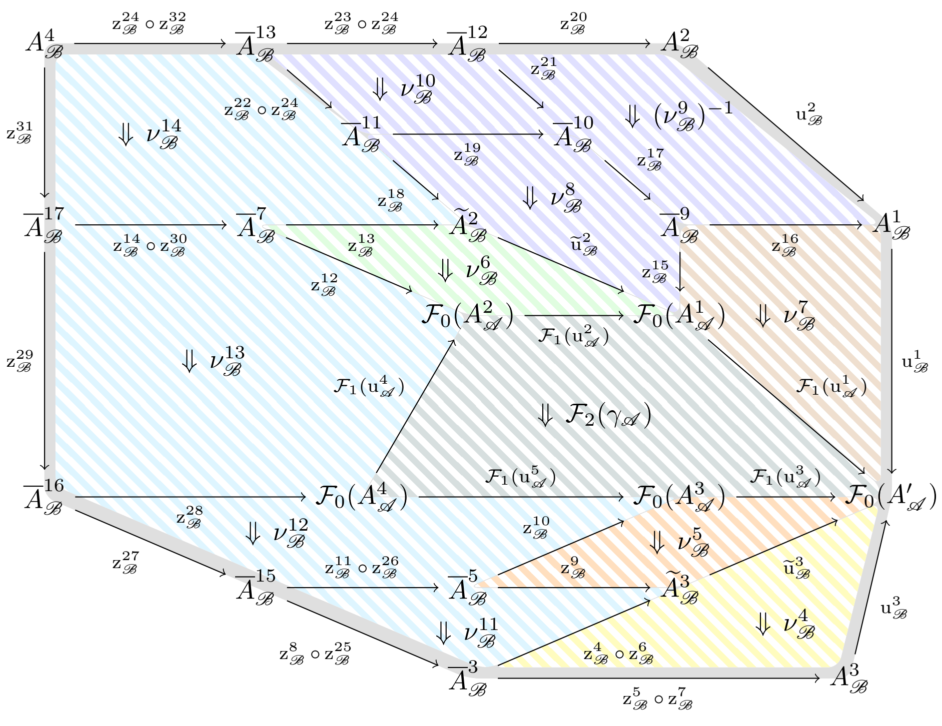


Then using (4.18), (4.15) and [T1, Proposition 0.2], we get that $G^1_{\mathscr{B}} \odot G^2_{\mathscr{B}} \odot G^3_{\mathscr{B}} \odot G^4_{\mathscr{B}} \odot G^5_{\mathscr{B}}$ is represented by the following diagram



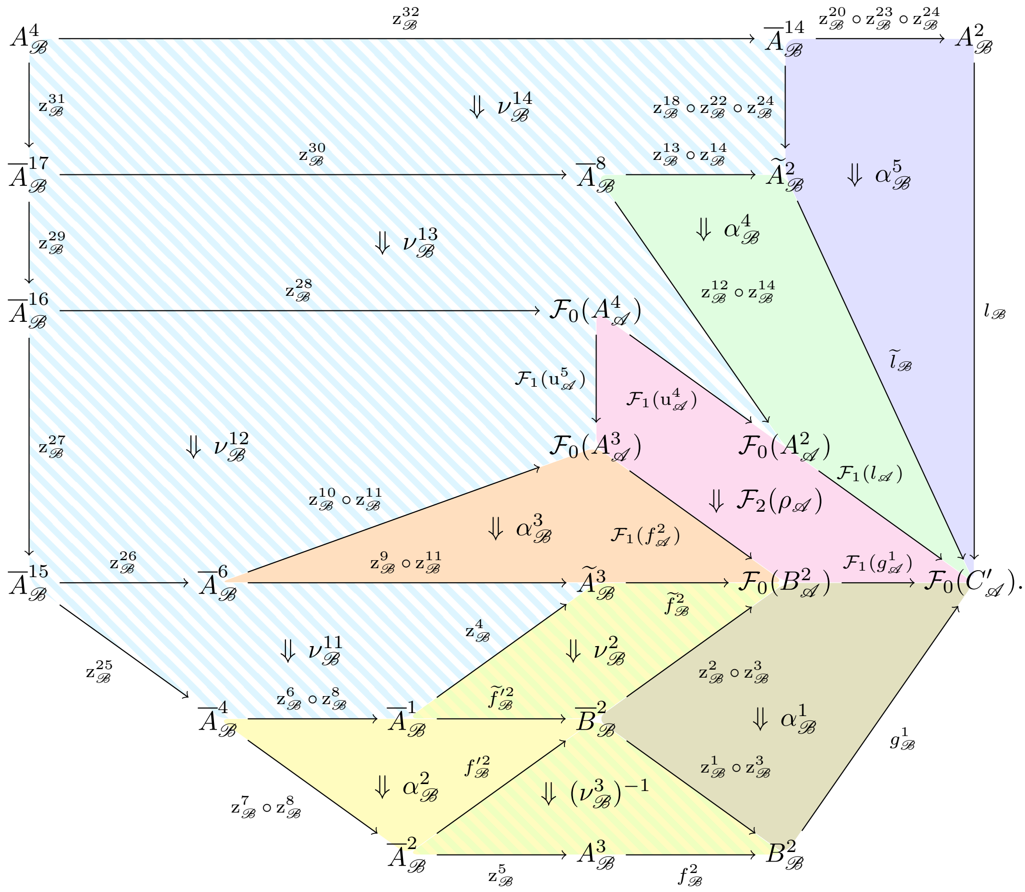
(4.19)

where $\gamma_{\mathscr{B}}$ is the following composition



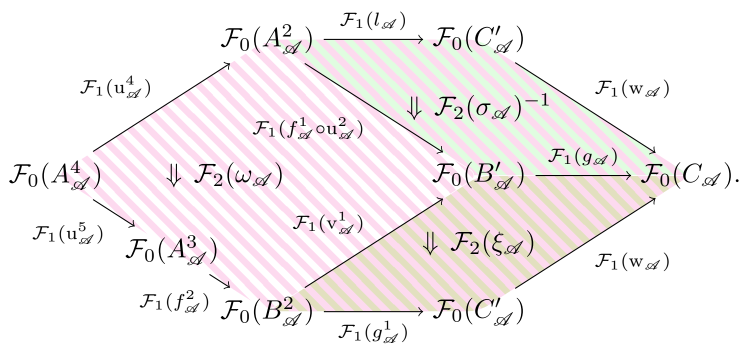
(4.20)

and $\rho_{\mathscr{B}}$ is the following composition:



(4.21)

In order to conclude, we want to apply Proposition 1.1 on $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$ (instead of $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$). In order to do that, as a first step we compute $i_{\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})} * \rho_{\mathscr{B}}$ (and compare it with the condition stated in (F3)). For that, first of all we recall that by construction $\rho_{\mathscr{A}}$ satisfies condition (F3) (for $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$), hence the term $i_{\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})} * \mathcal{F}_2(\rho_{\mathscr{A}}) = \mathcal{F}_2(i_{\mathrm{w}_{\mathscr{A}}} * \rho_{\mathscr{A}})$ in $i_{\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})} *$(4.21) coincides with the following composition:



(4.22)

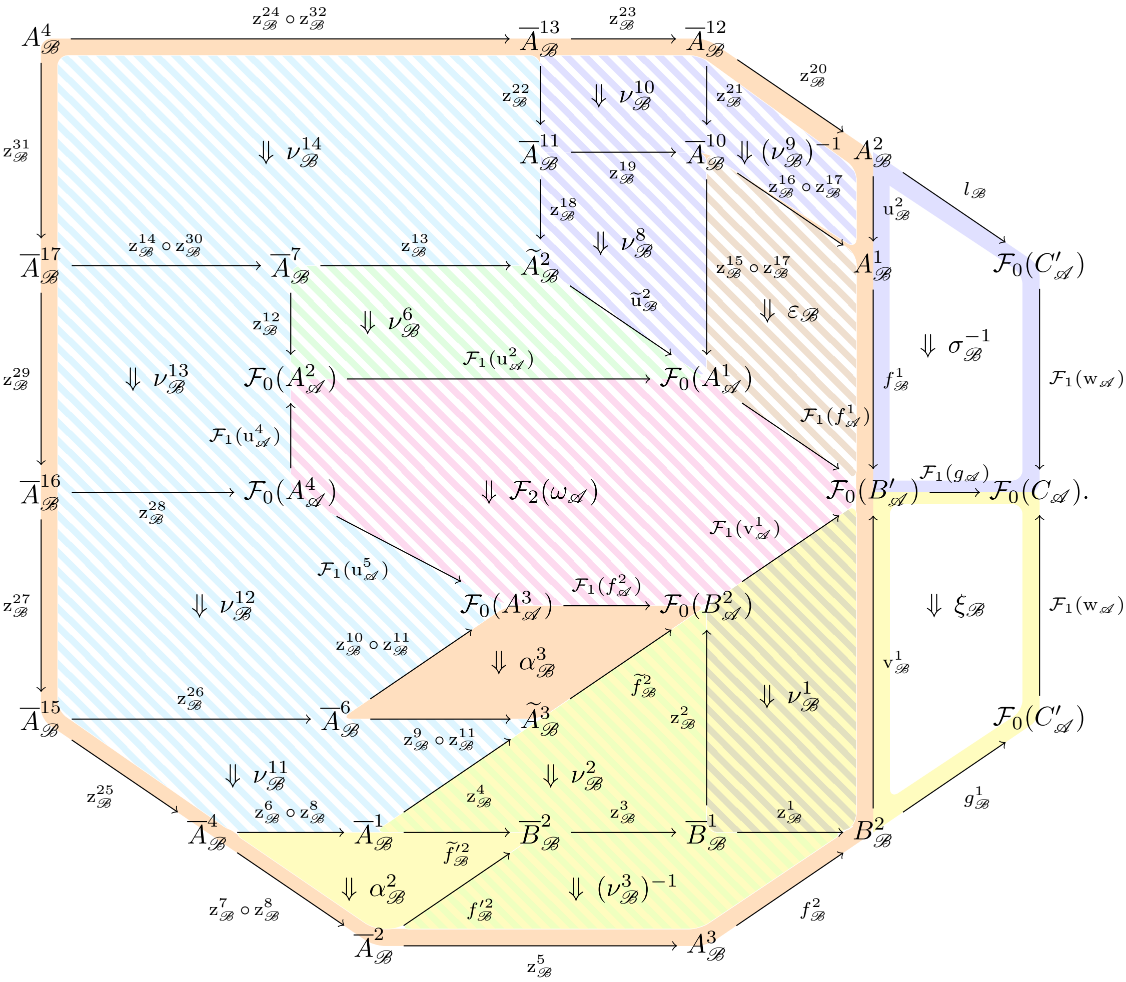
Using also the explicit expressions for the terms

$$i_{\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})} * \alpha^1_{\mathscr{B}}, \quad i_{\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})} * \alpha^4_{\mathscr{B}} \quad \text{and} \quad i_{\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})} * \alpha^5_{\mathscr{B}}$$

computed above in (4.4), (4.11) and (4.14) respectively, in the expression of $i_{\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})}*(4.21)$ we can simplify the following terms:

- $\widetilde{\sigma}_{\mathscr{B}}$ (from (4.14)) with its inverse (from (4.11));
- $\mathcal{F}_2(\xi_{\mathscr{A}})$ (from (4.22)) with its inverse (from (4.4));
- $\mathcal{F}_2(\sigma_{\mathscr{A}})$ (from (4.11)) with its inverse (from (4.22)).

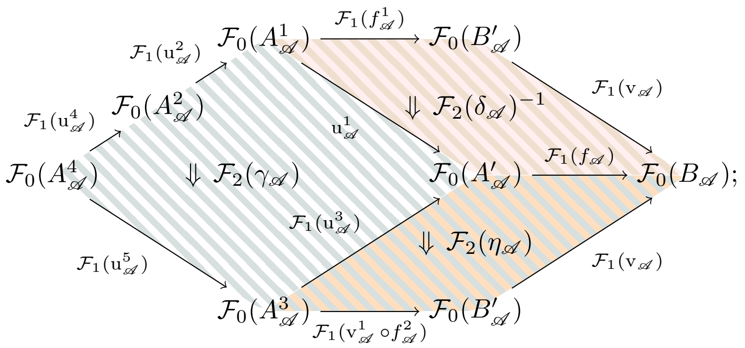
After doing that, we conclude that $i_{\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})} * \rho_{\mathscr{B}} = i_{\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})}*(4.21)$ coincides with the following composition:



(4.23)

We denote by $\omega_{\mathscr{B}}$ the composition of the 2-morphisms bordered in orange (the bigger part of the diagram above). By comparing (4.23) with (1.10) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ (instead of $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$), we conclude that $\rho_{\mathscr{B}}$ **satisfies condition (F3) in the bicategory** $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$ (instead of $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$).

Then we want to verify condition (F2), so we need to compute $i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})} * \omega_{\mathscr{B}}$. In order to do that, we recall that by construction $\omega_{\mathscr{A}}$ satisfies condition (F2) (for $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$), hence the term $i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})} * \mathcal{F}_2(\omega_{\mathscr{A}}) = \mathcal{F}_2(i_{\mathrm{v}_{\mathscr{A}}} * \omega_{\mathscr{A}})$ coincides with the following composition:



(4.24)

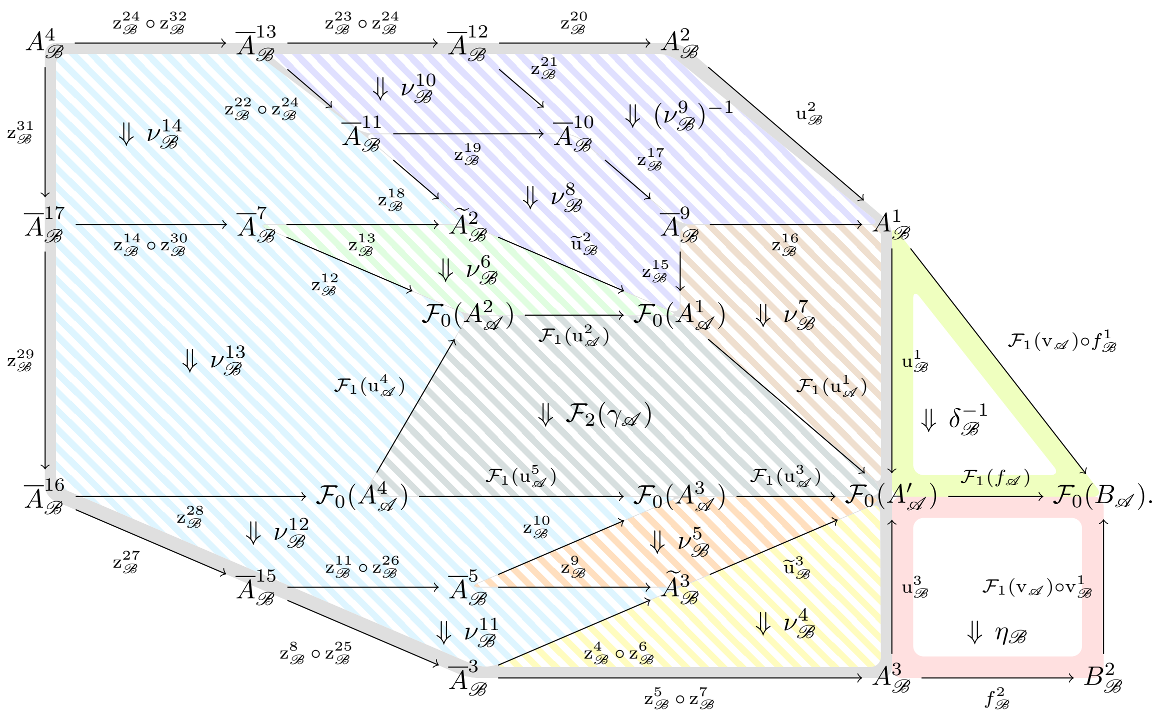
Using also the explicit expressions for

$$i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}}) \circ \mathrm{z}^2_{\mathscr{B}} \circ \mathrm{z}^3_{\mathscr{B}}} * \alpha^2_{\mathscr{B}}, \quad i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}})} * \alpha^3_{\mathscr{B}} \quad \text{and} \quad i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})} * \varepsilon_{\mathscr{B}}$$

computed in (4.6), (4.8) and (4.13) respectively, in the expression of $i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})} * \omega_{\mathscr{B}}$ we can simplify the following terms:

- $\mathcal{F}_2(\delta_{\mathscr{A}})$ (from (4.13)) with its inverse (from (4.24));
- $\mathcal{F}_2(\eta_{\mathscr{A}})$ (from (4.24)) with its inverse (from (4.8));
- $\nu^1_{\mathscr{B}}$ (from $\omega_{\mathscr{B}}$) with its inverse (from (4.6));
- $\nu^2_{\mathscr{B}}$ (from $\omega_{\mathscr{B}}$) with its inverse (from (4.6));
- $\widetilde{\eta}_{\mathscr{B}}$ (from (4.8)) with its inverse (from (4.6));
- $\nu^3_{\mathscr{B}}$ (from (4.6)) with its inverse (from $\omega_{\mathscr{B}}$).

After doing that, we conclude that $i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})} * \omega_{\mathscr{B}}$ coincides with the following composition:



(4.25)

Here the part bordered in gray is again the 2-morphism $\gamma_{\mathscr{B}}$ defined in (4.20). By comparing (4.25) with (1.9) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ (instead of $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$), we conclude that $\omega_{\mathscr{B}}$ **satisfies condition (F2) in the bicategory** $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$ (instead of $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$) if we set:

$$\mathrm{u}^4_{\mathscr{B}} := \mathrm{z}^{20}_{\mathscr{B}} \circ \mathrm{z}^{23}_{\mathscr{B}} \circ \mathrm{z}^{24}_{\mathscr{B}} \circ \mathrm{z}^{32}_{\mathscr{B}} :\ A^4_{\mathscr{B}} \longrightarrow A^2_{\mathscr{B}},$$
$$\mathrm{u}^5_{\mathscr{B}} := \mathrm{z}^{5}_{\mathscr{B}} \circ \mathrm{z}^{7}_{\mathscr{B}} \circ \mathrm{z}^{8}_{\mathscr{B}} \circ \mathrm{z}^{25}_{\mathscr{B}} \circ \mathrm{z}^{27}_{\mathscr{B}} \circ \mathrm{z}^{29}_{\mathscr{B}} \circ \mathrm{z}^{31}_{\mathscr{B}} :\ A^4_{\mathscr{B}} \longrightarrow A^3_{\mathscr{B}}.$$

We recall that we have already defined a pair of invertible 2-morphisms $\gamma_{\mathscr{B}}$ and $\mu_{\mathscr{B}}$ satisfying conditions (F1) and (F3) in the bicategory $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$ (instead of $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$). Then it suffices to apply Proposition 1.1 for $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$ (instead of $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$) in order to conclude that diagram (4.19) is a representative for the associator

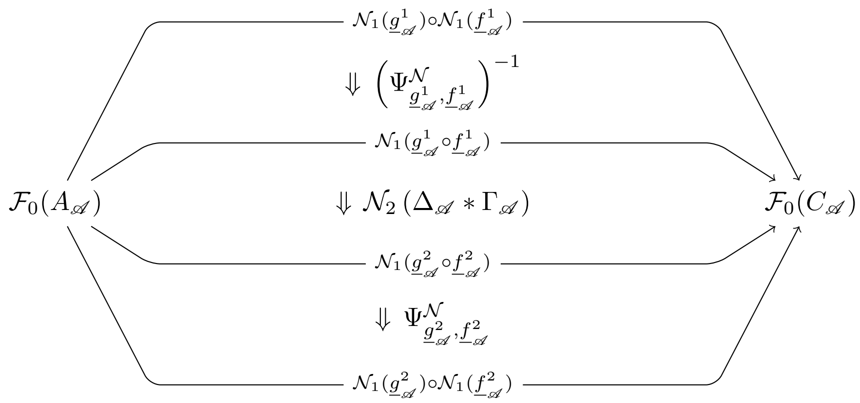
$$\Theta^{\mathscr{B},\mathbf{W}_{\mathscr{B},\mathrm{sat}}}_{\mathcal{N}_1(\underline{h}_{\mathscr{A}}),\mathcal{N}_1(\underline{g}_{\mathscr{A}}),\mathcal{N}_1(\underline{f}_{\mathscr{A}})}.$$

□

5. $\mathcal{N}_2$ IS COMPATIBLE WITH HORIZONTAL COMPOSITIONS

In this section we will prove the following lemma.

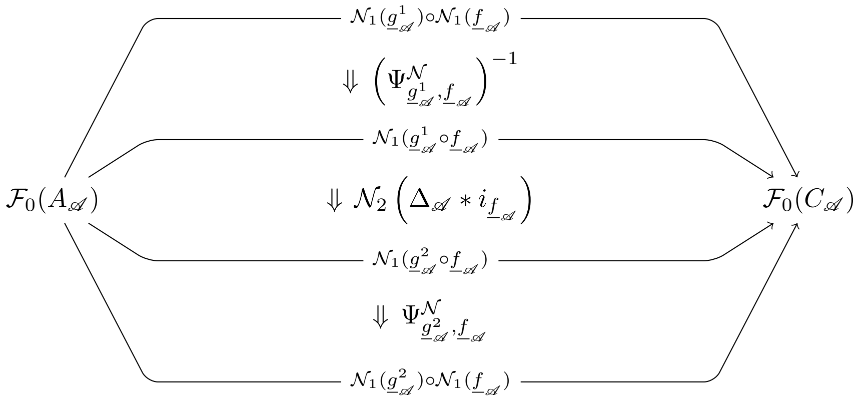
Lemma 5.1. *Let us fix any triple of objects $A_{\mathscr{A}}, B_{\mathscr{A}}, C_{\mathscr{A}}$, any pair of morphisms $\underline{f}^m_{\mathscr{A}} : A_{\mathscr{A}} \to B_{\mathscr{A}}$ for $m = 1,2$ and any pair of morphisms $\underline{g}^m_{\mathscr{A}} : B_{\mathscr{A}} \to C_{\mathscr{A}}$ for $m = 1,2$ in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$. Moreover, let us fix any 2-morphism $\Gamma_{\mathscr{A}} : \underline{f}^1_{\mathscr{A}} \Rightarrow \underline{f}^2_{\mathscr{A}}$ and any morphism 2-morphism $\Delta_{\mathscr{A}} : \underline{g}^1_{\mathscr{A}} \Rightarrow \underline{g}^2_{\mathscr{A}}$. Then the following composition*



coincides with $\mathcal{N}_2(\Delta_{\mathscr{A}}) * \mathcal{N}_2(\Gamma_{\mathscr{A}})$.

For that, let us suppose that we have already proved the following 2 lemmas.

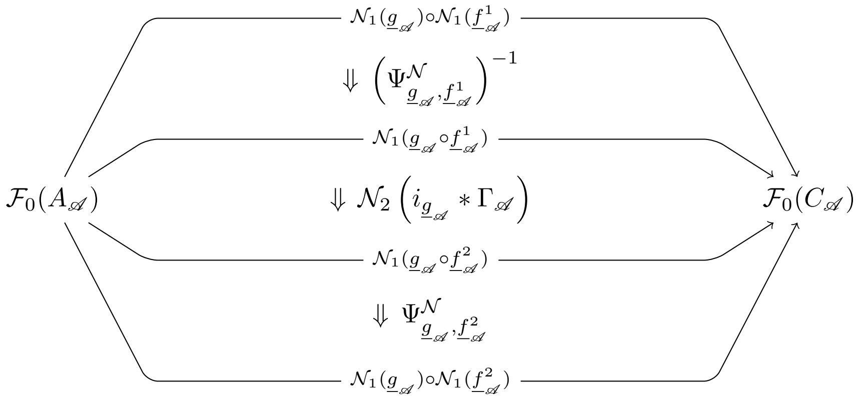
Lemma 5.2. *Let us fix any morphism $\underline{f}_{\mathscr{A}} : A_{\mathscr{A}} \to B_{\mathscr{A}}$, any pair of morphisms $\underline{g}^m_{\mathscr{A}} : B_{\mathscr{A}} \to C_{\mathscr{A}}$ for $m = 1,2$ and any 2-morphism $\Delta_{\mathscr{A}} : \underline{g}^1_{\mathscr{A}} \Rightarrow \underline{g}^2_{\mathscr{A}}$ in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$. Then the following composition*



(5.1)

coincides with $\mathcal{N}_2(\Delta_{\mathscr{A}}) * i_{\mathcal{N}_1(\underline{f}_{\mathscr{A}})}$.

Lemma 5.3. *Let us fix any morphism* $\underline{g}_{\mathscr{A}} : B_{\mathscr{A}} \to C_{\mathscr{A}}$*, any pair of morphisms* $\underline{f}^m_{\mathscr{A}} : A_{\mathscr{A}} \to B_{\mathscr{A}}$ *for* $m = 1, 2$ *and any* 2*-morphism* $\Gamma_{\mathscr{A}} : \underline{f}^1_{\mathscr{A}} \Rightarrow \underline{f}^2_{\mathscr{A}}$ *in* $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$. *Then the following composition*



(5.2)

coincides with $i_{\mathcal{N}_1(\underline{g}_{\mathscr{A}})} * \mathcal{N}_2(\Gamma_{\mathscr{A}})$.

Then Lemma 5.1 follows easily:

- we use Lemma 5.2 with $\underline{f}_{\mathscr{A}} := \underline{f}^1_{\mathscr{A}}$;
- we use Lemma 5.3 with $\underline{g}_{\mathscr{A}} := \underline{g}^2_{\mathscr{A}}$;
- we simplify $\Psi_{\underline{g}^2_{\mathscr{A}},\underline{f}^1_{\mathscr{A}}}$ and its inverse;
- we use Lemma 3.1 and we replace $\mathcal{N}_2(i_{\underline{g}^2_{\mathscr{A}}} * \Gamma_{\mathscr{A}}) \odot \mathcal{N}_2(\Delta_{\mathscr{A}} * i_{\underline{f}^1_{\mathscr{A}}})$ with

$$\mathcal{N}_2\left(\left(i_{\underline{g}^2_{\mathscr{A}}} * \Gamma_{\mathscr{A}}\right) \odot \left(\Delta_{\mathscr{A}} * i_{\underline{f}^1_{\mathscr{A}}}\right)\right) = \mathcal{N}_2\left(\Delta_{\mathscr{A}} * \Gamma_{\mathscr{A}}\right).$$

Therefore, we have only to prove separately Lemmas 5.2 and 5.3. We divide the initial problem in those 2 smaller problems because the definition of horizontal composition in [Pr] is given on diagrams like that, so we can follow closely the prescriptions of [Pr].

5.1. $\mathcal{N}_2$ **is compatible with compositions with** 1**-arrows on the left.** This subsection is devoted to the following:

Proof of Lemma 5.2. First of all, let us fix any representative for $\Delta_{\mathscr{A}}$ as below:

$$
\begin{array}{ccccc}
 & & B^1_{\mathscr{A}} & & \\
 & \swarrow^{\mathrm{v}^1_{\mathscr{A}}} & \uparrow{\scriptstyle \mathrm{u}^1_{\mathscr{A}}} & \searrow^{g^1_{\mathscr{A}}} & \\
B_{\mathscr{A}} & \Downarrow \alpha_{\mathscr{A}} & B^3_{\mathscr{A}} & \Downarrow \beta_{\mathscr{A}} & C_{\mathscr{A}} \\
 & \nwarrow_{\mathrm{v}^2_{\mathscr{A}}} & \downarrow{\scriptstyle \mathrm{u}^2_{\mathscr{A}}} & \nearrow_{g^2_{\mathscr{A}}} & \\
 & & B^2_{\mathscr{A}} & &
\end{array}
$$

and let us suppose that $\underline{f}_{\mathscr{A}} = (A'_{\mathscr{A}}, \mathrm{w}_{\mathscr{A}}, f_{\mathscr{A}})$. Then we suppose that for each $m = 1, 2$, choices $\mathrm{C}(\mathbf{W}_{\mathscr{A}})$ give data as in the upper part of the following diagram, with $\mathrm{v}'^m_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\sigma^m_{\mathscr{A}}$ invertible:

$$
\begin{array}{ccccc}
 & & \overline{A}^m_{\mathscr{A}} & & \\
 & \swarrow^{\mathrm{v}'^m_{\mathscr{A}}} & \sigma^m_{\mathscr{A}} \atop \Rightarrow & \searrow^{f'^m_{\mathscr{A}}} & \\
A'_{\mathscr{A}} & \xrightarrow[f_{\mathscr{A}}]{} & B_{\mathscr{A}} & \xleftarrow[\mathrm{v}^m_{\mathscr{A}}]{} & B^m_{\mathscr{A}},
\end{array}
\tag{5.3}
$$

so that by [Pr, § 2.2] for each $m = 1, 2$ we have:

$$
\underline{g}^m_{\mathscr{A}} \circ \underline{f}_{\mathscr{A}} = \Big(\overline{A}^m_{\mathscr{A}}, \mathrm{w}_{\mathscr{A}} \circ \mathrm{v}'^m_{\mathscr{A}}, g^m_{\mathscr{A}} \circ f'^m_{\mathscr{A}}\Big). \tag{5.4}
$$

Then we use (BF3) for $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ in order to get data as in upper part of the following diagram with $\mathrm{u}'^m_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\gamma^m_{\mathscr{A}}$ invertible for each $m = 1, 2$:

$$
\begin{array}{ccccc}
 & & \widetilde{A}^m_{\mathscr{A}} & & \\
 & \swarrow^{\mathrm{u}'^m_{\mathscr{A}}} & \gamma^m_{\mathscr{A}} \atop \Rightarrow & \searrow^{f''^m_{\mathscr{A}}} & \\
\overline{A}^m_{\mathscr{A}} & \xrightarrow[f'^m_{\mathscr{A}}]{} & B^m_{\mathscr{A}} & \xleftarrow[\mathrm{u}^m_{\mathscr{A}}]{} & B^3_{\mathscr{A}}
\end{array}
\qquad m = 1, 2.
$$

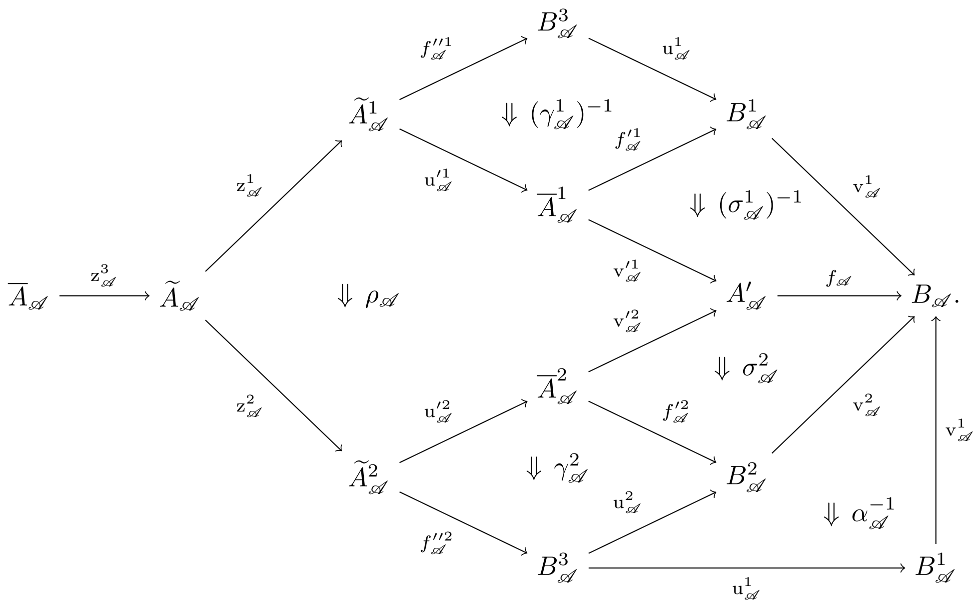
Moreover, we use (BF3) for $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ in order to get data as in the upper part of the following diagram, with $\mathrm{z}^1_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\rho_{\mathscr{A}}$ invertible:

$$
\begin{array}{ccccc}
 & & \widetilde{A}_{\mathscr{A}} & & \\
 & \swarrow^{\mathrm{z}^1_{\mathscr{A}}} & \rho_{\mathscr{A}} \atop \Rightarrow & \searrow^{\mathrm{z}^2_{\mathscr{A}}} & \\
\widetilde{A}^1_{\mathscr{A}} & \xrightarrow[\mathrm{v}'^1_{\mathscr{A}} \circ \mathrm{u}'^1_{\mathscr{A}}]{} & A'_{\mathscr{A}} & \xleftarrow[\mathrm{v}'^2_{\mathscr{A}} \circ \mathrm{u}'^2_{\mathscr{A}}]{} & \widetilde{A}^2_{\mathscr{A}}.
\end{array}
$$

Then we use (BF4a) and (BF4b) for $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ in order to get an object $\overline{A}_{\mathscr{A}}$, a morphism $\mathrm{z}^3_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and an invertible 2-morphism

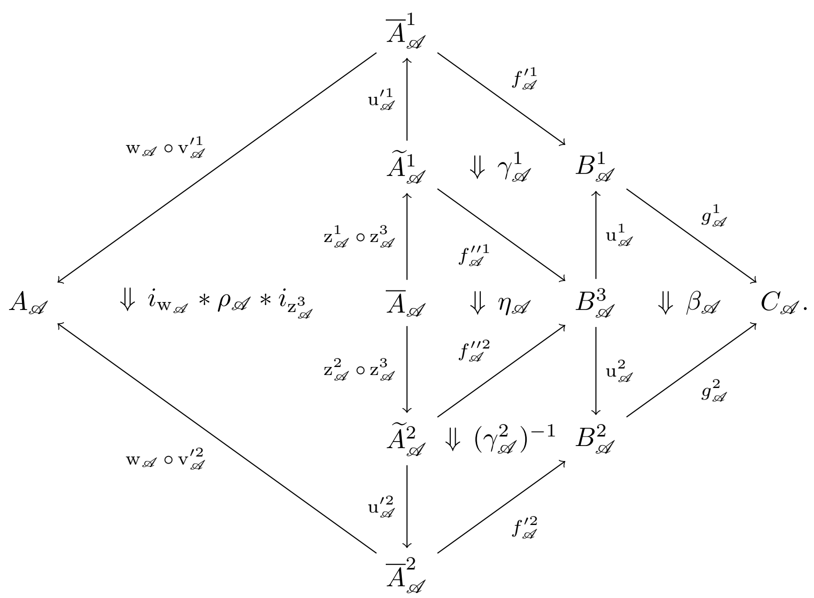
$$\eta_{\mathscr{A}} : f''^1_{\mathscr{A}} \circ \mathrm{z}^1_{\mathscr{A}} \circ \mathrm{z}^3_{\mathscr{A}} \Longrightarrow f''^2_{\mathscr{A}} \circ \mathrm{z}^2_{\mathscr{A}} \circ \mathrm{z}^3_{\mathscr{A}},$$

such that $i_{\mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}^1_{\mathscr{A}}} * \eta_{\mathscr{A}}$ coincides with the following composition:

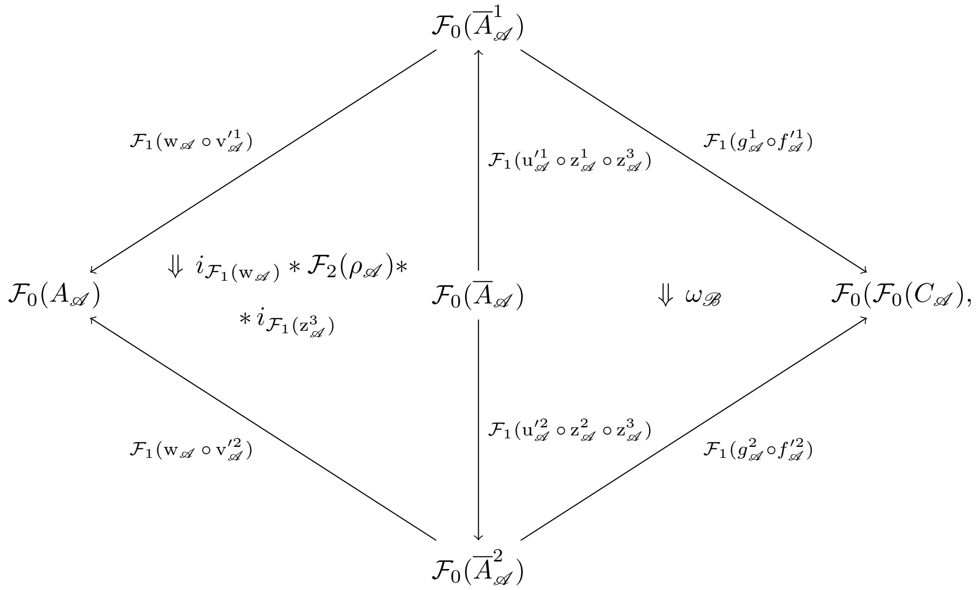


(5.5)

Then by [T1, Proposition 0.3], $\Delta_{\mathscr{A}} * i_{\underline{f}_{\mathscr{A}}}$ is represented by the following diagram:

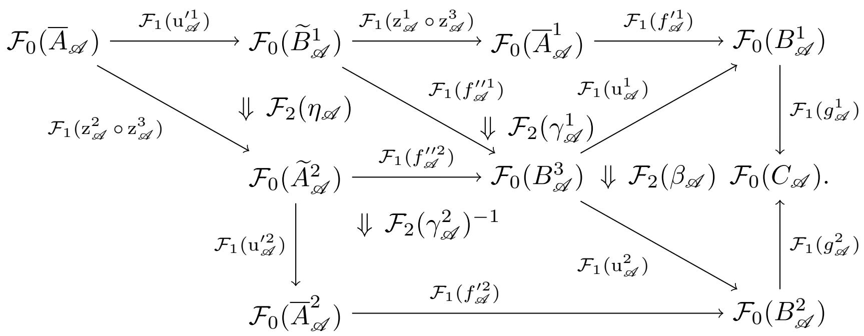


Therefore, $\mathcal{N}_2(\Delta_{\mathscr{A}} * i_{\underline{f}_{\mathscr{A}}})$ is represented by the following diagram:

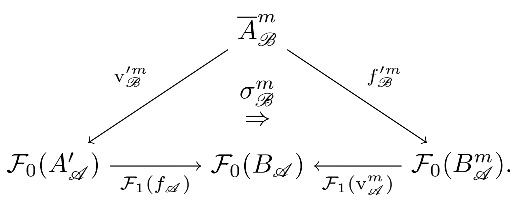


(5.6)

where $\omega_{\mathscr{B}}$ is the following composition



Now we compute $\Psi^{\mathcal{N}}_{\underline{g}^m_{\mathscr{A}}, \underline{f}_{\mathscr{A}}}$ for $m = 1, 2$. We suppose that for each $m = 1, 2$ choices $\mathrm{C}(\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ give data as in the upper part of the following diagram, with $\mathrm{v}'^m_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\sigma^m_{\mathscr{B}}$ invertible:



(5.7)

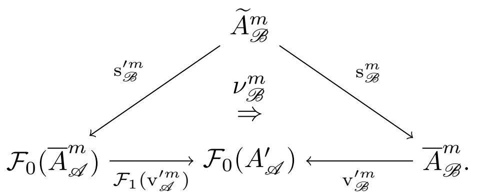
Therefore, from (5.3) we have for each $m = 1, 2$:

$$\mathcal{N}_1\left(\underline{g}^m_{\mathscr{A}} \circ \underline{f}_{\mathscr{A}}\right) := \left(\mathcal{F}_0(A_{\mathscr{A}}) \xleftarrow{\mathcal{F}_1(\mathrm{w}_{\mathscr{A}} \circ \mathrm{v}'^m_{\mathscr{A}})} \mathcal{F}_0(\overline{A}^m_{\mathscr{A}}) \xrightarrow{\mathcal{F}_1(g^m_{\mathscr{A}} \circ f'^m_{\mathscr{A}})} \mathcal{F}_0(C_{\mathscr{A}})\right)$$

and from (5.7) we have for each $m = 1, 2$:

$$\mathcal{N}_1\left(\underline{g}^m_{\mathscr{A}}\right)\circ\mathcal{N}_1\left(\underline{f}_{\mathscr{A}}\right):=\left(\mathcal{F}_0(A_{\mathscr{A}})\xleftarrow{\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})\circ\mathrm{v}'^m_{\mathscr{B}}}\overline{A}^m_{\mathscr{B}}\xrightarrow{\mathcal{F}_1(g^m_{\mathscr{A}})\circ f'^m_{\mathscr{B}}}\mathcal{F}_0(C_{\mathscr{A}})\right).$$

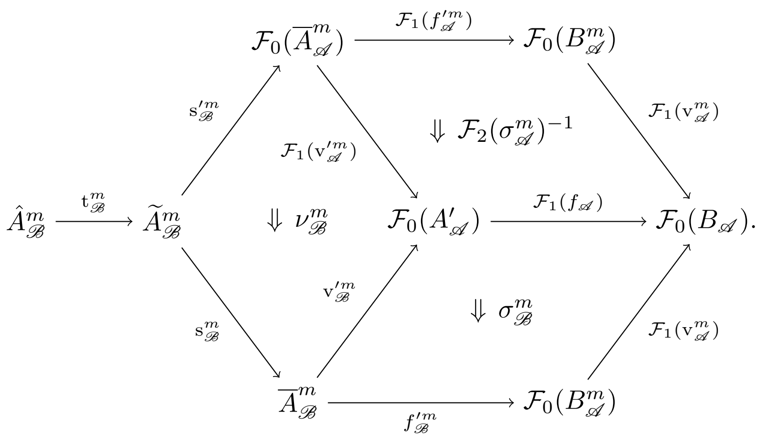
For each $m=1,2$, we use (BF3) for $(\mathscr{B},\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get data as in the upper part of the following diagram, with $\mathrm{s}'^m_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\nu^m_{\mathscr{B}}$ invertible:



Using (BF4a) and (BF4b) for $(\mathscr{B},\mathbf{W}_{\mathscr{B},\mathrm{sat}})$, for each $m=1,2$ there are an object $\hat{A}^m_{\mathscr{B}}$, a morphism $\mathrm{t}^m_{\mathscr{B}}:\hat{A}^m_{\mathscr{B}}\rightarrow\widetilde{A}^m_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and an invertible 2-morphism

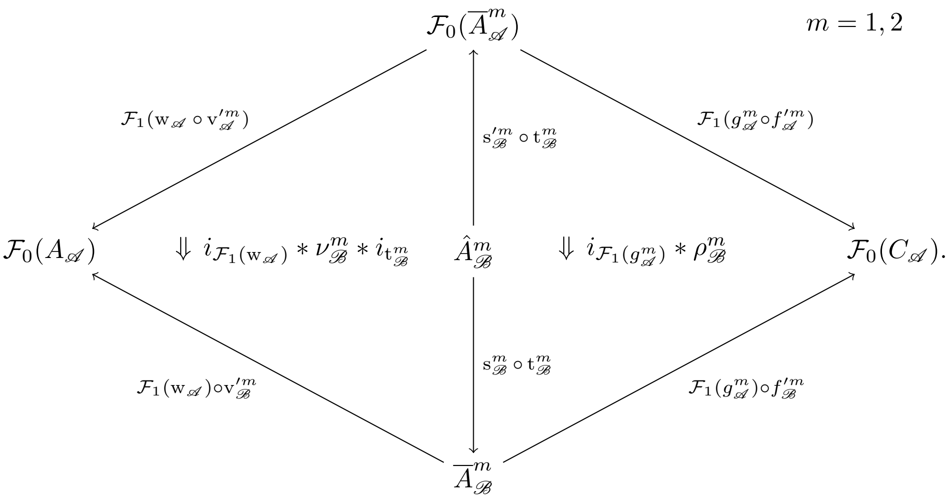
$$\rho^m_{\mathscr{B}}:\,\mathcal{F}_1(f'^m_{\mathscr{A}})\circ\mathrm{s}'^m_{\mathscr{B}}\circ\mathrm{t}^m_{\mathscr{B}}\Longrightarrow f'^m_{\mathscr{B}}\circ\mathrm{s}^m_{\mathscr{B}}\circ\mathrm{t}^m_{\mathscr{B}},$$

such that $i_{\mathcal{F}_1(\mathrm{v}^m_{\mathscr{A}})}*\rho^m_{\mathscr{B}}$ coincides with the following composition:



(5.8)

Since we are assuming for simplicity that $\mathcal{F}$ is a strict pseudofunctor and that $\mathscr{B}$ is a 2-category (instead of a bicategory), then by Lemma 2.1 we get that for each $m=1,2$, $\Psi^{\mathcal{N}}_{\underline{g}^m_{\mathscr{A}},\underline{f}_{\mathscr{A}}}$ is represented by the following diagram:

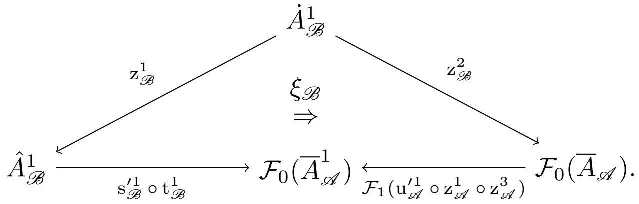


(5.9)

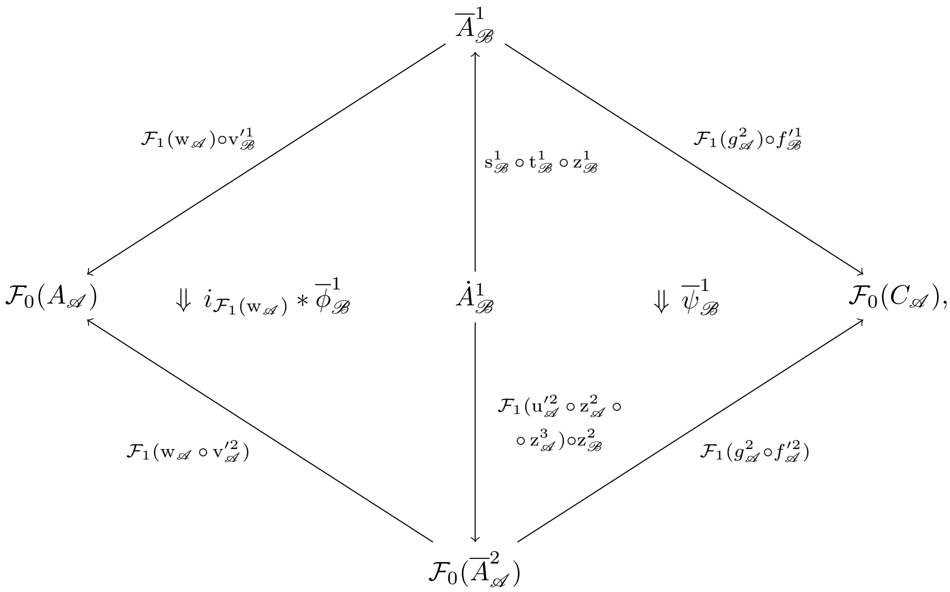
Now we need to compute the composition (5.1). First of all, we compute the composition

$$\mathcal{N}_2(\Delta_{\mathscr{A}} * i_{\underline{f}_{\mathscr{A}}}) \odot \left(\Psi^{\mathcal{N}}_{\underline{g}^1_{\mathscr{A}}, \underline{f}_{\mathscr{A}}}\right)^{-1}. \tag{5.10}$$

For that, we use (BF3) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get data as in the upper part of the following diagram, with $\mathrm{z}^1_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\xi_{\mathscr{B}}$ invertible:

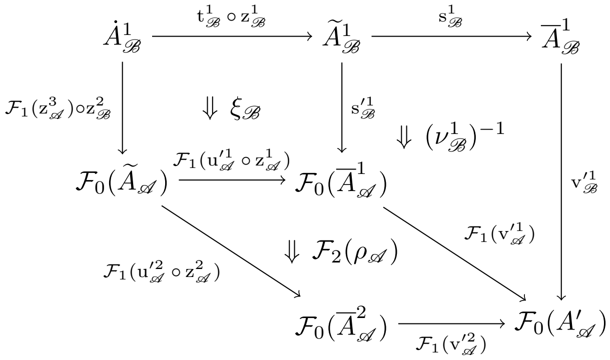


Then using the inverse of (5.9) for $m = 1$, (5.6) and [T1, Proposition 0.2], we get that (5.10) is represented by the following diagram:



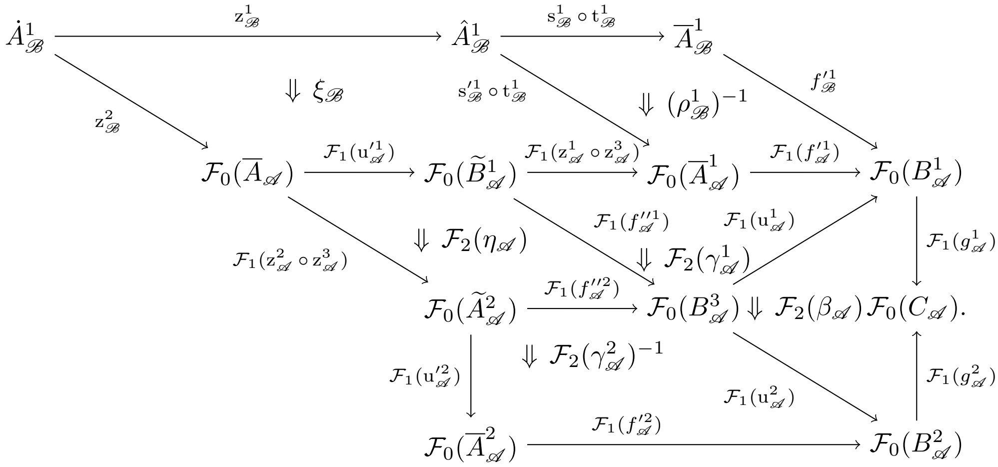
(5.11)

where $\overline{\phi}^1_{\mathscr{B}}$ is given by the following composition

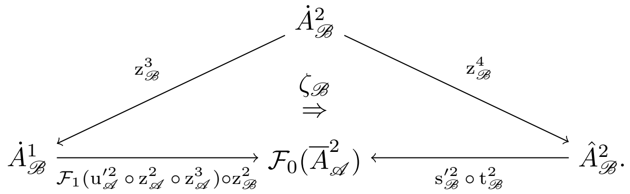


(5.12)

and $\overline{\psi}^1_{\mathscr{B}}$ is given by the following composition:



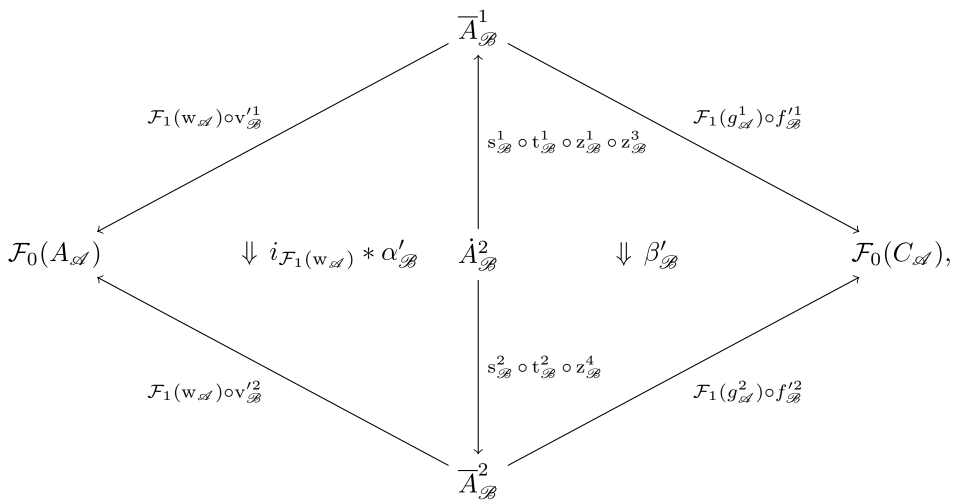
Now we use again (BF3) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get data as in the upper part of the following diagram, with $\mathrm{z}^3_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\zeta_{\mathscr{B}}$ invertible:



Then using (5.9) for $m = 2$, (5.11) and [T1, Proposition 0.2], we get that

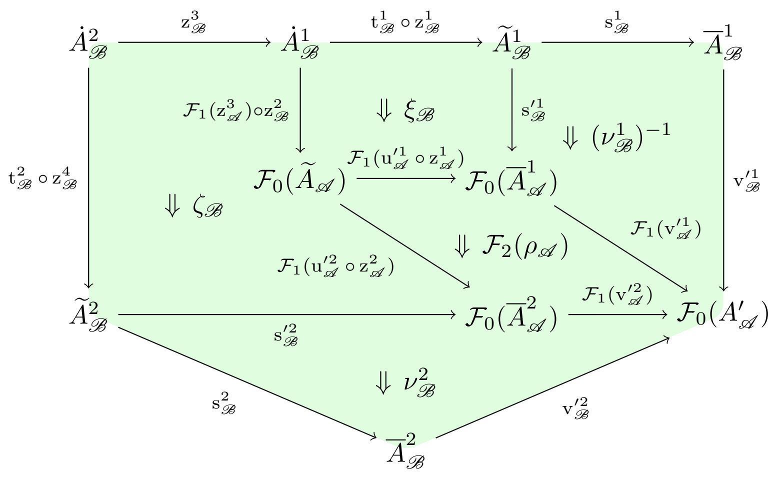
$$\Psi^{\mathcal{N}}_{\underline{g}^2_{\mathscr{A}}, \underline{f}_{\mathscr{A}}} \odot \mathcal{N}_2(\Delta_{\mathscr{A}} * i_{\underline{f}_{\mathscr{A}}}) \odot \left(\Psi^{\mathcal{N}}_{\underline{g}^1_{\mathscr{A}}, \underline{f}_{\mathscr{A}}}\right)^{-1}$$

is represented by the following diagram:



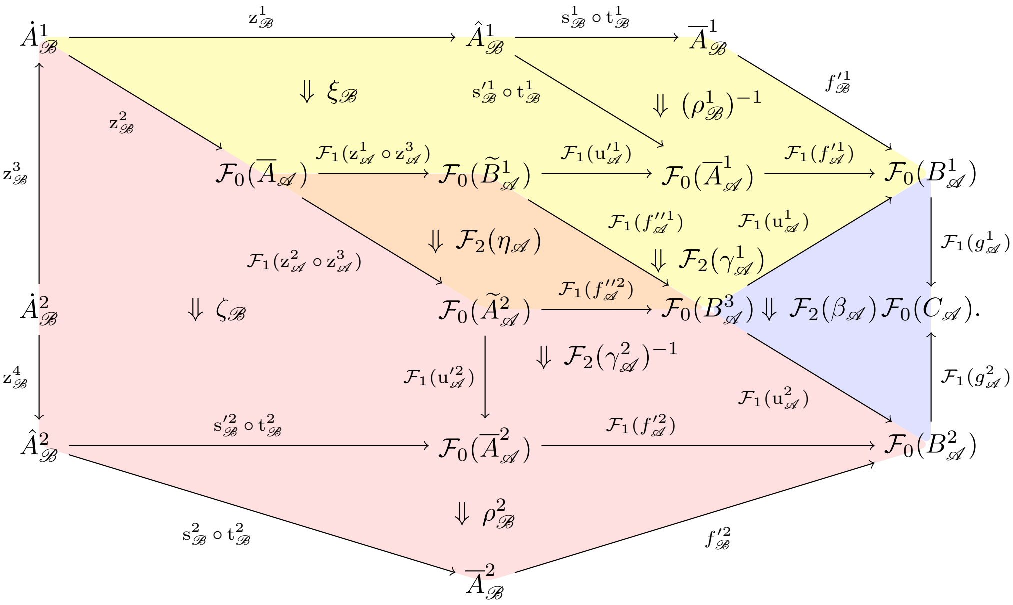
(5.13)

where $\alpha'_{\mathscr{B}}$ is given by the following composition



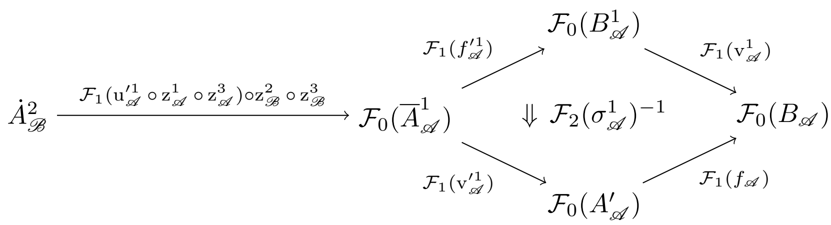
(5.14)

and $\beta'_{\mathscr{B}}$ is given by the following composition:



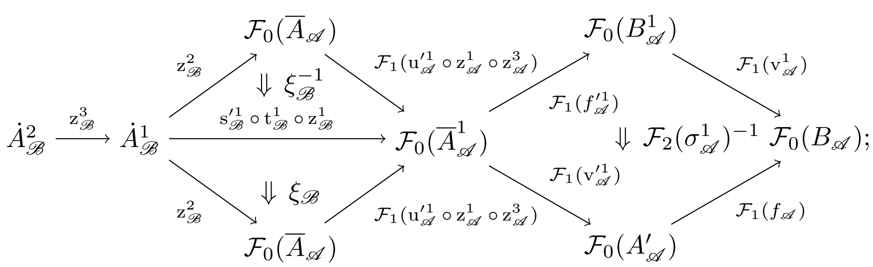
(5.15)

Now we have easily that

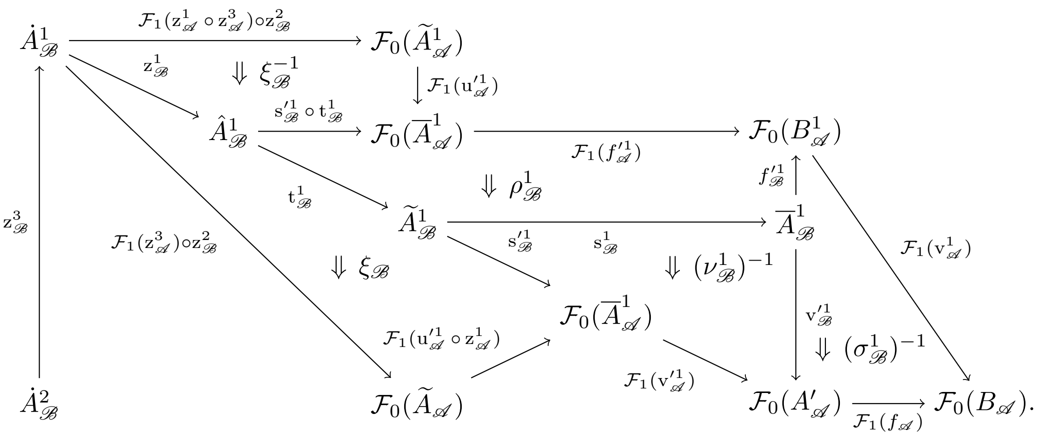


(5.16)

coincides with

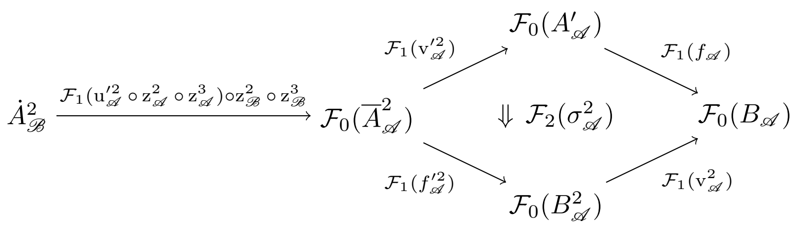


so using (5.8) for $m = 1$, (5.16) coincides with the following composition:



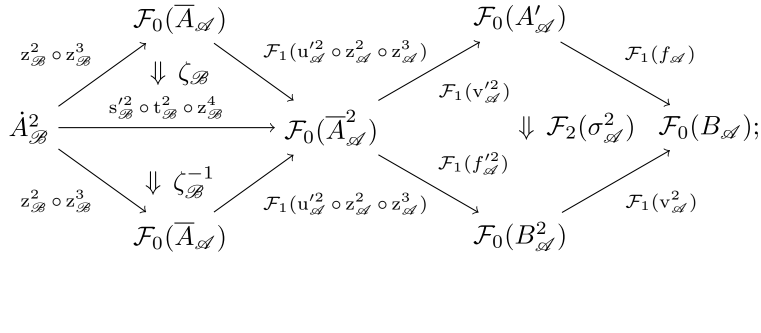
(5.17)

Moreover, we have that

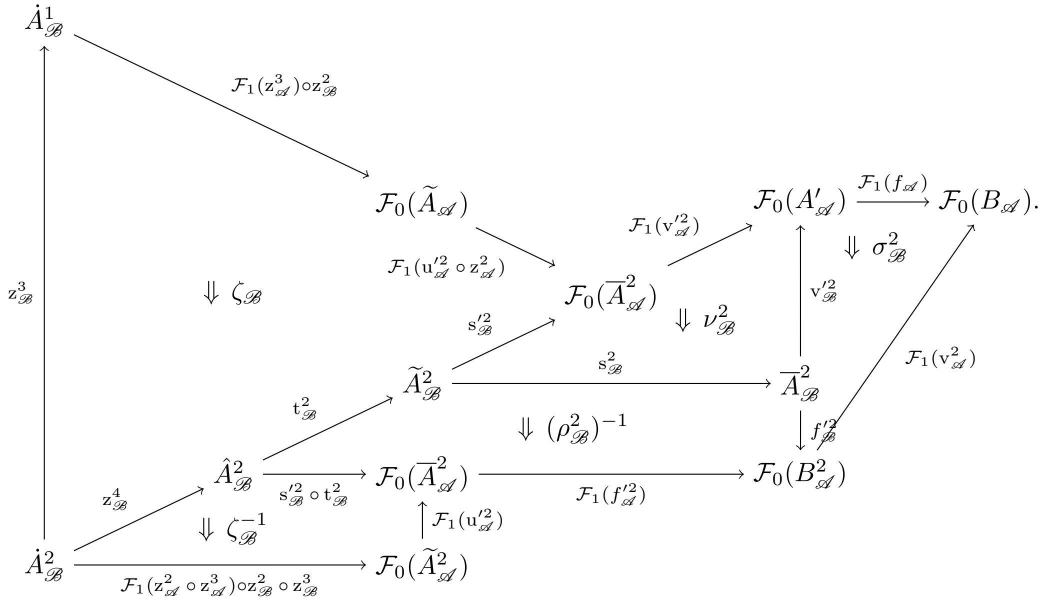


(5.18)

coincides with

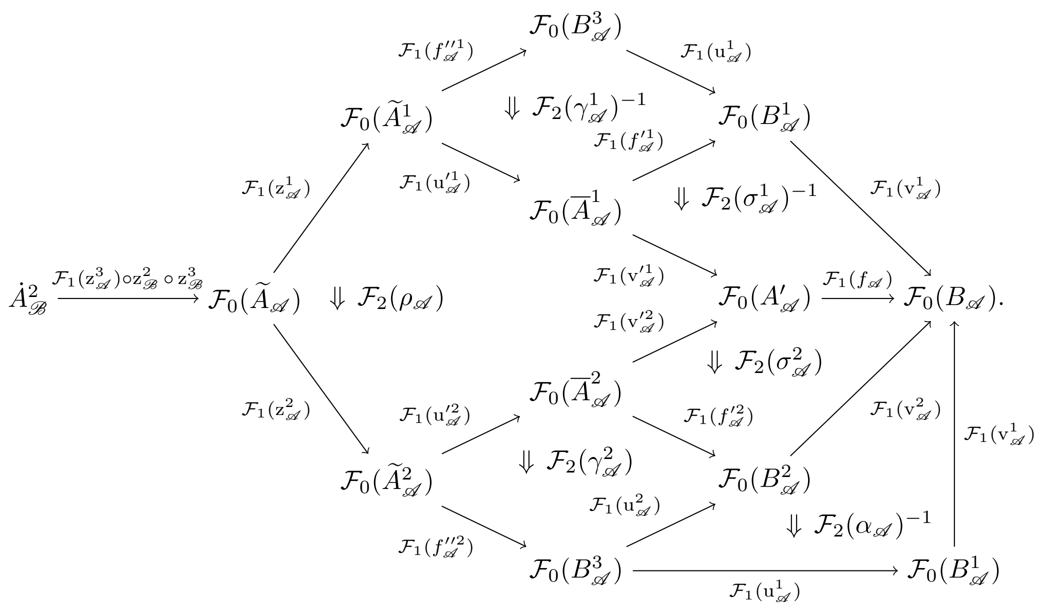


so using (5.8) for $m = 2$, (5.18) coincides with the following composition:

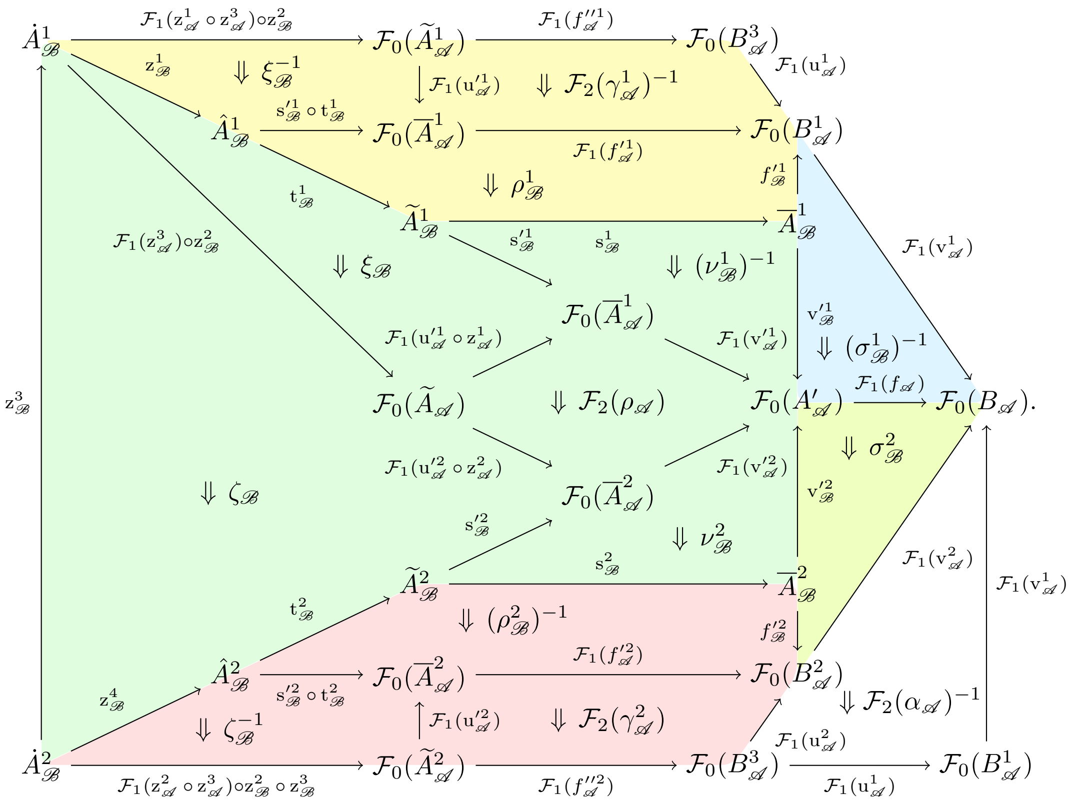


(5.19)

Now from (5.5) we get that $i_{\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}}\circ\mathrm{u}^1_{\mathscr{A}})}*\mathcal{F}_2(\eta_{\mathscr{A}})*i_{\mathrm{z}^2_{\mathscr{B}}\circ\mathrm{z}^3_{\mathscr{B}}}$ coincides with the following composition:



Using the identity of (5.16) with (5.17) and of (5.18) with (5.19), the previous composition is equal to the next one:



(5.20)

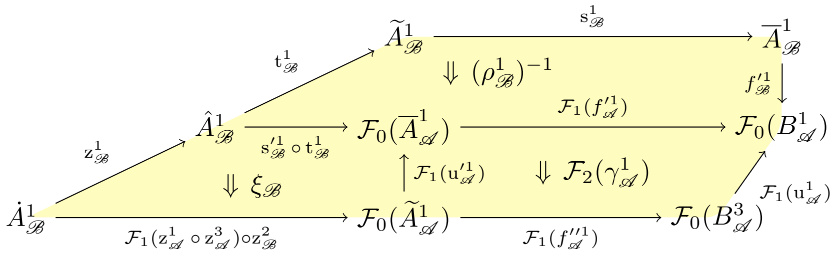
Now we want to apply [T1, Proposition 0.3] for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ and for the composition of $\mathcal{N}_2(\Delta_{\mathscr{A}})$ with

$$i_{\mathcal{N}_1(\underline{f}_{\mathscr{A}})} = i_{\big(\mathcal{F}_0(A'_{\mathscr{A}}),\mathcal{F}_1(\mathrm{w}_{\mathscr{A}}),\mathcal{F}_1(f_{\mathscr{A}})\big)}.$$

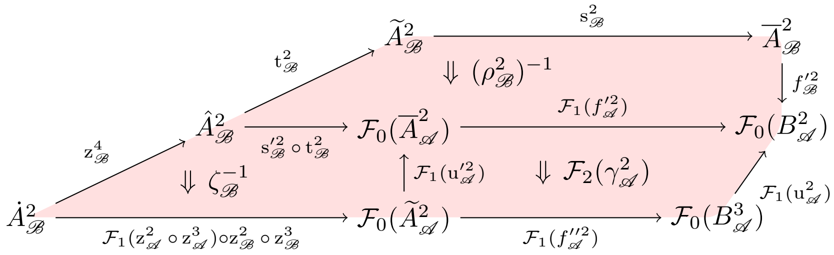
For that, we set:

$$\alpha := \mathcal{F}_2(\alpha_{\mathscr{A}}) \quad \text{and} \quad \beta := \mathcal{F}_2(\beta_{\mathscr{A}})$$

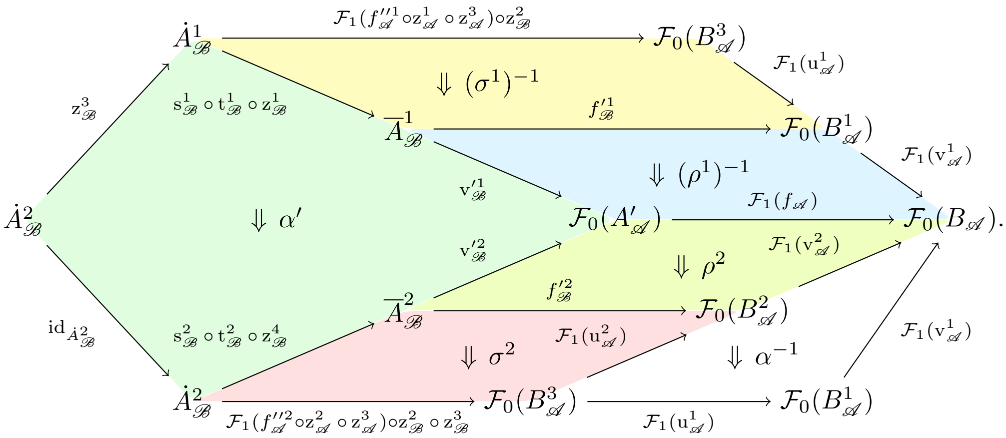
By hypothesis, diagram (5.7) gives the fixed choices $\mathrm{C}(\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ for the pair $(\mathcal{F}_1(f_{\mathscr{A}}),$ $\mathcal{F}_1(\mathrm{v}^m_{\mathscr{A}})$ for each $m = 1, 2$. Therefore, for each $m = 1, 2$, the 2-morphism ρ^m mentioned in [T1, Proposition 0.3] coincides with $\sigma^m_{\mathscr{B}}$. Then we have to fix a set of choices (F5) – (F7) as in [T1, Proposition 0.3]. For that, we define σ^1 as the following composition



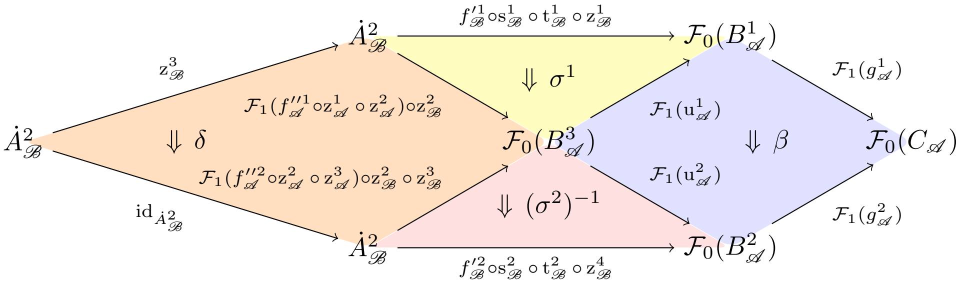
and σ^2 as the following composition



Moreover, we define $\alpha' := \alpha'_{\mathscr{B}}$ (as in (5.15)); lastly we set $\delta := \mathcal{F}_2(\eta_{\mathscr{A}}) * i_{\mathrm{z}^2_{\mathscr{B}} \circ \mathrm{z}^3_{\mathscr{B}}}$. Then (5.20) proves that $i_{\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}^1_{\mathscr{A}})} * \delta$ coincides with the following composition:



Using the previous choices, the 2-morphism β' defined in [T1, Proposition 0.3] as the following composition



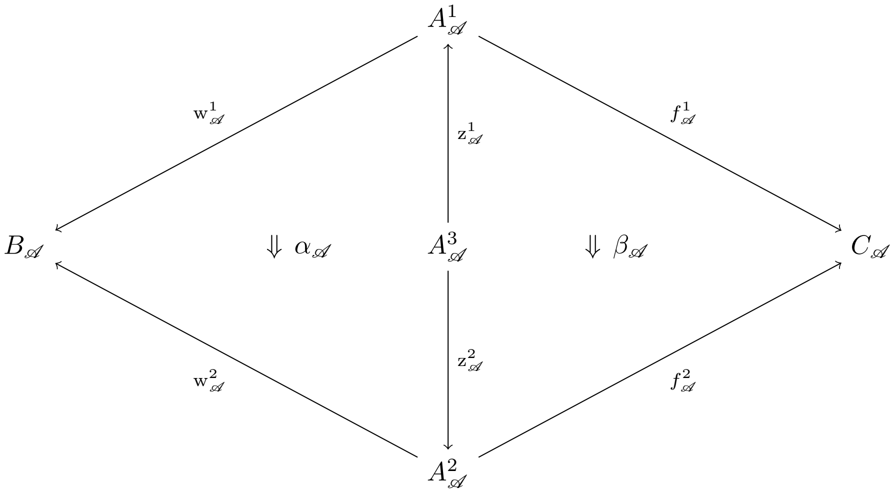
coincides with $\beta'_{\mathscr{B}}$ (see (5.14)). Therefore by [T1, Proposition 0.3] we conclude that (5.13) is a representative for $\mathcal{N}_2(\Delta_{\mathscr{A}}) * i_{\mathcal{N}_1(\underline{f}_{\mathscr{A}})}$, so

$$\mathcal{N}_2(\Delta_{\mathscr{A}}) * i_{\mathcal{N}_1(\underline{f}_{\mathscr{A}})} = \Psi^{\mathcal{N}}_{\underline{g}^2_{\mathscr{A}}, \underline{f}_{\mathscr{A}}} \odot \mathcal{N}_2\left(\Delta_{\mathscr{A}} * i_{\underline{f}_{\mathscr{A}}}\right) \odot \left(\Psi^{\mathcal{N}}_{\underline{g}^1_{\mathscr{A}}, \underline{f}_{\mathscr{A}}}\right)^{-1}.$$

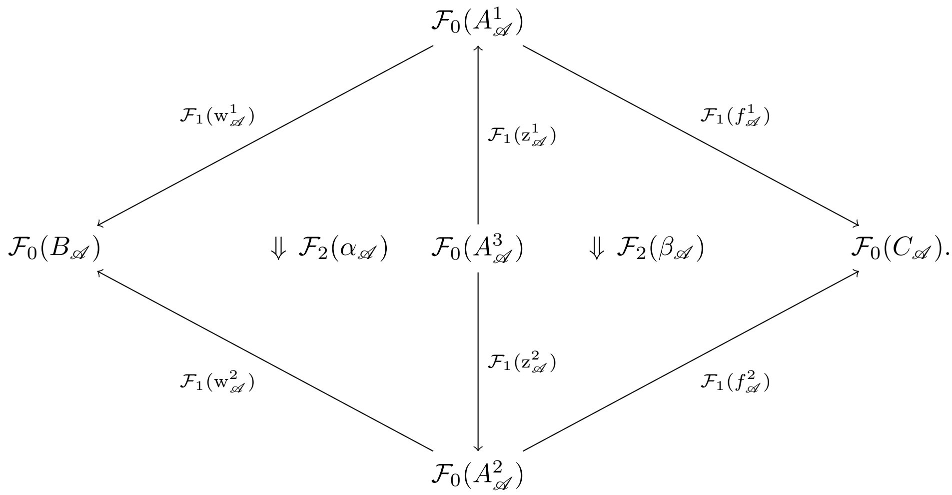
□

5.2. $\mathcal{N}_2$ **is compatible with compositions with** 1**-arrows on the right.** This subsection is devoted to the following

Proof of Lemma 5.3. First of all, let us fix any representative for $\Gamma_{\mathscr{A}}$ as below:

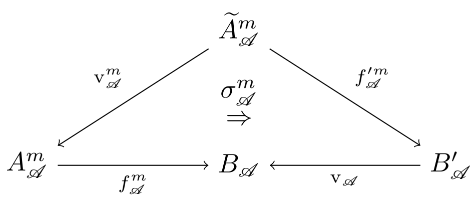


and let us suppose that $\underline{g}_{\mathscr{A}} = (B'_{\mathscr{A}}, \mathrm{v}_{\mathscr{A}}, g_{\mathscr{A}})$; then $\mathcal{N}(\underline{g}_{\mathscr{A}}) = (\mathcal{F}_0(B'_{\mathscr{A}}), \mathcal{F}_1(\mathrm{v}_{\mathscr{A}}), \mathcal{F}_1(g_{\mathscr{A}}))$ and $\mathcal{N}_2(\Gamma_{\mathscr{A}})$ is represented by the following diagram:



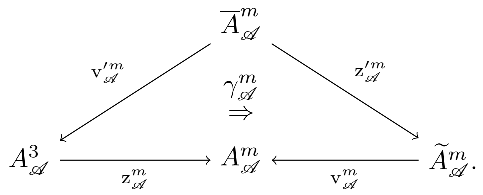
(5.21)

Now we suppose that choices $\mathrm{C}(\mathbf{W}_{\mathscr{A}})$ give data as in the upper part of the following diagram, with $\mathrm{v}^m_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\sigma^m_{\mathscr{A}}$ invertible for each $m = 1, 2$

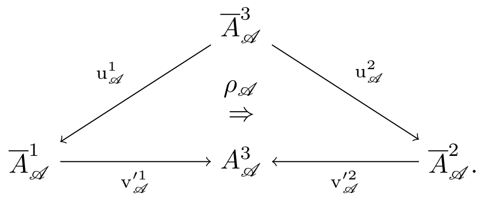


$m = 1, 2.$

By (BF3) for $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$, there are data as in the upper part of the following diagram, with $\mathrm{v}'^m_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\gamma^m_{\mathscr{A}}$ invertible for each $m = 1, 2$:



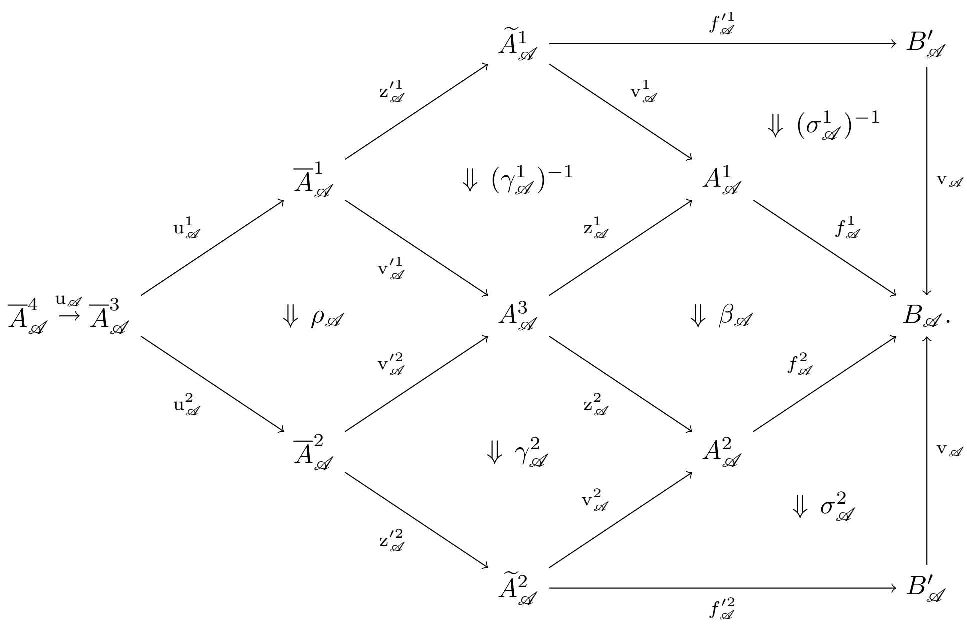
Again by (BF3) for $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$, there are data as in the upper part of the following diagram, with $\mathrm{u}^1_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\rho_{\mathscr{A}}$ invertible:



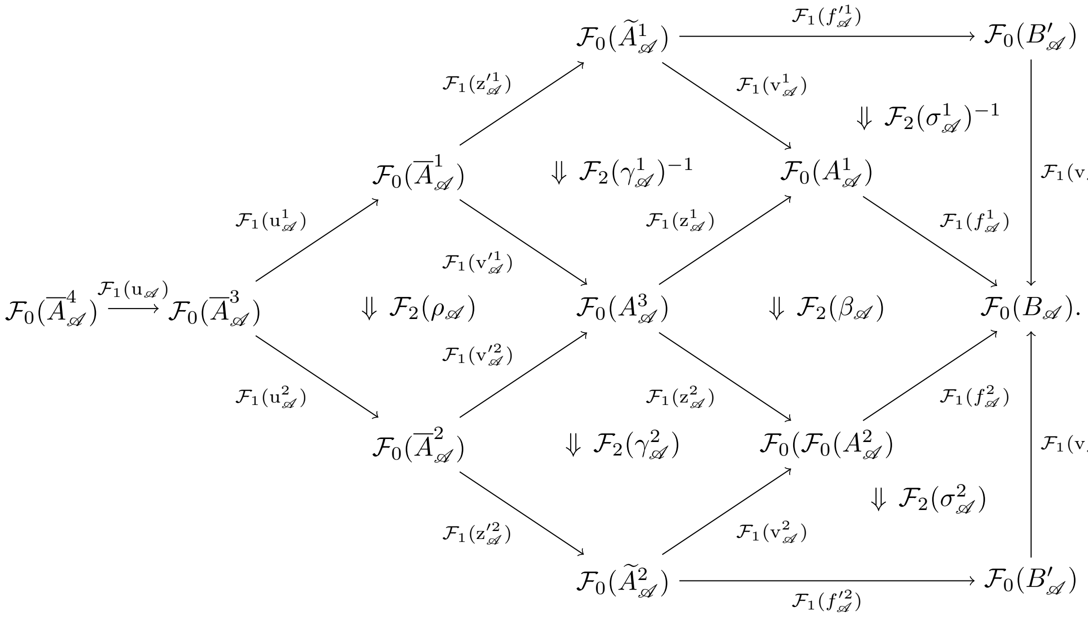
Using (BF4a) for $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$, we get an object $\overline{A}^4_{\mathscr{A}}$, a morphism $\mathrm{u}_{\mathscr{A}} : \overline{A}^4_{\mathscr{A}} \to \overline{A}^3_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and a 2-morphism

$$\eta_{\mathscr{A}} : f'^1_{\mathscr{A}} \circ \mathrm{z}'^1_{\mathscr{A}} \circ \mathrm{u}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}} \Longrightarrow f'^2_{\mathscr{A}} \circ \mathrm{z}'^2_{\mathscr{A}} \circ \mathrm{u}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}$$

(in general not invertible), such that $i_{\mathrm{v}_{\mathscr{A}}} * \eta_{\mathscr{A}}$ coincides with the following composition:

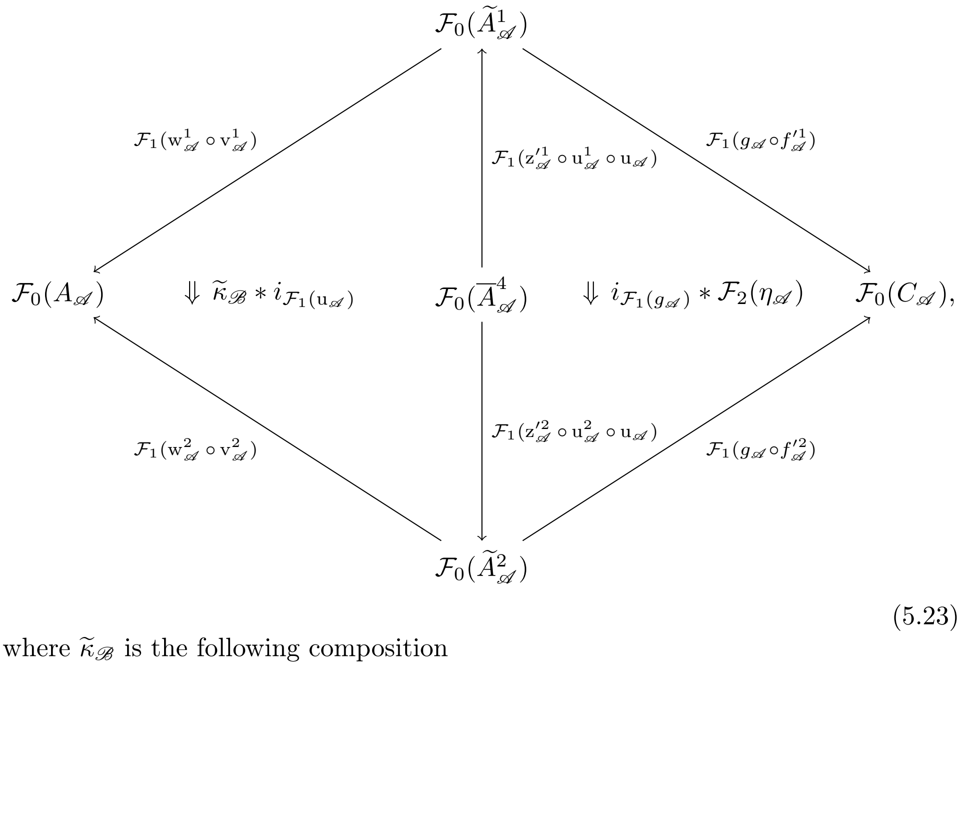


Then $i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})} * \mathcal{F}_2(\eta_{\mathscr{A}}) = \mathcal{F}_2(i_{\mathrm{v}_{\mathscr{A}}} * \eta_{\mathscr{A}})$ coincides with the following composition



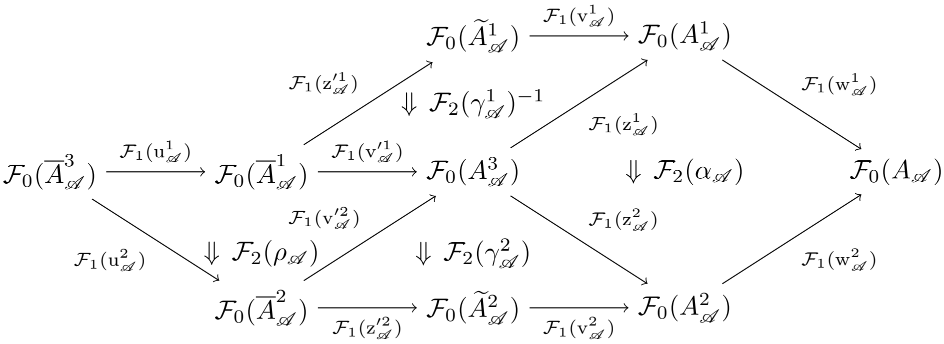
(5.22)

Moreover, by [T1, Proposition 0.4] and (2.1), $\mathcal{N}_2(i_{\underline{g}_{\mathscr{A}}} * \Gamma_{\mathscr{A}})$ is represented by the following diagram:

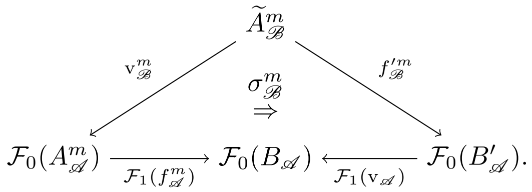


(5.23)

where $\widetilde{\kappa}_{\mathscr{B}}$ is the following composition

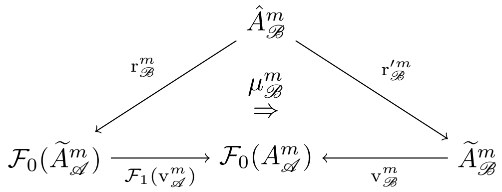


Now we want to compute $\Psi^{\mathcal{N}}_{\underline{g}_{\mathscr{A}},\underline{f}^m_{\mathscr{A}}}$ for $m = 1, 2$. Following the description leading to (2.2), first of all we suppose that choices $\mathrm{C}(\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ give data as in the upper part of the following diagram, with $\mathrm{v}^m_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\sigma^m_{\mathscr{B}}$ invertible for each $m = 1, 2$:



(5.24)

Then for each $m = 1, 2$ we use choices (BF3) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to obtain data as in the upper part of the following diagram, with $\mathrm{r}^m_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\mu^m_{\mathscr{B}}$ invertible:

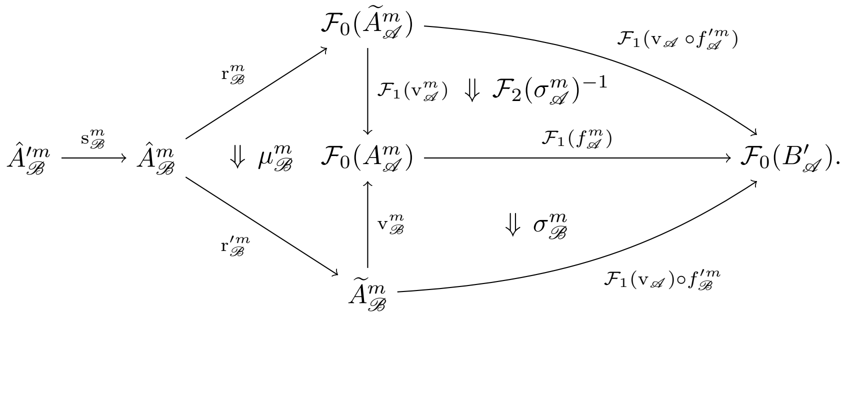


$m = 1, 2.$

Using (BF4a) and (BF4b) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$, for each $m = 1, 2$ there are an object $\hat{A}'^m_{\mathscr{B}}$, a morphism $\mathrm{s}^m_{\mathscr{B}} : \hat{A}'^m_{\mathscr{B}} \to \hat{A}^m_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and an invertible 2-morphism

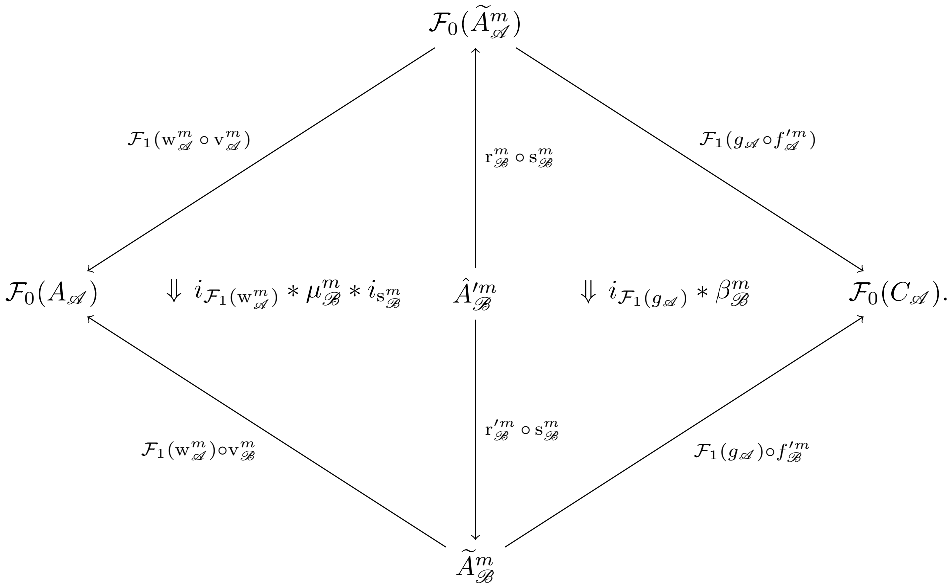
$$\beta^m_{\mathscr{B}} : \mathcal{F}_1(f'^m_{\mathscr{A}}) \circ \mathrm{r}^m_{\mathscr{B}} \circ \mathrm{s}^m_{\mathscr{B}} \Longrightarrow f'^m_{\mathscr{B}} \circ \mathrm{r}'^m_{\mathscr{B}} \circ \mathrm{s}^m_{\mathscr{B}},$$

such that $i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})} * \beta^m_{\mathscr{B}}$ coincides with the following composition:



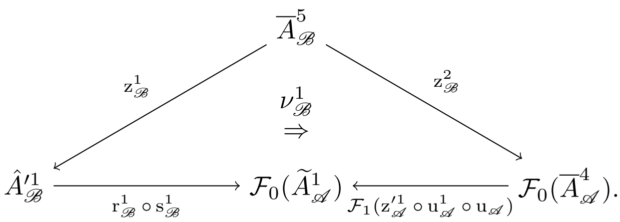
(5.25)

Then by Lemma 2.1, for each $m = 1, 2$ the 2-morphism $\Psi^{\mathcal{N}}_{\underline{g}_{\mathscr{A}}, \underline{f}^m_{\mathscr{A}}}$ is represented by the following diagram:



(5.26)

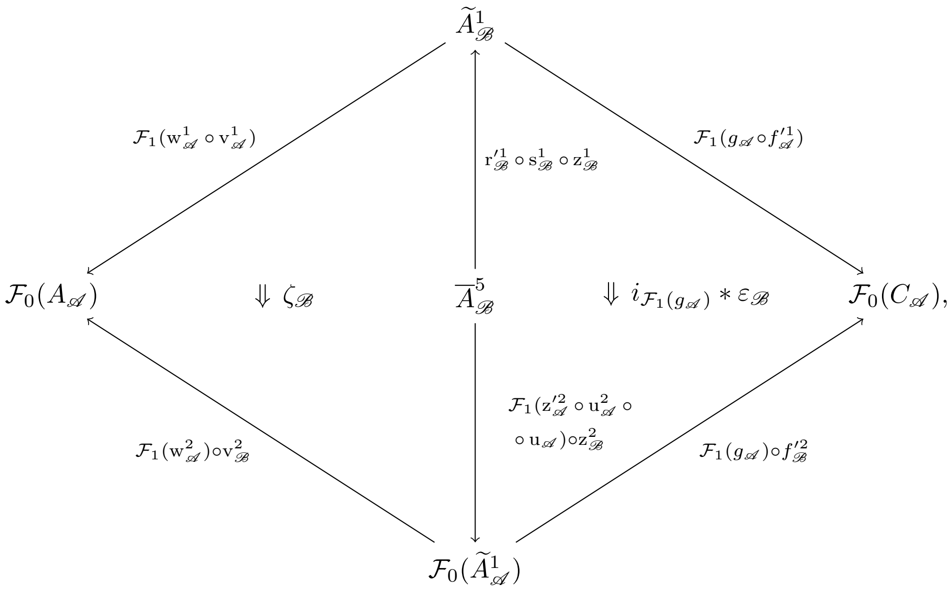
Now we use (BF3) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get data as in the upper part of the following diagram, with $\mathrm{z}^1_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\nu^1_{\mathscr{B}}$ invertible:



Then using (5.23), the inverse of (5.26) for $m = 1$ and [T1, Proposition 0.2], we get that

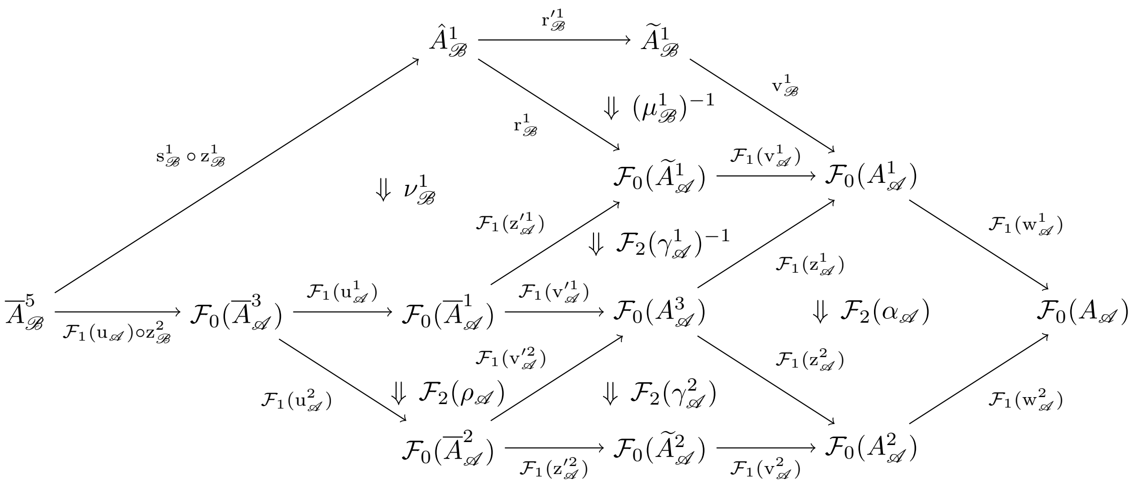
$$\mathcal{N}_2\left(i_{\underline{g}_{\mathscr{A}}} * \Gamma_{\mathscr{A}}\right) \odot \left(\Psi^{\mathcal{N}}_{\underline{g}_{\mathscr{A}}, \underline{f}^1_{\mathscr{A}}}\right)^{-1}$$

is represented by the following diagram:

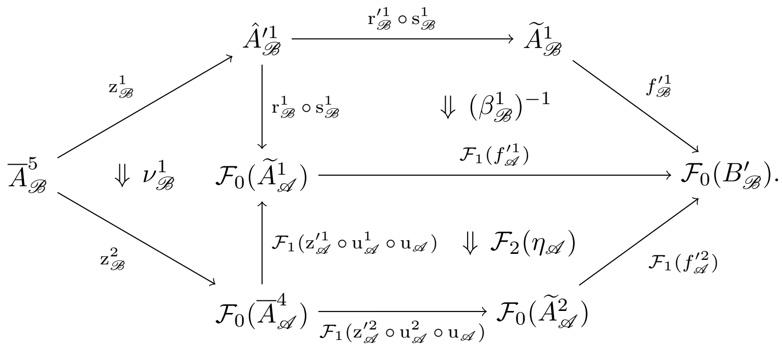


(5.27)

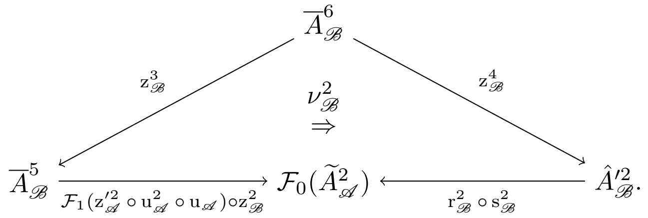
where $\zeta_{\mathscr{B}}$ is the following composition:



and $\varepsilon_{\mathscr{B}}$ is the following composition:



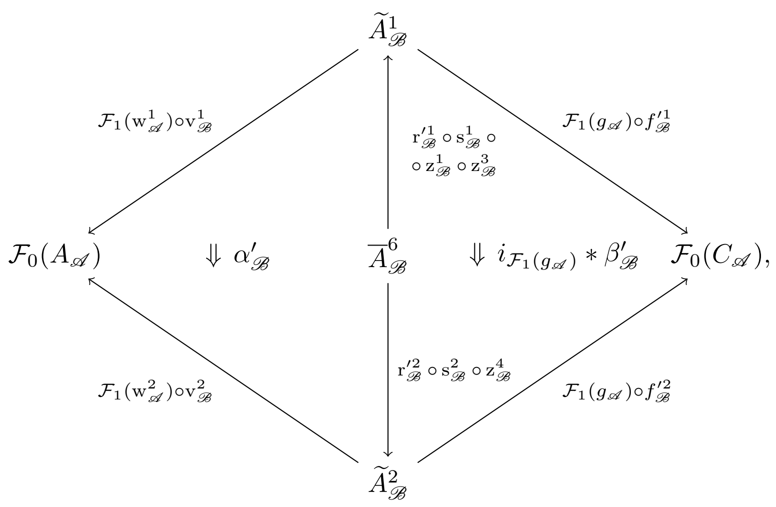
Now we use again (BF3) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ in order to get data as in the upper part of the following diagram, with $\mathrm{z}^3_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\nu^2_{\mathscr{B}}$ invertible:



Then using (5.27), (5.26) for $m = 2$ and [T1, Proposition 0.2], we get that

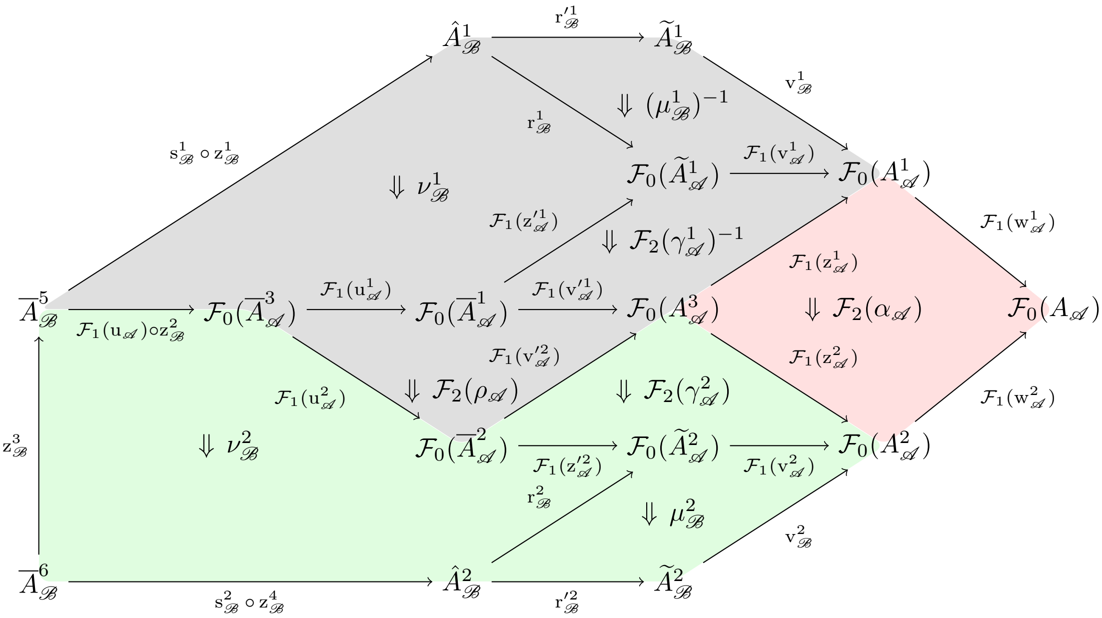
$$\Psi^{\mathcal{N}}_{\underline{g}_{\mathscr{A}}, \underline{f}^2_{\mathscr{A}}} \odot \mathcal{N}_2 \Big(i_{\underline{g}_{\mathscr{A}}} * \Gamma_{\mathscr{A}} \Big) \odot \Big(\Psi^{\mathcal{N}}_{\underline{g}_{\mathscr{A}}, \underline{f}^1_{\mathscr{A}}} \Big)^{-1}$$

is represented by the following diagram:



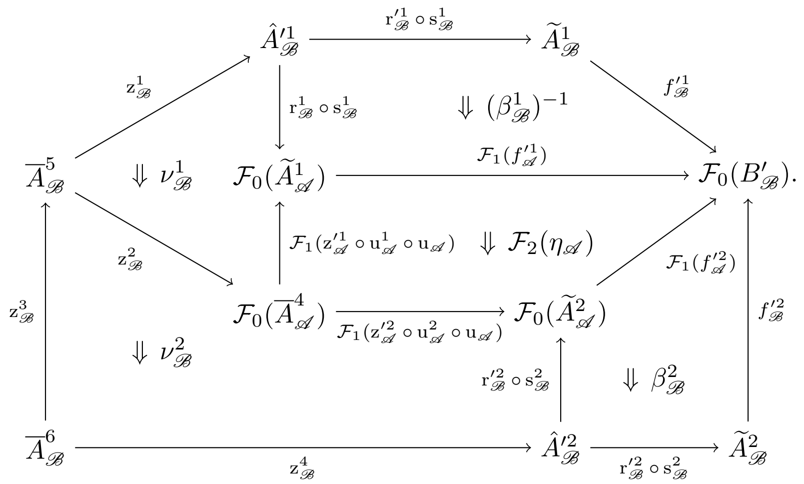
(5.28)

where $\alpha'_{\mathscr{B}}$ is the following composition



(5.29)

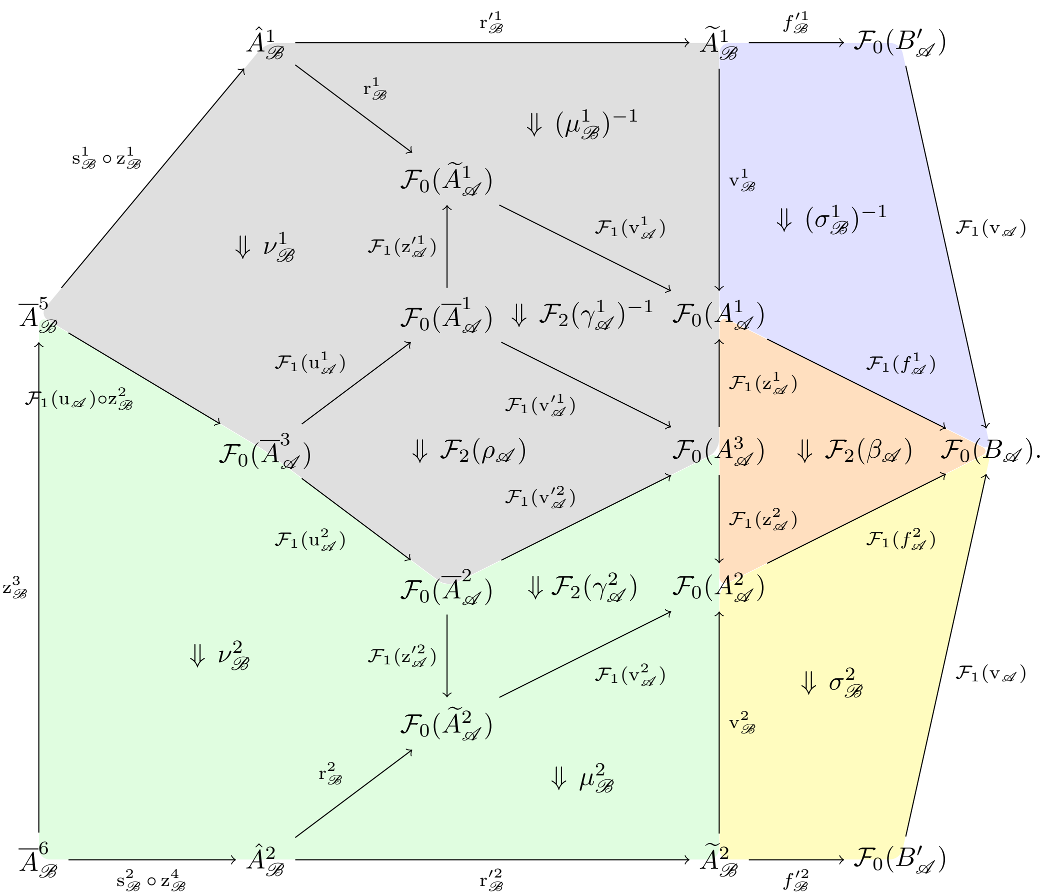
and $\beta'_{\mathscr{B}}$ is the following composition:



Now we want to compute $i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})} * \beta'_{\mathscr{B}}$. In order to do that,

- we replace $i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})} * (\beta^1_{\mathscr{B}})^{-1}$ with the inverse of diagram (5.25) for $m = 1$;
- we replace $i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})} * \mathcal{F}_2(\eta_{\mathscr{A}})$ with diagram (5.22);
- we replace $i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})} * \beta^2_{\mathscr{B}}$ with diagram (5.25) for $m = 2$;
- we simplify $\mathcal{F}_2(\sigma^2_{\mathscr{A}})$ (from (5.22)) with its inverse (from diagram (5.25) for $m = 2$) ;
- we simplify $\mathcal{F}_2(\sigma^1_{\mathscr{A}})$ (from the inverse of diagram (5.25) for $m = 1$) with its inverse (from (5.22)).

So we get that $i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})} * \beta'_{\mathscr{B}}$ coincides with the following composition:



(5.30)

Now we want to apply [T1, Proposition 0.4] for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ and for the data

$$\alpha := \mathcal{F}_2(\alpha_{\mathscr{A}}), \quad , \beta := \mathcal{F}_2(\beta_{\mathscr{A}}).$$

For that, we use the fact that (5.24) gives choices $\mathrm{C}(\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ for the pair $(\mathcal{F}_1(f^m_{\mathscr{A}}), \mathcal{F}_1(\mathrm{v}_{\mathscr{A}}))$ for each $m = 1, 2$, so we have to set $\rho^m := \sigma^m_{\mathscr{B}}$ for each $m = 1, 2$. Moreover, we choose η^1 as the inverse of the composition of the 2-morphisms in gray above, and η^2 as the composition of the 2-morphisms in green above. In addition, we choose η^3 as a 2-identity. Then we are exactly in the setup of [T1, Proposition 0.4] if we set $\alpha' := \alpha'_{\mathscr{B}}$ and $\beta' := \beta'_{\mathscr{B}}$. So by that result we conclude that (5.28) represents $i_{\mathcal{N}_1(\underline{g}_{\mathscr{A}})} * \mathcal{N}_2(\Gamma_{\mathscr{A}})$, hence

$$\Psi^{\mathcal{N}}_{\underline{g}_{\mathscr{A}}, \underline{f}^2_{\mathscr{A}}} \odot \mathcal{N}_2\left(i_{\underline{g}_{\mathscr{A}}} * \Gamma_{\mathscr{A}}\right) \odot \left(\Psi^{\mathcal{N}}_{\underline{g}_{\mathscr{A}}, \underline{f}^1_{\mathscr{A}}}\right)^{-1} = i_{\mathcal{N}_1(\underline{g}_{\mathscr{A}})} * \mathcal{N}_2(\Gamma_{\mathscr{A}}).$$

□

References

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[T1] Matteo Tommasini, *Some insights on bicategories of fractions - I*, (2014), arXiv: math.CT 1410.3990v2;

[T2] Matteo Tommasini, *Some insights on bicategories of fractions - II*, (2014) arXiv: math.CT 1410.5075v2.

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